STABLE NORM ON HOMOLOGY AND GEODESICS ON TRANSLATION SURFACES

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Comunicación Técnica No I-02-18/03-09-2002
(MB/CIMAT)

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ABSTRACT. We study the stable norm on the homology of a closed oriented surface endowed with a possibly singular Riemannian metric. We apply our results to the asymptotic counting of the simple homology classes and the parallel bands of closed geodesics on translation surfaces.

1. INTRODUCTION

The stable norm on the homology of a manifold depends on the choice of a Riemannian metric [Fe 69, G-L-P 81]. It has been extensively used in geometry and analysis. See [Ba 94, BI94, McS-R 95 I, Mt 97]. For convenience of the reader, we briefly recall the basic definitions.

Let (M, g) be a Riemannian manifold. We denote by $\ell(\gamma)$ the length, with respect to the metric g, of a connected, rectifiable curve. By multicurves we will mean formal linear combinations of curves. If $\gamma = \sum_i r_i\gamma_i$ is a multicurve, we set $\|\gamma\| = \sum_i |r_i|\ell(\gamma_i)$. Let $h \in H_1(M, \mathbb{Z})$ be a homology class. Its stable norm is $\|h\| = \min_\gamma \|\gamma\|$, where γ runs through the geodesic multicurves in the class h.

An alternative definition of the stable norm is due to H. Federer [Fe 69]. It is based on the notion of the mass of a Lipschitz current. The stable norm of $h \in H_1(M, \mathbb{R})$ is then the minimal mass of a Lipschitz current in the homology class h. Federer's approach allows to minimize over more general objects than the multicurves or laminations. However, J. Mather proved that the minimizing currents are supported by geodesic laminations. See [M 91].

Since the stable norm is defined by the length, it makes sense for the singular Riemannian metrics. The latter arise in several contexts. In particular, flat Riemannian surfaces with singular points, and their geodesic flows, are closely related to the billiards in polygons. See [GJ 00] and the bibliography there.

The theme of the present work is the stable norm on possibly singular Riemannian surfaces. Our goal is to use the stable norm to study the closed geodesics on certain special, singular Riemannian surfaces. These are the so-called translation surfaces [GJ 00], and they are closely related to rational polygons. This relationship, and its applications to the polygonal billiard dynamics, which go back to [ZK 75], are well represented in the modern literature. See, e. g., [EsMa 01, Gu 84, GHT 01, KS 00, Ma 88, Ma 90, Ve 89, Vo 96] and the survey [Gu 96]. However, we believe that by bringing

Date: August 26, 2002.

in the stable norm on the homology of flat surfaces, we open up a new and promising aspect in this relationship.

We will now describe the contents of the paper in more detail.

In section 2 we briefly discuss the relationship between the periodic billiard orbits in rational polygons and the closed geodesics on translation and half-translation surfaces. The latter are equivalent to Riemann surfaces with quadratic differentials [Ga 95]. Both can be viewed as Riemannian surfaces with singular, flat metrics. H. Masur has obtained in [Ma 88, Ma 90] the quadratic lower and upper bounds on the counting function for the families of closed geodesics on these surfaces. See Theorem 1 for the precise formulation. Masur's proof does not offer any geometric interpretation of the constants that arise in this respect, and may have a complicated dependence on the surface. These are the so-called quadratic constants. We refer the reader to [GJ 00, Ve 89, Vo 96] and [EsMa 01] for more information. The need for the geometrically meaningful constants in Masur's bounds leads to the problem of estimating the growth rate for the number of simple homology classes for these surfaces. In Proposition 3 we obtain a geometric lower bound on the latter. The bound is quadratic, and the corresponding quadratic constant involves the unit ball of the stable norm.

In section 3 we study the structure of the unit ball of the stable norm on translation surfaces. The real homology of a translation surface, M, contains a particular plane. It is determined by the metric, and is called the holonomy plane of M. See [GJ 00] and [KS 00]. In particular, we prove that the stable norm, restricted to the holonomy plane, is euclidean. See Theorem 8. Suppose, for concreteness, that the genus of M is greater than one. Then Theorem 8 is in contrast with the situation for non-singular metrics. See [Mt 97] and [Mt 00]. On a heuristical level, the results of [Mt 97, Mt 00] mean that the better a minimizing current fills out the surface, the smoother is the stable norm at the corresponding homology class.

Some of the work on the paper was done in the Summer of 1998 while the authors were visiting the University of Freiburg. It is a pleasure to thank Victor Bangert for the invitation and the faculty and staff of the Mathematics Institute for their hospitality.

2. Growth rate for simple homology classes

To motivate the material in this, and the following section, we slightly digress from the stable norm, and briefly discuss polyonal billiards. Several basic questions about billiard orbits in polygons are open. See [Gu 96] and the bibliography there. Let P be a polygon, and denote by $e_P(x)$ the number of parallel bands of periodic billiard orbits in P, of length less than x. Some of the open questions concern the asymptotics of $e_P(x)$, as x goes to infinity. If P is rational, then $e_P(x)$ has asymptotic quadratic bounds, from both above and below. This is a direct consequence of the results of H. Masur on the growth rate of closed orbits for quadratic differentials [Ma 88, Ma 90]. For convenience of the reader, we formulate these results below.

Theorem 1. [H. Masur] *Let (M, g) be a Riemann surface with a quadratic differential. Denote by $f(x)$ the number of cylinders of closed leaves of length less than x. Then there exist constants $0 < c_1 \leq c_2 < \infty$ such that for x*

sufficiently large we have

$$c_1 x^2 \leq f(x) \leq c_2 x^2. \tag{1}$$

The proof of Theorem 1 in [Ma 88, Ma 90] yields no information on the constants in inequality (1). In what follows we use the stable norm on the homology to give a geometric interpretation of the lower bound in equation (1).

Definition 2. *Let (M, g) be a closed surface with a (possibly singular) Riemannian metric. A homology class $h \in H_1(M, \mathbb{Z})$ is* simple *if it has a simple closed curve as a length-minimizing representative.*

Note that the set $SH = SH(M, g) \subset H_1(M, \mathbb{Z})$ of simple homology classes depends on the metric. If $h \in SH$, then any of its minimal representatives is a simple closed geodesic. We denote by $n(x)$ the number of simple homology classes of stable norm less than x. The main result of this section is a quadratic lower bound on $n(x)$.

Let $s \geq 1$ be the genus of the surface M. The vector space $H^1(M, \mathbb{R}) \simeq \mathbb{R}^{2s}$ has a natural symplectic structure. Transferred by the Poincare duality to $H_1(M, \mathbb{R})$, it is given by the intersection number of homology classes. We will refer to it as the symplectic intersection form on the homology. It induces the symplectic volume form on $H_1(M, \mathbb{R})$. Note that all these structures do not depend on the metric. Let $\mathcal{B} \subset H_1(M, \mathbb{R})$ be the unit ball with respect to the stable norm, and let $\mathrm{vol}(\mathcal{B})$ be its symplectic volume. The positive number $\mathrm{vol}(\mathcal{B})$ does depend on the metric.

Proposition 3. *Let (M, g) be an oriented, closed surface with a (possibly singular) Riemannian metric. Let s be the genus of M. Then for any $c < \sqrt[s]{s!\mathrm{vol}(\mathcal{B})}$ the inequality*

$$cx^2 \leq n(x) \tag{2}$$

holds for all sufficiently large x.

Proof. Suppose first that the metric g is nonsingular. Then every integer homology class has a length-minimizing representative which is a union of at most s simple closed geodesics. See, for instance, [Mt 97] for a proof. Hence the number of elements $h \in H_1(M, \mathbb{Z})$ with stable norm less than x is bounded above by $n(x)^s/s!$. Set $\mathcal{B}(x) = x \cdot \mathcal{B} \subset H_1(M, \mathbb{R})$, the ball of radius x. Thus

$$|\mathcal{B}(x) \cap H_1(M, \mathbb{Z})| \leq \frac{n(x)^s}{s!}. \tag{3}$$

But the cardinality $|\mathcal{B}(x) \cap H_1(M, \mathbb{Z})|$ grows asymptotically as $\mathrm{vol}(\mathcal{B})x^{2s}$. Let f and g be positive functions on $\mathbb{R}_+$. We will use the notation $f(x) \sim g(x)$ to indicate that the ratio converges to one, as x goes to infinity. Then

$$|\mathcal{B}(x) \cap H_1(M, \mathbb{Z})| \sim \mathrm{vol}(\mathcal{B})x^{2s}. \tag{4}$$

Combining equations (3) and (4), we obtain

$$\sqrt[s]{s!\mathrm{vol}(\mathcal{B})} \leq \liminf_{x \to \infty} \frac{n(x)}{x^2}. \tag{5}$$

This implies our claim in the nonsingular case. To prove the inequality (2) for metrics with singularities, we use the fact that if (M, g) is a Riemannian surface with cone points, then it is a limit, in the Hausdorff-Lipschitz topology (see [G-L-P 81], 3.19, p. 42), of a sequence, (M, g_k), of nonsingular surfaces. The claim follows. See the next subsection. □

2.1. **Approximation in the Hausdorff-Lipschitz topology.**

Lemma 4. *The injectivity radius is continuous with respect to the Hausdorff-Lipschitz distance.*

Proof. Take $\epsilon > 0$. We want a $\delta > 0$ such that for any two surfaces M, M', if the Hausdorff-Lipschitz distance $d_{HL}(M, M')$ is less than δ, then the difference between the injectivity radii $|r_{inj}(M) - r_{inj}(M')|$ is less than ϵ.

For any $\delta > 0$, $d_{HL}(M, M') \leq \delta$ implies that there exists a homeomorphism $\phi\colon M \longrightarrow M'$ with Lipschitz constant $\mathrm{Lip}(\phi)$ such that $\log(\mathrm{Lip}(\phi)) + \log(\mathrm{Lip}(\phi^{-1})) \leq \delta$ (this is where we use the Lipschitz part of the Hausdorff-Lipschitz distance). Then for any embedded disc in M, with radius r, its image under ϕ is an embedded disc in M', of radius at least $\exp(-\delta)r$. Hence $r_{inj}(M') \geq \exp(-\delta)r \times r_{inj}(M)$ and symetrically $r_{inj}(M) \geq \exp(-\delta)r \times r_{inj}(M')$ so

$$|r_{inj}(M) - r_{inj}(M')| \leq (\exp(\delta) - 1)\max(r_{inj}(M), r_{inj}(M'))$$

which proves the lemma. □

Lemma 5. *Let M be a Riemannian surface. There exists a number $C(M)$ such that for any $\epsilon > 0$, there exists $\alpha > 0$ such that for any M' with $d_{HL}(M, M') \leq \alpha$, for any integer homology class h, we have*

$$|\|h\|_M - \|h\|_{M'}| \leq C(M)\|h\|_M \epsilon.$$

Proof. Take $\epsilon > 0$. By the above lemma there exists δ_1 such that

$$d_{HL}(M, M') \leq \delta \text{ implies } 2r_{inj}(M) \geq r_{inj}(M') \geq \frac{1}{2} r_{inj}(M).$$

Let h be an integer homology class, and γ a minimising representative of h.

Cut γ into N pieces of length less than half the injectivity radius of M. Let $x_i, i \in \mathbb{Z}/N\mathbb{Z}$ be the endpoints of the pieces in cyclic order along γ. Here we consider the manifolds M and M' as identified so the points x_i belong to any of them. Since the distance between x_i and x_{i+1} is less than the injectivity radius of M', there is a unique geodesic segment in M' joining x_i to x_{i+1} and the broken geodesic γ' made up with those segments is homotopic to γ.

Now take δ_2 such that for $d_{HL}(M, M') \leq \delta_2$ we have,

$$\text{for all } x, x' \in M,\ |d_M(x, x') - d_{M'}(x, x')| \leq \epsilon$$

(this is where we use the Hausdorff part of the Hausdorff-Lipschitz distance). Assume M and M' are δ-close with $\alpha = \min(\delta_1, \delta_2)$. Then if the length, in M', of γ' is less than

$$\begin{aligned} \sum_{i \in Z/N\mathbb{Z}} d_{M'}(x_i, x_{i+1}) &\leq \sum_{i \in \mathbb{Z}/N\mathbb{Z}} (d_M(x_i, x_{i+1} + \epsilon \\ &\leq l_M(\gamma) + N\epsilon \end{aligned}$$

so there exists in M' a closed geodesic homologic to γ, of length less than $l_M(\gamma) + N\epsilon$. □

Remark

- The constant $C(M)$ above only depends on the injectivity radius, of M, which is bounded below on compact sets of the space of Riemannian surfaces. So what we have proved really is that the stable norm is uniformly continuous on the space of Riemannian surfaces, the topology at the source being the Hausdorff-Lipschitz one, and the topology on stable norms being that of uniform convergence on compact subsets of $H_1(M.\mathbb{R})$.
- Uniform continuity allows us to extend the above Lemma to the closure, with respect of the Hausdorff-Lpischitz distance, of the space of Riemannian surfaces.

Corollary 6. *The volume of the unit ball of the stable norm is uniformly continuous, with respect to the Hausdorff-Lipschitz topology.*

Theorem 7. *Let M be a metric surface in the Hausdorff-Lipschitz closure of the set of Riemannian surfaces. Then the counting function of M is asymptotically quadratic, with constant the colume of the unit ball of the stable norm of M.*

Proof. Let M_n be a sequence of Riemannian surfaces converging to M. Let c_n, c be the volumes of the stable norms of M_n, M respectively. Take $\epsilon > 0$. Pick N_ϵ and T_ϵ so that for any $n \geq N_\epsilon$, $T \geq T_\epsilon$ we have

- $|c_n - c| \leq \epsilon$
- for any integer homology class h, $|\|h\|_M - \|h\|_{M'}| \leq C(M)\|h\|_M\epsilon$
- $$c_N + \epsilon \geq \frac{N_{M_N}(T)}{T^2} \geq c_N - \epsilon.$$

Under these assumptions we have, for any large enough T,

$$\begin{aligned} N_M(T) \geq N_{M_N}(T(1 - c_{M_N}\epsilon)) &\geq (c_N - \epsilon)(1 + c_{M_N}\epsilon)^2 T^2 \\ &\geq (c - 2\epsilon)(1 + (c - \epsilon)\epsilon)^2 T^2 \end{aligned}$$

and likewise,

$$(c_N + \epsilon)T^2 \geq N_{M_N}(T) \geq N_M(T(1 - c\epsilon))$$

whence

$$\frac{(c + 2\epsilon)}{[1 - (c - \epsilon)\epsilon]^2} \geq \frac{N_M(T)}{T^2} \geq (c - 2\epsilon)[1 - (c - \epsilon)\epsilon]^2.$$

□

3. Applications to periodic and singular orbits on translation surfaces

A translation surface is a closed Riemann surface with a holomorphic differential. See [GJ 00]. Let M be the surface, let ζ be the differential, and let $\Sigma \subset M$ be the set of zeroes of ζ. We cover $M \setminus \Sigma$ with contractible patches $U_i : 1 \leq i \leq n$. Integrating ζ in U_i yields complex coordinates

$z_i : U_i \to \mathbb{C}$. If U_i and U_j overlap, then the two coordinates in $U_i \cap U_j$ are related by

$$z_j = z_i + c_{ij}. \tag{6}$$

If we replace a linear differential with a quadratic differential, then the preceding construction of local coordinates still applies. However, instead of equation (6), we have

$$z_j = \pm z_i + c_{ij}. \tag{7}$$

Equations (6,7) explain the names "translation surface" and "half-translation surface". See [GJ 96, GJ 00], where equations (6,7) were the starting point. The study of quadratic differentials belongs to the classical complex analysis [Ga 95]. By taking a 2-to-1 covering, if necessary, any half-translation surface becomes a translation surface. This observation often allows to extend the results, obtained for translation surfaces only, to half-translation surfaces. In view of this, and because translation surfaces naturally arise in the study of polygonal billiards (see [Gu 96]), here we will restrict our attention to translation surfaces only.

Let $\Sigma \subset M$ be the finite set of zeros of ζ. Set $\zeta = \omega + i\,{}^\star\omega$, and for $v \in T_xM \setminus \Sigma$ set $\|v\|_g = \sqrt{\omega(v)^2 + {}^\star\omega(v)^2}$. This defines a flat Riemannian metric on M, with singularities in Σ, and the results of sections 2, 3 apply. The local coordinates on $M \setminus \Sigma$ induced by ω, ${}^\star\omega$ yield a local isometry of $M \setminus \Sigma$ into the Euclidean plane. In these coordinates $X_\omega = (1,0)$, $X_{{}^\star\omega} = (0,1)$. We also have

$$d\mathrm{vol} = \omega \wedge {}^\star\omega, \quad \mathrm{comass}(\omega) = 1, \ \|\omega\|_{L^2} = \sqrt{\mathrm{vol}(M)}. \tag{8}$$

For arbitrary $a, b \in \mathbb{R}$, set $\eta_{(a,b)} = a\omega + b\star\omega$. We regard $\eta_{(a,b)}$ as an element in $H^1(M,\mathbb{R})$ and denote by $h_{(a,b)} \in H_1(M,\mathbb{R})$ the corresponding homology class, by Poincare duality. Let $L \subset H_1(M,\mathbb{R})$ be the plane spanned by these classes. Following [GJ 00] and [KS 00], we say that L is the *holonomy plane* of the translation surface M.

Theorem 8. *Let M be a translation surface. The restrictions of stable norm and the L^2-norm on the homology to the holonomy plane are proportional. More precisely*

$$\|h_{(a,b)}\| = \sqrt{vol(M)}\|h_{(a,b)}\|_{L^2} = vol(M)\sqrt{a^2+b^2}. \tag{9}$$

Proof. From the rotational invariance of the metric and equation (8), we have $\mathrm{comass}(\eta_{(a,b)}) = \sqrt{a^2+b^2}$ and $\|h_{(a,b)}\|_{L^2} = \sqrt{\mathrm{vol}(M)}\sqrt{a^2+b^2}$. In view of Theorem 1 of [GM], it suffices to prove the inequalty

$$\|h_{(a,b)}\| \geq \mathrm{vol}(M)\sqrt{a^2+b^2}. \tag{10}$$

Let η be any differential 1-form, and let ϕ be an arbitrary current. Then

$$<\phi, \eta> \leq \mathrm{mass}(\phi)\ \mathrm{comass}(\eta).$$

Let $k \in H_1(M,\mathbb{R})$, and let $\phi_{\min}(k)$ be a minimizing current in the homology class k. Specializing the inequality above, we obtain

$$<\phi_{\min}(k), \eta> \leq \|k\|\ \mathrm{comass}(\eta). \tag{11}$$

Set $k = h_{(a,b)}$. Its minimizing current corresponds to the vector field $X_{(a,b)} = aX_\omega + bX_{\star\omega}$. Setting $\eta = \eta_{(a,b)}$ in equation (11), taking into account that

$$< X_{(a,b)}, \eta_{(a,b)} > = \operatorname{vol}(M)(a^2 + b^2),$$

and using the expression above for comass$(\eta_{(a,b)})$, we obtain the inequality (10). □

The theorem says that the L^2-ball of radius $(\operatorname{vol}(M))^{-1/2}$ is contained in the unit ball of the stable norm, the two having the same intersection with the holonomy plane. This intersection is a Euclidean disc which we denote D. Actually the unit ball of the stable norm is, in turn, contained in the orthogonal cylinder over D, that is, the inverse image of D under the L^2-orthogonal projection to the holonomy plane. Indeed this cylinder is defined by the inequations $\eta_{(a,b)}(h) \leq 1$ for all $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$. Now by equation 11 we have $|\eta_{(a,b)}(h)| \leq \|h\|$ for all $h \in H_1(M, \mathbb{R})$.

The intersection of the boundary of the orthogonal cylinder over D with the unit ball of the stable norm does not consist only of D. Indeed, if h is the homology class of a parallel band of periodic orbit, then the class $h/\|h\|$ belongs to this intersection. This is because, if $h_{(a,b)}$ is the homology class of the singular geodesic foliation with direction that of h, then $|\eta_{(a,b)}(h)| = \|h\|$. The same applies to the homology class of a closed curve made of saddle connections with the same direction.

The difference between such a class and that of a geodesic in a periodic band is that in the latter case the unit sphere of the stable norm contains a segment with extremity $h/\|h\|$ and containing $h_{(a,b)}$ in its interior . This is because, if h is the homology class of a parallel band of periodic orbit, and $w(h)$ is the width of that parallel band, then the class $\operatorname{vol}(M)h_{(a,b)} - w(h)h$, divided by its norm, also belongs to intersection of boundary of the orthogonal cylinder over D with the unit ball of the stable norm, and the segment joining it to $h/\|h\|$ contains $h_{(a,b)}$ in its interior.

The simplest translation surfaces are flat tori. Natural mappings of translation surfaces are the coverings. See [GJ 00, Vo 96, Gu 00]. Hence, it is natural to study translation surfaces that admit a covering of a flat torus. In particular, this class contains the *arithmetic translation surfaces.* For instance, if S admits a covering, $p : S \to T$, over a flat torus, whose branch locus is a single point, then S is an arithmetic translation surface. See [GJ 00]. There are several characterizations of the arithmetic translation surfaces in the literature, both from the geometric and from the dynamical viewpoints. See [GJ 00],[Vo 96], and [GHT 01]. This motivates the following definition.

Definition 9. *We say that a translation surface is* almost arithmetic *if it admits an affine branched covering of a flat torus.*

Obviously, the class of almost arithmetic translation surfaces is wider than the class of arithmetic translation surfaces. Intuitively, a translation surface is almost arithmetic if it can be represented by a polygon, whose vertices belong to a lattice in the plane. An arithmetic translation surface can be represented by a polygon, drawn on a planar lattice. See [Gu 84].

Proposition 10. *A translation surface is almost arithmetic if and only if its holonomy plane is spanned by the integer homology classes.*

Proof. Let (S,ω) be any translation surface, and let $L_S \subset H_1(S,\mathbb{R})$ be its holonomy plane. Denote by $L_S^* \subset H^1(S,\mathbb{R})$ the image of L_S under the Poincare duality. By definition, L_S^* is spanned by the cohomology classes of the closed forms $\Re(\omega)$ and $\Im(\omega)$. In what follows we call L_S^* the *coholonomy plane* of the translation surface S. Note that L_S is spanned by the integer homology classes if and only if L_S^* is spanned by the integer cohomology classes.

Let now (S,ω) be a translation surface, and let $p : S \to T$ be a covering of a flat torus (T,α). Then $\omega = p^*(\alpha)$ and $L_S^* = p^*(L_T^*)$. Slightly abusing notation, we denote by the same symbol a form and its cohomology class. Since L_T^* is spanned by the integer classes, the same holds for $p^*(L_T^*)$.

Recall that the group $SL(2,\mathbb{R})$ naturally acts on the space of translation surfaces. See, e. g., [GJ 00]. Although, this action changes only the translation structure, and does not physically change the surface itself, we will denote it by $(S,\omega) \mapsto (g\cdot S, g\cdot\omega)$, where $g \in SL(2,\mathbb{R})$. We say that S and $g\cdot S$ are affinely equivalent translation surfaces. The action of $SL(2,\mathbb{R})$ does not change the holonomy plane of a translation surface : $L_{g\cdot S} = L_S$.

Suppose now that L_S is spanned by the integer classes. Then so is the plane L_S^*. Set $\alpha = \Re(\omega)$, $\beta = \Im(\omega)$, i. e., $\omega = \alpha + \beta i$. Replacing S by a $g\cdot S$, if nesessary, we may assume that $\alpha, \beta \in H^1(S,\mathbb{Z})$. Therefore the integrals of ω over the closed curves in S belong to $\mathbb{Z}+i\mathbb{Z}$. Hence the standard procedure of the integration of ω over the curves joining $s \in S$ with a reference point, yields a mapping, $p\ : S \to \mathbb{C}/\mathbb{Z}^2$. This mapping is the desired covering. Compare with the proof of Theorem 5.5 in [GJ 00]. □

In the course of our proof of Proposition 10 we have obtained the following statement.

Corollary 11. *Let S be a translation surface. Then the following properties are equivalent :*
1. S admits an affine covering over a torus ;
2. The holonomy plane of S is spanned by the integer homology classes ;
3. The coholonomy plane of S is spanned by the integer cohomology classes.

Note that if L_S is spanned by the integer classes, then its L^2-orthogonal $L_S^\perp$ is also spanned by the integer classes. Indeed $L_S^\perp = \mathrm{Ker} p_*$ but p_* sends the integer lattice in $H_1(S,\mathbb{R})$ to the integer lattice in $H_1(T,\mathbb{R})$ so its kernel must be an integer subspace.

On a translation surface S there is a well-defined notion of direction. We call closed geodesic a geodesic that either belongs to a periodic strip, or is made up with saddle connections, all with the same direction. We call length spectrum of S, and denote $LS(S)$, the set of lengths of closed geodesics of S. The growth function of S, denoted $N_S(T)$, is the number of elements of $LS(S)$ which are less than or equal to T. Our main result is

Theorem 12. *The growth function of an almost arithmetic translation surface is asymptotically quadratic.*

Remarks.

1. This property is proved in [Ve 89] for surfaces with a large group of affine automorphisms. Generically an almost arithmetic surface has

a trivial group of automorphisms since the branch points are setwise fixed.

2. our theorem ignores multiplicities in the sense that there may be several (at most $3(\text{genus}(M)-1)$) periodic geodesics which yield the same point in the holonomy plane. Besides, we count some singular closed orbits along with non-singular ones.

Proof. Call $\pi : H_1(M,\mathbb{R}) \longrightarrow L_S$ the orthogonal projection with respect to the L_2 Euclidean metric on $H_1(M,\mathbb{R})$. If γ is a closed geodesic with direction θ and length l, then $\pi([\gamma]) = lh_\theta$.

Besides, note that $\pi([\gamma])$ belongs to the image under π of $H_1(M,\mathbb{Z})$, which is a lattice in L_S. Conversely, take a primitive point h in $\pi(H_1(M,\mathbb{Z}))$. Its direction is completely periodic by our previous theorem. Hence some integer homology above h in $H_1(M,\mathbb{Z})$ contains a closed geodesic, or combination thereof. The latter is impossible since h is primitive.

Therefore $N_S(T)$ equals the growth function of the lattice $\pi(H_1(M,\mathbb{Z}))$, which is quadratic by Minkovski's theorem, with constant π divided by the determinant of the lattice. □

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