A GENERAL *N*-DIMENSIONAL QUADRATURE TRANSFORM AND ITS APPLICATION TO INTERFEROGRAM DEMODULATION

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Abstract

Quadrature operators are useful to obtain the modulating phase ϕ in interferometry and temporal signals in electrical communications. In carrier frequency interferometry and electrical communication one uses the Hilbert transform to obtain the quadrature of the signal. In these cases the Hilbert transform gives the desired quadrature because the modulating phase is monotonically increasing. Here we propose an n-dimensional quadrature operator which transforms $\cos[\phi]$ into $-\sin[\phi]$ regardless of the frequency spectrum of the signal. Having the quadrature of the phase modulated signal one can easily calculate the value of ϕ over all the domain of interest. Our quadrature operator is composed of two n-dimensional vector fields: one is related to the gradient of the image normalized with respect to local frequency magnitude and the other is related to the sign of the local frequency of the signal. The inner product of these two vector

fields gives us the desired quadrature signal. This quadrature operator is derived in the image space using differential vector calculus and in the frequency domain using a n-dimensional generalization of the Hilbert transform. A robust numerical algorithm is given to find the modulating phase of two-dimensional single-image closed-fringe interferograms using the ideas put forward in this paper.

2650 Fringe Analysis, 3180 Interferometry, 5050 Phase Measurement, 6160 Speckle Interferometry

I.- Introduction

Recently a number of researchers [1-5] have contributed to the understanding and development of methods to estimate the modulating phase of a single-image interferogram having closed fringes. As it is well understood today, this is a problem for which a unique solution does not exist; so, additional constraints must be added in order to find a unique solution. The most widely used additional constraint is to consider the modulating phase to be smooth. Using this smoothness constraint one is able to demodulate a single fringe pattern by minimizing a non-linear global cost function as in [3] or by locally minimizing a non-linear cost function and then propagating its solution following the fringes of the interferogram as in the case of the regularized phase tracking (RPT) algorithm [2]. In the RPT case the local phase and frequency are estimated simultaneously.

The problem of finding the quadrature signal of a single closed-fringe interferogram may also be achieved by factorizing the whole task into two separate problems, namely, finding the orientation of the fringes and their local frequency. As far as we know this was first proposed by Marroquin *et. al.* [4] where two separate cost functions were proposed to obtain the orientation and the local frequency independently. The phase estimation problem may also be factorized using the RPT by scanning the interferogram following the path defined by its fringes [2].

By doing this, one is able to somewhat decouple the local phase estimation (which is almost constant along the fringes of the image) from its local frequency (fringe orientation). This two dimensional scanning strategy is shown to be more robust than following a scanning strategy independent of the form of the interferogram's fringes, such as a row by row scanning.

On the other hand, efforts have been made for estimating the modulating phase of a single-image closed-fringe interferogram by the use of a linear operator [1,5]. This seems to be a very good approach, since if it were possible to find such an operator , this process would be as easy as demodulating a carrier frequency interferogram [6]. Unfortunately as we show in this paper, such linear system does not exist. In contrast, what does exist [5] is a linear operator which gives the quadrature signal in a "direct" way once and only once, the orientation of the fringes is already estimated (which is not a trivial process).

The first well known work that uses a linear operator, (the Hilbert transform) to obtain the modulating phase of a single-image closed-fringe interferogram was made by Kreis [1]. He uses a two-dimensional generalization (2D) Hilbert operator $H_2'\{.\}$ to find the modulating phase of the interferogram. As we will see in this paper, the trouble with this approach is twofold. Firstly, the recovered phase has spurious ringing effects along paths where the magnitude of the phase gradient is close to zero, due to the hard discontinuity of the $H_2'\{.\}$ operator at the origin. Secondly, the recovered phase is always monotonically increasing even though the actual modulating phase may be non-monotonic.

In this paper we present a quadrature operator which is composed by two vector fields; one is related to the image gradient normalized with respect to the local frequency magnitude , and the other, which we call $\boldsymbol{n}_\phi$, to the orientation angle of the fringes of the interferogram. We also present examples of the application of this operator to the demodulation of computer simulated fringe patterns and to an experimentally obtained interferogram. Finally, we compare the performance of our numerical algorithm with the vortex operator proposed by Larkin *et. al.* [5].

II The 1D Quadrature Transform

The Hilbert transform in one dimension (1D) $H_1\{.\}$ is a very useful mathematical tool to obtain the quadrature of a single frequency sinusoidal signal. According to [7] , the Hilbert transform of a cosine and sine functions with a linear increasing phase are:

$$\begin{aligned} H_1[\cos(\omega_0 x)] &= -\sin(\omega_0 x), \quad \textit{for} \quad \omega_0>0 \\ H_1[\sin(\omega_0 x)] &= \cos(\omega_0 x) \end{aligned} \tag{1}$$

because of this, we may have the (wrong) impression that the Hilbert transform $H_1\{.\}$always gives the quadrature of a cosine signal. The transforming properties of the Hilbert operator become clear by looking at the form of the frequency response of $H_1\{.\}$ as applied to a real function $g(x)$=cos[ϕ(x)], which is [7]:

$$\mathscr{F}\Big\{H_1[g(x)]\Big\} = -i\ sign(u)\ \hat{g}(u) = -i\frac{u}{|u|}\hat{g}(u) \tag{2}$$

where the Fourier transform is represented by $\mathscr{F}\{.\}$ and,

$$\hat{g}(u) = \mathscr{F}\Big\{g(x)\Big\} \tag{3}$$

The Hilbert transform still renders the expected result (the quadrature signal) when it is applied to a more complicated carrier frequency signal, that is:

$$H_1\{\cos(\omega_0 x + \psi(x))\} = -\sin(\omega_0 x + \psi(x)), \tag{4}$$

provided the local frequency does not change sign, *i.e.*

$$\omega_0 + \frac{d\psi(x)}{dx} > 0 \quad , \quad \forall x \tag{5}$$

in this case this cosine signal has its two spectral lobes well separated, and the $H_1\{.\}$ operator may be used to recover the modulating phase $\psi(x)$ of the signal. This is the reason why the Fourier method is so widely used to find the modulating phase of carrier frequency interferograms.

Unfortunately, Eq.4 will not hold if condition (5) is not satisfied, i.e., if the phase modulating signal is not a monotonically increasing function of x. As it is demonstrated in the next section, without a carrier one needs to know the sign of the local frequency to obtain the expected quadrature signal. Therefore, in general, to obtain the one dimensional quadrature of $\cos(\phi)$ one needs to use the following formula which relates the one dimensional quadrature transform $Q_1\{.\}$ to the one dimensional Hilbert transform $H_1\{.\}$

$$Q_1\{\cos(\phi)\} = \frac{d\phi/dx}{|d\phi/dx|} H_1\{\cos(\phi)\} = -\sin(\phi) \tag{6}$$

where the x dependence of $\phi(x)$ was omitted for clarity. The result of applying $Q_1\{.\}$ reduces then to $H_1\{.\}$ only when the local frequency $\omega = d\phi/dx$ of the signal is everywhere greater than zero, so that the ratio $\omega/|\omega|$ equals one all over the whole domain of interest. In general, to obtain the quadrature signal of the cosine of a non-monotonic function $\phi(x)$ one needs to know the sign of the local spatial frequency to correct the sign of the signal obtained by the application of the Hilbert transform.

III.- The 2-D Quadrature Transform for Carrier Frequency Interferograms

A straight forward generalization for the 1D Hilbert transform has been used in optical signal processing as a component of the Schlieren phase analysis method [8]. This 2D Hilbert transform generalization is also the one used by Kreis [1] to find the modulating phase of a closed-fringe interferogram.

To describe it, let us start by considering a two-dimensional fringe pattern with carrier frequency. This is normally represented as:

$$I_a(x,y) = a(x,y) + b(x,y)\cos[\psi(x,y) + u_0 x + v_0 y] \quad \textbf{(7)}$$

This equation represents a fringe pattern $I_a(x,y)$ that depends on a two dimensional phase $\psi(x,y)$ plus a phase plane. The function $a(x,y)$ is a low frequency signal that represents the background illumination. The function $b(x,y)$ is also a low frequency signal that represents the slowly varying contrast of the fringes. The two carrier frequencies u_0 and v_0 are assumed to be greater than the maximum spatial frequency of the modulating signal along the x and y directions, that is,

$$u_0 > \left|\frac{\partial\psi(x,y)}{\partial x}\right| \quad , \quad v_0 > \left|\frac{\partial\psi(x,y)}{\partial y}\right| \quad , \quad \forall\ (x,y) \quad \textbf{(8)}$$

Now, let us assume that we can eliminate the background illumination $a(x,y)$ so our fringe pattern will be reduced to:

$$I(x,y) = b(x,y)\cos[\ \psi(x,y) + u_o x + v_0 y\,] \quad \textbf{(9)}$$

this fringe pattern may be rewritten as,

$$I(x,y) = \frac{b(x,y)}{2}\left[\exp\Big(i\,[-\psi(x,y) - u_o x - v_0 y]\Big) + \exp\Big(i\,[\psi(x,y) + u_o x + v_0 y]\Big)\right] \quad \textbf{(10)}$$

The Fourier transform of this signal may be represented as,

$$\hat{I}(u,v) = \frac{1}{2}\left[\,f_-(u,v) + f_+(u,v)\right] \quad \textbf{(11)}$$

Where $f_+(u,v)$ and $f_-(u,v)$ are the Fourier transforms of the corresponding complex exponential terms multiplied by the contrast function $b(x,y)$. Because of condition

(8), this signal is composed of two well separated spectral lobes, corresponding to the signal $f_+(u,v)$, which lies exclusively in the positive side of the half plane $u_0\, u+v_0\, v=0$, and $f_-(u,v)$, which lies in the negative side. One may obtain the quadrature of the interferogram by filtering this signal using the following operator,

$$H_2^{\prime}\{\, I(x,y)\,\} \;=\; -b(x,y)\sin[\psi(x,y)+u_o x+v_0 y] = \mathscr{F}^{-1}\left\{\frac{-i\,\boldsymbol{\omega_0}\bullet\boldsymbol{q}}{|\boldsymbol{\omega_0}\bullet\boldsymbol{q}|}[f_+(u,v)+f_-(u,v)]\right\} \qquad \textbf{(12)}$$

where

$$\boldsymbol{\omega_0} = u_0\,\boldsymbol{e_1} + v_0\,\boldsymbol{e_2}\,,\quad \boldsymbol{q} = u\,\boldsymbol{e_1} + v\,\boldsymbol{e_2} \qquad \textbf{(13)}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are the unit vectors along the u and v spectral coordinates. The result of applying this 2D form of the Hilbert transform $H_2'\{.\}$ to the fringe pattern coincides with the expected quadrature signal because the input interferogram has its two spectral lobes well separated, and the 2D Hilbert transform reduces in one dimension to the one given in the last section. If this spectral separation condition is not satisfied, however, the result of applying $H_2'\{.\}$ will not give the expected quadrature signal.

The Schlieren transform $S\{.\}$ uses $H_2'\{.\}$ in the following way:

$$S\{\,\hat{I}(u,v)\,\} = \frac{1}{2}\Big[\,1 + H_2^{\prime}\{\,\hat{I}(u,v)\,\}\Big] = U(u,v) + i\,V(u,v) \qquad \textbf{(14)}$$

where the (u,v) space is the Fourier transform space of the $\boldsymbol{R}^2 =(x,y)$ plane. In the optical laboratory this filter $S\{.\}$ is easily implemented simply by blocking out (using a knife edge for example), half of the spectral domain along the line given by $\boldsymbol{\omega_0}\bullet\boldsymbol{q}=0$. The estimated phase $\psi(x,y)$ is then obtained from,

$$\psi(x,y) + u_0 x + v_0 y = \tan^{-1}\left[\frac{Im(A)}{Re(A)}\right] \quad \textbf{(15)}$$

where,

$$A(x,y) = \mathscr{F}^{-1}\{\, U(u,v) + i\, V(u,v)\,\} \quad \textbf{(16)}$$

Note that, because of condition (8), the phase obtained by this relation is always monotonic due to the phase carrier. Therefore, when the Schlieren operator $S\{.\}$ is used to obtain the phase of a closed-fringe interferogram the estimated phase does not represent the actual non-monotonic modulating phase; also, the estimated phase in this case has serious spurious ringing effects along the path where this phase is stationary. However in the case of carrier frequency interferogram phase demodulation using the Schlieren operator, $S\{.\}$ works perfectly well and its estimated phase is free of undesirable phase distortions.

IV.- The General *n*-Dimensional Quadrature Transform

In this section we will present a general n-dimensional quadrature transform that works well for closed-fringes, as well as for carrier frequency interferograms. Moreover, the result of this section will also permit us to obtain as special case the one dimensional result stated in section II.

The aim of an n-dimensional quadrature operator $Q_n\{.\}$ is to transform a given fringe pattern into its quadrature, which may be represented by:

$$Q_2\{\, b(\boldsymbol{r})\cos[\phi(\boldsymbol{r})]\,\} = -b(\boldsymbol{r})\sin[\,\phi(\boldsymbol{r})\,] \quad \textbf{(17)}$$

where $\boldsymbol{r}=(x_1,...,x_n)$ is the n-dimensional vector position. Using this quadrature signal one can easily determine the phase $\phi(\boldsymbol{r})$ modulo 2π over the whole domain of interest.

The first step towards obtaining the quadrature signal will be to obtain the gradient of the fringe pattern, which is:

$$\nabla I(\boldsymbol{r}) = \cos[\,\phi(\boldsymbol{r})\,]\,\nabla b(\boldsymbol{r}) + b(\boldsymbol{r})\nabla\Big[\cos[\phi(\boldsymbol{r})]\Big] \tag{18}$$

knowing that in most practical situations the contrast $b(\boldsymbol{r})$ is a low frequency signal the first term in this equation may be neglected with respect to the second one to obtain:

$$\nabla I(\boldsymbol{r}) \approx b(\boldsymbol{r})\nabla\Big[\cos[\phi(\boldsymbol{r})]\Big] \tag{19}$$

hereafter we will use this approximation as valid so the approximation sign will be replaced by an equal sign. Of course for the special case of a constant contrast $b(\boldsymbol{r})=b_0$ the above mathematical relation is exact. Applying the chain rule for differentiation we obtain,

$$\nabla I(\boldsymbol{r}) = -b(\boldsymbol{r})\sin[\phi(\boldsymbol{r})]\,\nabla\phi(\boldsymbol{r}) \tag{20}$$

if it were possible to know the actual sign and magnitude of the local frequency $\nabla\phi(\boldsymbol{r})$ one could use this information as,

$$\nabla I(\boldsymbol{r})\bullet\nabla\phi(\boldsymbol{r}) = -b(\boldsymbol{r})\sin[\phi(\boldsymbol{r})]\,|\nabla\phi(\boldsymbol{r})|^2 \tag{21}$$

and the quadrature signal would be obtained dividing both sides of this equation by the squared magnitude of the local frequency $|\nabla\phi(\boldsymbol{r})|^2$:

$$Q_n\{\,b(\boldsymbol{r})\cos[\,\phi(\boldsymbol{r})\,]\,\} = \frac{\nabla\phi(\boldsymbol{r})}{|\nabla\phi(\boldsymbol{r})|^2}\bullet\nabla I(\boldsymbol{r}) = -b(\boldsymbol{r})\sin[\phi(\boldsymbol{r})] \tag{22}$$

We start by noticing that the equation for $Q_n\{.\}$ is a little bit tricky because, as far as we know, there is no a linear system that applied to our fringe pattern $I(\boldsymbol{r})$ give us $\nabla\phi(\boldsymbol{r})$ in a direct way. In the following we propose some techniques to calculate the gradient of the modulating phase. Eq.22 may be rewritten as

$$Q_n\{\, b(\boldsymbol{r})\cos[\,\phi(\boldsymbol{r})\,]\,\} = \frac{\nabla\phi(\boldsymbol{r})}{|\nabla\phi(\boldsymbol{r})|} \bullet \frac{\nabla I(\boldsymbol{r})}{|\nabla\phi(\boldsymbol{r})|} = \boldsymbol{n}_\phi \bullet \frac{\nabla I(\boldsymbol{r})}{|\nabla\phi(\boldsymbol{r})|} \qquad \textbf{(23)}$$

where $\boldsymbol{n}_\phi = \boldsymbol{n}_\phi(\boldsymbol{r})$ is a unit vector normal to the corresponding isophase contour, which points in the direction of $\nabla\phi$. This mathematical relation is the desired quadrature operator and the main result of this paper. The remaining of the paper will discuss some properties of this operator as well as several techniques that may be used to numerically calculate it as a computer program that can be used to demodulate a single or multiple closed fringe interferograms.

In 2D, the vector $\boldsymbol{n}_\phi$ points in the direction $\theta_{2\pi}$, which we call the fringe orientation angle, which is given by,

$$\tan[\theta_{2\pi}(x,y)] = \frac{\omega_y(x,y)}{\omega_x(x,y)} \qquad \textbf{(24)}$$

with

$$\omega_x = \frac{\partial\phi(x,y)}{\partial x} \quad , \quad \omega_y = \frac{\partial\phi(x,y)}{\partial y} \qquad \textbf{(25)}$$

from this relation one can obtain the cosine and sine of the local fringe orientation:

$$\cos[\theta_{2\pi}(x,y)]\,\boldsymbol{i} + \sin[\theta_{2\pi}(x,y)]\,\boldsymbol{j} = \frac{\omega_x\,\boldsymbol{i}}{\sqrt{\omega_x^2+\omega_y^2}} + \frac{\omega_y\,\boldsymbol{j}}{\sqrt{\omega_x^2+\omega_y^2}} = \frac{\nabla\phi(x,y)}{|\nabla\phi(x,y)|} \qquad \textbf{(26)}$$

This relation may be generalized to higher dimensions. For example in three dimensions one obtains,

$$\boldsymbol{n}_\phi = \frac{\nabla\phi(x,y,z)}{|\nabla\phi(x,y,z)|} = \cos\alpha_{2\pi}(x,y,z)\ \boldsymbol{i} + \cos\beta_{2\pi}(x,y,z)\ \boldsymbol{j} + \cos\gamma_{2\pi}(x,y,z)\ \boldsymbol{k} \qquad \textbf{(27)}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$, are the unit vectors along the x, y and z coordinates, and α, β and γ are the direction cosines of the normal vector $\boldsymbol{n}_\phi$. As can be seen the use of direction cosines permits one to extend the concept of 2D fringe orientation to higher dimensions. Knowing the following property of direction cosines,

$$\cos^2[\alpha_{2\pi}(x,y,z)] + \cos^2[\beta_{2\pi}(x,y,z)] + \cos^2[\gamma_{2\pi}(x,y,z)] = 1 \qquad \textbf{(28)}$$

any two direction cosines or equivalently; two orientation angles are needed to obtain the third one. In general in an n-dimensional space one would need to obtain the orientation of $(n$-1) angles. Conversely, knowing the magnitude and sign of the phase gradient one can find the direction cosines of the fringe's orientation. Unfortunately, there is no direct way (e.g., a linear transformation) to obtain $\boldsymbol{n}_\phi$ from a single-image fringe pattern containing closed fringes. That is because there is a fundamental ambiguity in the global sign of the recovered phase. So, in general, one decides arbitrarily the sign of the phase at a given seed point, and afterwards one propagates that phase solution throughout the entire interferogram.

One can see that the quadrature operator of Eq. 23 reduces in one dimension to the form for $Q_1\{.\}$ stated in section II. In the next section (IV-1) we give a numerical technique to calculate the quadrature operator of a closed fringe interferogram $I(x,y)$. In the following section (IV-2) we show another alternative to calculate $\nabla I/|\nabla\phi|$ as a linear operator whose Fourier transform corresponds to n Reisz transforms [9] along each spectral coordinate. We denote this operator by $\mathbf{H}_n\{.\}$, given that this is another possible generalization of the 1D Hilbert transform. This generalization of $H_1\{.\}$ is different from the 2D generalization used in the Schlieren method mentioned in section III.

IV-1 A Numerical Method to Calculate the Quadrature of a 2D Signal $Q_2\{I(x,y)\}$

In this section we will use the ideas put forward in this paper to propose a practical numerical method to demodulate a single closed-fringe interferogram. For the reader's convenience let us first recall the form of our quadrature operator in 2D,

$$Q_2\{ I(x,y) \} = \frac{\nabla\phi(x,y)}{|\nabla\phi(x,y)|} \bullet \frac{\nabla I(x,y)}{|\nabla\phi(x,y)|} \tag{29}$$

we may see that there are several ways for calculating $Q_2\{.\}$. The orientation term $\nabla\phi/|\nabla\phi|$ as we mention, is by far the most difficult term to estimate. To calculate $\nabla\phi/|\nabla\phi|$ we recommend to use the Quiroga *et. al.* [10] technique outlined in section IV-3. In contrast the term $\nabla I/|\nabla\phi|$ is easier to compute. A possible way to find $\nabla I/|\nabla\phi|$ is to estimate $|\nabla\phi|$ and ∇I directly from the fringe pattern. Another possibility which is very easy to implement and is the one used in this section is to calculate just ∇I instead of calculating $\nabla I/|\nabla\phi|$ by using centered first order differences. Obviously ∇I and $\nabla I/|\nabla\phi|$ will differ when $|\nabla\phi|\neq 1.0$ so a final adjustment will be necessary. By doing this one obtains,

$$\boldsymbol{n}_{\phi}(x,y) \bullet \nabla I(x,y) = |\nabla\phi(x,y)| \sin[\phi(x,y)] \tag{30}$$

where we have assumed that the fringe pattern is normalized so that $b(x,y)=1$ and its fringe orientation modulo 2π $(\nabla\phi/|\nabla\phi|)$ has been already estimated. We can see that the signal in the left hand side of Eq. 30 is proportional to the quadrature signal; therefore, the phase obtained by:

$$\phi^0(x,y) = \tan^{-1}\left\{\frac{\boldsymbol{n}_{\phi}(x,y) \bullet \nabla I(x,y)}{I(x,y)}\right\} = \tan^{-1}\left\{\frac{|\nabla\phi(x,y)| \sin[\phi(x,y)]}{\cos[\phi(x,y)]}\right\} \tag{31}$$

may have a small phase error over the two dimensional regions where $|\nabla\phi|\neq 1$, but fortunately it is in the correct branch of the $\tan^{-1}(.)$ function. The correct phase value may therefore be found as the solution to the non-linear equation:

$$f(x,y) = \cos[\phi(x,y)] - I(x,y) = 0 \tag{32}$$

that is closest to the phase value $\phi^0(x,y)$ obtained above. The value for $\phi^0(x,y)$ is used as an initial guess in an iterative algorithm that moves closer to the real modulating phase. This may be done pixel by pixel using the very efficient iterative Halley method [14] which is an improved version of the Newton-Raphson technique; the iteration is given by:

$$\phi^{k+1}(x,y) = \phi^k(x,y) - \frac{2\, f(x,y)\, f_\phi(x,y)}{2\, f_\phi^2(x,y) + f_{\phi\phi}(x,y)\, f(x,y)}\ , \quad k=0,1,2,\ldots \tag{33}$$

where $f_\phi(x,y)$ and $f_{\phi\phi}(x,y)$ are the first and second derivative of $f(x,y)$ with respect to ϕ, which gives

$$\phi^{k+1}(x,y) = \phi^k(x,y) + \frac{2\, f(x,y)\, \sin[\phi^k(x,y)]}{2\, \sin^2[\phi^k(x,y)] - \cos[\phi^k(x,y)]\, f(x,y)}\ , \quad k=0,1,2,\ldots \tag{34}$$

This recursive formula is very stable and extremely efficient given that the value $\phi^{k+1}(x,y)$ approaches the searched solution $\phi(x,y)$ tripling the number of accurate significant figures at each iteration. So in the average about two iterations per pixel are needed to correct the estimated phase.

One might think that to solve Eq.32 one only needs to evaluate,

$$\phi(x,y) = \cos^{-1}[\, I(x,y)\,] \tag{35}$$

the problem with this solution is that our resultant $\phi(x,y)$ will be obtained modulo π and this is not what we are looking for, so we end up at the beginning of our problem. In contrast what we need is the non-monotonic solution for $\phi(x,y)$ modulo 2π which is obtained by the proposed method.

IV-2 A Fourier Transform Method to Calculate the Vector Field $\nabla I/|\nabla\phi|$

In this section we show how the vector field $\nabla I/|\nabla\phi|$ may also be calculated using the Fourier transform. This Fourier operator acting on $I(x,y)$ may be seen as a n-dimensional generalization of the 1D Hilbert transform. We will show this in two dimensions but it may be readily extended to n-dimensions. We start by writing a mathematical generalization of the 2D Hilbert transform and we then demonstrate that this Fourier transformed kernel is approximately equal to our operator $\nabla I/|\nabla\phi|$. The 2D generalization of the Hilbert transform that we will use is,

$$\mathbf{H}_2\left\{b\cos(\phi)\right\} = \mathscr{F}^{-1}\left\{\left[\frac{-i\,u}{\sqrt{u^2+v^2}}\,\boldsymbol{e}_1 + \frac{-i\,v}{\sqrt{u^2+v^2}}\,\boldsymbol{e}_2\right]\mathscr{F}\{b\cos(\phi)\}\right\} \tag{36}$$

where we define $\mathscr{F}^{-1}(a\,\boldsymbol{e}_1 + b\,\boldsymbol{e}_2) = \mathscr{F}^{-1}(a)\,\boldsymbol{e}_1 + \mathscr{F}^{-1}(b)\,\boldsymbol{e}_2$. Note that the operator $\mathbf{H}_2\{.\}$ is a vector field. This linear operator is the vectorial sum of two Reisz transforms [9] along each coordinate in the spectral space. This $\mathbf{H}_2\{.\}$ operator is equivalent to the spiral operator presented by Larkin *et. al.* [5] when the vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are replaced by 1 and i , respectively (note that if this substitution is performed , $\mathbf{H}_2\{.\}$ would be complex valued, but it is not a vector field). We have found, however, that from a computational viewpoint, equation 36 is better behaved because of the numerical cross-talk between real and imaginary parts that takes place when a single Fourier transform is performed. Eq.36 may be rewritten as,

$$\mathbf{H}_2\{\, I(x,y)\,\} = \mathscr{F}^{-1}\left[\frac{\mathscr{F}[\nabla I(x,y)]}{\sqrt{u^2+v^2}}\right] \tag{37}$$

using the approximation stated in appendix A, we may write the last equation as,

$$\mathbf{H}_2\{\, I(x,y)\,\} \approx \mathscr{F}^{-1}\left[\mathscr{F}\left[\frac{\nabla I(x,y)}{\sqrt{\omega_x^2(x,y)+\omega_y^2(x,y)}}\right]\right] \tag{38}$$

which finally gives,

$$\mathbf{H}_2\{\ I(x,y)\ \} \approx \frac{\nabla I(x,y)}{|\nabla\phi(x,y)|} \tag{39}$$

given that this relation is stated in vectorial form it is valid in any dimension in the Euclidean space. As can be seen from appendix A, the approximation used to transform Eq.37 into Eq.38 is more accurate when the modulating phase $\phi(x,y)$ varies slowly. The result stated in Eq.39 may be used to demonstrate that the vortex operator [5] is an approximation to the quadrature transform $Q_2\{.\}$ proposed in this paper. As shown in apendix A, this approximation is only valid for a smooth modulating phase.

According to what has been presented in this section, we have at least the following two alternative ways to calculate this particular n-dimensional generalization of the Hilbert transform; these are:

$$\mathbf{H}_n\{I(\boldsymbol{r})\} \approx \frac{\nabla I(\boldsymbol{r})}{|\nabla\phi(\boldsymbol{r})|} \tag{40}$$

and,

$$\mathbf{H}_n\{I(\boldsymbol{r})\} = \mathscr{F}^{-1}\left[\frac{-\boldsymbol{q}}{|\boldsymbol{q}|}\,\hat{I}(\boldsymbol{u})\right] \tag{41}$$

where $\boldsymbol{u}$ is a point in the n-dimensional spectral space. The variables $\boldsymbol{r}$ and $\boldsymbol{q}$ are the position vectors in the spectral and the image domains respectively, which are given by

$$\boldsymbol{r} = (x_1,\ldots,x_n)\ , \qquad \boldsymbol{q} = (u_1,\ldots,u_n) \tag{42}$$

Note that this $\mathbf{H}_n\{.\}$ generalization transform also reduces to the 1D Hilbert transform when it is applied to a 1D signal.

IV-3 A Numerical Method to Calculate the Vector Field $\boldsymbol{n}_\phi(x,y)$

As we have shown in the previous section the most critical part in computing the quadrature of a fringe pattern $I(\boldsymbol{r})$ is the estimation of the fringe orientation field $\boldsymbol{n}_\phi=\nabla\phi/|\nabla\phi|$. As mentioned, it is impossible to find a linear system to calculate this vector field directly from the interferogram. The reason is that $\phi(x,y)$ is wrapped by the observation process ($\cos[\phi]$), so we do not have direct access to it (otherwise our problem would be solved). What we do have access from a single image interferogram $I(x,y)$ is the fringe orientation modulo π. This orientation angle must be unwrapped to obtain the searched orientation modulo 2π.

For the reader's convenience in this section we briefly outline the main ideas behind the technique proposed by Quiroga *et. al.*[10] to unwrap the fringe orientation angle modulo π to obtain the required orientation angle modulo 2π. The orientation angle modulo π may be easily found from the interferogram irradiance by

$$\tan[\theta_\pi(x,y)] = \frac{\partial I(x,y)/\partial y}{\partial I(x,y)/\partial x} \tag{43}$$

this formula is valid provided the fringe pattern $I(x,y)$ has been previously normalized which means that the amplitude modulation term $b(x,y)$ is almost constant over the whole region of interest.

The relation between the fringe orientation angle θ_π modulo π with the modulo 2π orientation angle $\theta_{2\pi}$ is,

$$\theta_\pi = \theta_{2\pi} + k\pi \tag{44}$$

where k is an integer. Using this relation one may write,

$$2\theta_\pi = 2\theta_{2\pi} + 2k\pi = W[2\theta_{2\pi}] \tag{45}$$

this result states that the value for $2\theta_{\pi}$ is indistinguishable from that for the wrapped angle $W[2\theta_{2\pi}]$. Therefore it is possible to obtain the unwrapped $\theta_{2\pi}$ by unwrapping $W[2\theta_{\pi}]$ and finally dividing the unwrapped signal $2\theta_{2\pi}$ by 2.

However the unwrapping of $2\theta_{\pi}$ cannot be carried out by standard path independent techniques [11] , because in the presence of closed fringes it is not a consistent field: it may be easily verify that the sum of wrapped differences of $2\theta_{\pi}$ along any closed path that contains a center of closed fringes will be non-zero (in fact, it will be close to 4π) , which is a sufficient condition for path dependency in the unwrapping process [12]. One useful path dependent strategy is to unwrap the signal $2\theta_{\pi}$ following the fringes of the interferogram. In this way one normally encircles (avoids) the discontinuity point at the center of the closed fringes until its surrounding has been unwrapped [10]. Due to the large noise normally encountered in practice for $2\theta_{\pi}$ (due to the ratio of two derivatives) one must use robust path dependent strategies. The algorithm that best fits these requirements is the unwrapping algorithm based on the RPT [13]. A detailed account on fringe's orientation angle unwrapping along with interesting examples is given by Quiroga *et. al.* in Ref. [10].

V.- Discusion

The main contribution of this paper is the expression of the quadrature operator as the dot product of two vector fields (Eq.23). A particular case this idea is implicit in the Vortex transform proposed by Larkin [5] , although in that case it is treated as the product of two complex-valued fields. Here we generalize it in an important way: by giving explicit vectorial formulae for the characterization of these two fields, we not only make the extension to higher dimensions direct, but, more importantly, we open the possibility for using other methods for the computation of each one of these components, which may be more efficient and/or exhibit a better behavior. Thus, we show in this paper that there are several different ways to compute the vector field $\nabla I/|\nabla\phi|$. One possibility is by estimating ∇I and $|\nabla\phi|$ separately and then combining their outcome; a second one is to use ∇I as a coarse approximation to this field and then correct the phase estimate using a pixel-wise iterative procedure (as in sec. IV-1); a third way is by using Fourier transform techniques (as in sec. IV-2) which in the special two dimensional case is equivalent to Larkin's spiral filter [5]. We believe however that the vectorial representation is preferable, both from a theoretical viewpoint (since the generalization to higher dimensions becomes direct), and from a computational one; equation 36 is better behaved because of the numerical cross-talk between real and imaginary parts that takes place when a single Fourier transform is performed. As we will show in section V, the best results are obtained by the space-domain method of section IV-1; because of its mathematical form the $\mathbf{H}_2\{.\}$ operator given by Eq.39 gives unreliable results in places where the local frequency magnitude is close to zero; in contrast, these distortions are automatically and efficiently corrected by the pixel-wise Halley iterations of Eq.34.

The most difficult part of the whole process is the estimation of the orientation angle modulo 2π from its orientation modulo π, and this last step is the main reason why the estimation of the quadrature of single-image interferograms with closed fringes is most difficult. In any case it seems hopeless to reach the phase demodulation robustness of phase stepping or carrier frequency interferometry when a single closed-fringe interferogram is analyzed, the reason being that we have infinitely many solutions compatible with the observations in this case.

One might be tempted to say that the herein proposed quadrature operator $Q_n\{.\}$, as well as the vortex operator in [5] are "direct" methods to estimate the quadrature of a single image with closed fringes. We feel that the word "direct" is not accurate here because it might be interpreted as meaning that with these techniques one may demodulate a single closed-fringe interferogram as easily as in the case of carrier frequency interferometry, but unfortunately this is not the case. The reason is that the vortex operator as well as the quadrature operator presented in this paper readily give the quadrature of the signal once and only once, the orientation of the fringes modulo 2π, and hence $\boldsymbol{n}_\phi$, has been determined, and the estimation of the fringe orientation modulo 2π from a single closed-fringe pattern is a process that is far from "direct".

Once $\boldsymbol{n}_\phi$ is known, one may even use the form of the 2D Hilbert transform $H_2'\{.\}$ contained in the Schlieren operator $S\{.\}$ to obtain another 2D quadrature operator $Q_2'\{.\}$:

$$Q_2'\{\cos(\phi)\} = \frac{\boldsymbol{\omega_0} \bullet \boldsymbol{n}_\phi}{|\boldsymbol{\omega_0} \bullet \boldsymbol{n}_\phi|} H_2'\{\cos(\phi)\} = -\sin(\phi) \qquad \textbf{(46)}$$

where the vector $\boldsymbol{\omega_0} = u_0\, \boldsymbol{i} + v_0\, \boldsymbol{j}$ is the chosen filtering direction that is used by the operator $H_2'\{.\}$. This formula for $Q_2'\{.\}$ is just a two dimensional application of the one dimensional quadrature operator presented in section II. It has, however a drawback: the $H_2'\{.\}$ operator severely distorts the quadrature signal obtained because of the hard spectral discontinuity of the associated filter. As one may see, the n-dimensional quadrature operator is not unique; in our opinion, $Q_2'\{.\}$ is more difficult to evaluate numerically than $Q_2\{.\}$. The fact that several quadrature operators Q are possible, however, opens the possibility of finding other quadrature operators better suited for numerical estimation.

Finally one may realize that the solution to the problem of demodulating a single interferogram in 2D may also be obtained directly from Eq.20 (with $b(\boldsymbol{r})$=1.0), *i.e.* by solving the following set of non-linear coupled partial differential equations,

$$
\begin{aligned}
\sin(\phi)\,\frac{\partial\phi}{\partial x} + I_x = 0\\
\sin(\phi)\,\frac{\partial\phi}{\partial y} + I_y = 0
\end{aligned}
\tag{47}
$$

where $I_x(x,y)$ and $I_y(x,y)$ are the partials of the interferogram $I(x,y)$ with respect to coordinates x and y, the amplitude modulation of the interferogram $b(x,y) \approx 1.0$ has been normalized and the spatial (x,y) dependence was omitted for clarity purposes. The form of this set of partial differential equations may stimulate our imagination to find alternative ways or systems to demodulate interferograms. Among these demodulating systems one might think of powerful non-linear finite element methods to find an approximate solution to this set of equations.

VI Experimental results

We begin by considering the noiseless computer generated fringe pattern of Fig.1a. We show this noise free example to appreciate the form of the signals involved in the estimation of the modulating phase $\phi(x,y)$ of a single-image interferogram containing closed fringes. Figure 1(a) shows the input fringe pattern, Fig.1(b) shows its demodulated wrapped phase. Figures 1(c) and Fig.1(d) show the two centered first order differences I_x and I_y along the x and y coordinates respectively. The fringe orientation angle modulo π is shown in Fig.1(e), the black gray level correspond to 0 radians and the white gray level to π radians. This is the orientation angle that is directly observable from the interferogram $I(x,y)$. This orientation angle is wrapped modulo π which means that the fringe's orientation angle will have the same value at two points situated near and symmetrically away from the center of a given set of closed fringes. Its unwrapped fringe orientation angle (modulo 2π) is shown in Fig.1(f) obtained using the method proposed by

Quiroga *et. al.*.[10]. As can be seen from Fig.1(e) this fringe orientation angle cannot be unwrapped using path independent unwrapping techniques, because the orientation modulo π has three essential phase discontinuities [12]. These discontinuities must be preserved, and that is why an unwrapping strategy following the fringes is the best way to unwrap this signal.

Our next example is the demodulation of an experimentally obtained fringe pattern which is shown in Fig.2(a). These fringes were obtained from a shearing interferometer using Electronic Speckle Pattern Interferometry (ESPI). Figure 2(b) shows the fringe orientation modulo 2π that was obtained by unwrapping the orientation modulo π estimated from the ESPI image. The angle's orientation unwrapping was achieved using the algorithm by Quiroga *et. al.* [10], that is, the fringe orientation was unwrapped following the fringes of the interferogram. Figure 2 panels "c" and "d" show the signals obtained using first order centered differences to approximate the gradient $\nabla I(x,y)$. Figure 2(e) shows the recovered $\boldsymbol{n}_\phi \bullet \nabla I$ signal. Although this signal is orthogonal to the fringe pattern Fig.2(a) it is not everywhere in quadrature with it. As consequence the recovered phase from Figs.2(a) and 2(e) will have some small distortion that can be removed using the Halley algorithm discussed in section IV-1 to finally obtain the phase shown in Fig.2(f).

Finally we have calculated the quadrature signal of the noiseless computer generated fringe pattern $\sin(x^2+y^2)$ (*i.e.,* $\cos(x^2+y^2)$) using the method of section IV-1 to compare it with the vortex operator proposed by Larkin [5] and the quadrature operator using the Schlieren filter (Eq.46). In figure 3a we show the ideal quadrature signal. Figure 3b shows the quadrature signal obtained using the numerical algorithm outlined in section IV-1, one can see that this results is identical to the expected quadrature signal because of the point-wise "adjustment" using the Halley recursion formula. As one can see, the vortex operator distorts the recovered phase (Fig.3c) on regions of very low spatial frequency. Finally figure 3d shows the quadrature operator using the Schlieren filter, this quadrature formula renders the lowest quality result for the reasons explained above.

VII Conclusions

We have presented a quadrature operator that applied to a single interferogram gives its quadrature image $Q\{b\cos(\phi)\}=-b\sin(\phi)$. This result holds true even when the spectrum of the fringes is not well separated, i.e., for single closed-fringe interferograms . We have also shown that this quadrature operator $Q\{I\}$ can be expressed as the vectorial inner product of two vector fields. One of them, i.e., $\nabla I/|\nabla\phi|$, may be seen as an n-dimensional generalization of the Hilbert transform $\mathbf{H}_n\{.\}$ of the interferogram, and the other one is the orientation of the fringes $\boldsymbol{n}_\phi=\nabla\phi/|\nabla\phi|$. In this way our quadrature operator may be expressed as $Q\{I\}=\boldsymbol{n}_\phi\bullet\mathbf{H}_n\{I\}$ in any dimension.

We have also pointed out that the critical step to obtain the quadrature of the interferogram $Q\{I\}$ is the estimation of the fringe orientation $\boldsymbol{n}_\phi=\nabla\phi/|\nabla\phi|$. This operator can only be obtained modulo π directly from the interferogram irradiance, because the cosine function wraps the required signal ϕ. Given that the wrapped orientation modulo π has essential discontinuities that must be preserved, path dependent unwrapping processes are necessary to obtain the fringe's orientation modulo 2π .

Regarding $\mathbf{H}_n\{.\}$ we have presented two practical algorithms for its estimation; one is a frequency-domain method , which is closely related to the spiral operator presented in [5], while the other is space-domain method based on a pixel-wise correction of a phase estimate which is directly obtained from the image gradient. We have shown experimentally that this last method works quite well , even in the presence of moderately high measurement noise (*e.g.* the ESPI image of figure 2), and has the additional advantage of avoiding the phase distortions that frequency domain methods introduce places where low frequency dominates.

Appendix A

In this appendix we prove in an intuitive way the following approximation

$$H[\,\omega_x(x,y),\omega_y(x,y)\,]\,I(x,y) \;\approx\; \mathscr{F}^{-1}\Big\{H(u,v)\;\hat{I}(u,v)\Big\}, \tag{48}$$

where $\hat{I}(u,v)=\mathscr{F}\{I(x,y)\}$ is the two dimensional Fourier transform of $I(x,y)$ of the fringe pattern given by,

$$I(x,y) \;=\; b(x,y)\,\cos[\,\phi(x,y)\,] \tag{49}$$

and the signals $\omega_x(x,y)$ and $\omega_y(x,y)$ are are the local spatial frequencies of the fringe pattern given by:

$$\omega_x(x,y) = \frac{\partial\phi(x,y)}{\partial x}\,, \qquad \omega_y(x,y) = \frac{\partial\phi(x,y)}{\partial y} \tag{50}$$

Let us start by considering the following fringe signal decomposition as a sum of mutually exclusive squared regions, and the local phase $\phi(x,y)$ expanded as a linear phase using Taylor series expansion. This gives,

$$b(x,y)\cos[\phi(x,y)] \;\approx\; \sum_{n,m}\prod(x-n,y-m)\,b(n,m)\cos[\phi(n,m)+\omega_x(n,m)(x-n)+\omega_y(n,m)(y-m)] \tag{51}$$

where the symbol $\sqcap(x\text{-}n,y\text{-}m)$ is a squared window centered at coordinates (n,m) which equals one inside the window and zero otherwise. The value $b(n,m)$ is the fringe contrast evaluated at (n,m); the argument of the cosine function is a phase plane $p(x,y,n,m)$ oriented according to $[\omega_x(n,m),\omega_y(n,m)]$ within the spatial region around (n,m) and limited by $\sqcap(x\text{-}n,y\text{-}m)$. If we pass this signal through a linear spatial filter having a frequency response $H(u,v)$ we may write its response approximately as:

$$\mathscr{F}^{-1}[H(u,v)I(u,v)] \approx \sum_{n,m} \prod(x-n,y-m)\ b(n,m)\ H[\omega_x(n,m),\omega_y(n,m)]\ \cos[p(x,y,n,m)] \tag{52}$$

where $p(x,y,n,m)$ equals the linear Taylor expansion of $\phi(x,y)$ around the point (n,m)

$$p(x,y,n,m) = \phi(n,m)+\omega_x(n,m)(x-n)+\omega_y(n,m)(y-m) \tag{53}$$

An intuitive explanation of this is that inside of a given squared window $\prod(x\text{-}n,y\text{-}m)$ one has a spatially monochromatic fringe pattern, so its filtered amplitude depends only on its spatial frequency content along the x and y directions (ω_x,ω_y). Therefore,

$$\mathscr{F}^{-1}\{H(u,v)\mathscr{F}[I(x,y)]\} \approx H[\omega_x(x,y),\omega_y(x,y)]\ I(x,y) \tag{54}$$

This result is exact only if the fringe pattern is just a modulated phase plane. Conversely this result is less accurate when the modulated phase $\phi(x,y)$ changes abruptly within the domain of interest. This result may not surprise us given that this is the principle behind analog electronic spectrum analyzers and old broadcast FM demodulators using passive RLC electrical networks such the balanced slope FM detector [15].

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FIGURE 1. Demodulation steps using the algorithm proposed in section IV-1applied to a noiseless computer generated interferogram. (a) fringe pattern; (b) recovered phase; (c) and (d) are the two components of ∇I ; (e) fringe orientation angle modulo π ; (f) fringe orientation angle modulo 2π.

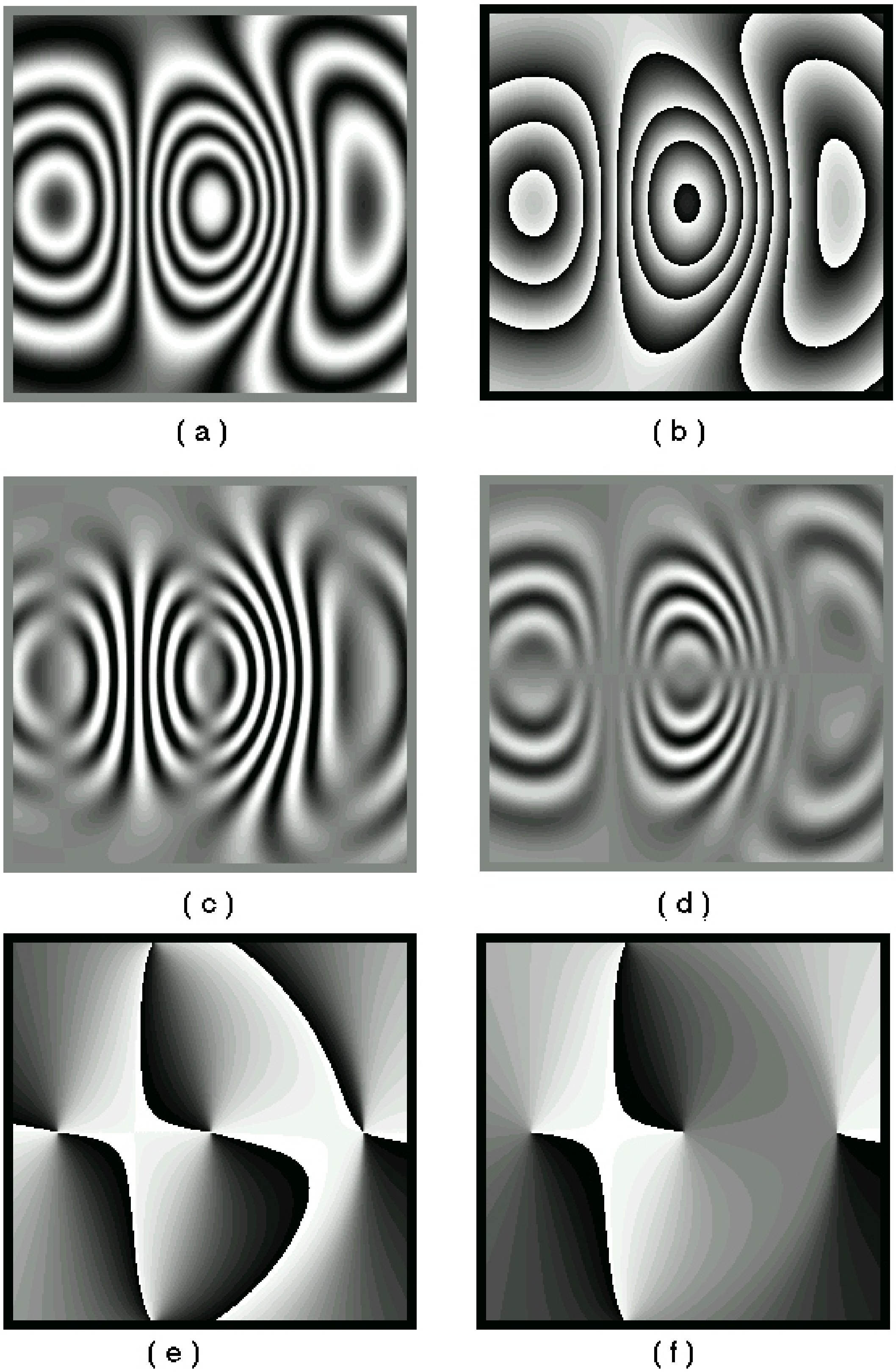

FIGURE 2. Experimentally specklegram. (a) The fringe pattern ; (b) the estimated fringe angle modulo 2π ; (c) and (d) are the two components of ∇I ; (e) shows the recovered signal $\boldsymbol{n}_{\phi} \bullet \nabla I$; (f) phase after the pixel-wise adjustment.

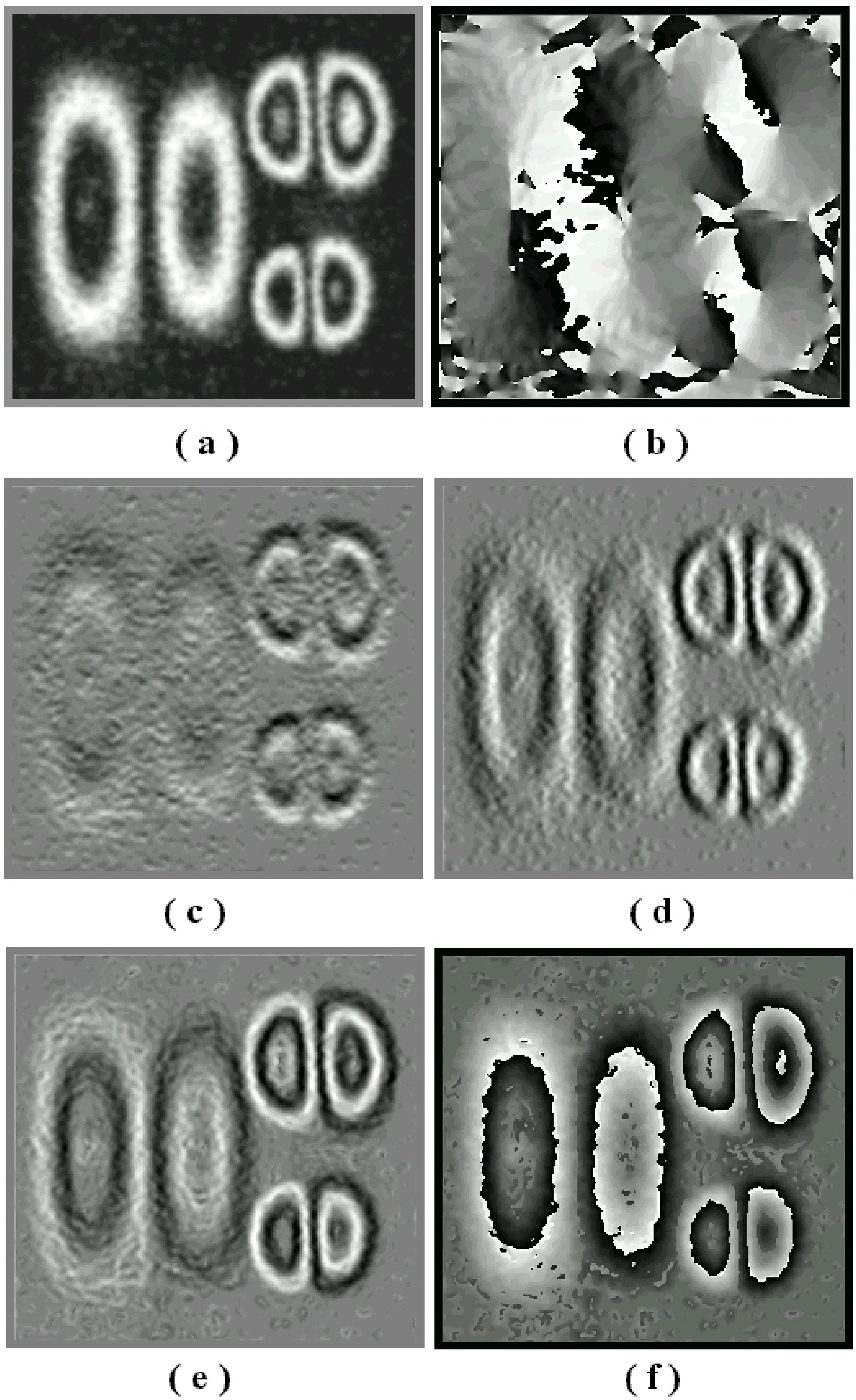

FIGURE 3. Errors obtained using the Vortex operator against the demodulation algorithm proposed using a noiseless computer generated interferogram. (a) The desired quadrature signal; (b) the obtained quadrature signal using the numerical algorithm proposed in section IV-1; (c) quadrature signal obtained using the Vortex operator; (d) quadrature signal obtained using the operator of Eq.46.