SUPER – BROWNIAN LOCAL TIME : A REPRESENTATION AND TWO APPLICATIONS

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Comunicación Técnica No I-02-21/01-10-2002
(PE/CIMAT)

Super-Brownian local time: a representation and two applications

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Abstract

Let $X = \{X_t : t \geq 0\}$ be a super-Brownian motion in $\mathbb{R}^d$ with $d \leq 3$. We give a short proof of existence of the local time L_t^x of X and deduce a semimartingale representation of L_t^x which allows us to prove that $\int_0^t X_s(f)\, ds = \int_{\mathbb{R}^d} f(x) L_t^x\, dx$ a.s. for all bounded measurable functions f. This implies that three different notions of super-Brownian local time known in the literature are equivalent. For $d = 2$ and any continuous function with compact support ψ, we give an easy proof of the weak convergence of $r^{-1} \int_0^{rt} X_s(\psi(\cdot - r^{1/2} y))ds$ to $\int_{\mathbb{R}^d} \psi(x) dx L_t^y$ as $r \to \infty$, a fact that was discovered by Cox and Griffeath [3] and Fleischmann and Gärtner [9].

1 Introduction

The local time of the super-Brownian motion $X := \{X_s,\ s \geq 0\}$ may be formally defined as $L_t^0 = \int_0^t \int_{\mathbb{R}^d} \delta_0(x) X_s(dx)\, ds$ (where δ_0 denotes the Dirac delta function at $0 \in \mathbb{R}^d$), and can be interpreted as a measure of the amount of time in the interval $[0, t]$ during which 0 belongs to the support of X.

A rigorous meaning of super-Brownian local time was given by Adler and Lewin in [1], where they showed that if $d \leq 3$ and $\{\varphi_\epsilon\}$ is a sequence of smooth functions converging to δ_0 in distributional sense, then the "approximating local times" $L_t^{0,\epsilon} = \int_0^t \int_{\mathbb{R}^d} \varphi_\epsilon(x) X_s(dx)\, ds$ converge in L^2 as $\epsilon \to 0$. The limit L_t^0 is independent of the particular choice of $\{\varphi_\epsilon\}$ and is called local time of super-Brownian motion.

Another way of defining the local time of super-Brownian motion is by means of the super-Brownian occupation time, a concept that was introduced by Iscoe [10] in the context of (α, d, β)-superprocess. The occupation time is defined by $Y_t(B) := \int_0^t X_s(B)\, ds$, for $t > 0$ and any Borel set $B \subset \mathbb{R}^d$. The local time $\mathfrak{L}_t$, when it exists,

*Supported in part by CONACyT (Mexico), Grant 32401E.
†Supported by PROMEP (Mexico), Grant UAAGS-99-04.

is defined as the density of $Y_t(dx)$ with respect to the Lebesgue measure $\lambda\,(\equiv dx)$ on $\mathbb{R}^d$. This concept of local time was studied, mainly, by Sugitani [15], Fleischmann [8] and Krone [11]. Dynkin introduced in [6] another concept of local time, l_t, and gave conditions for existence of l_t which are met by super-Brownian motion when $d \leq 3$.

In [7] Feldman and Iyer proved that the local times L_t and l_t are equivalent concepts. In this note we use a semimartingale representation of local time (Theorem 3.1) to show that the three notions of local time L_t, l_t and $\mathfrak{L}_t$ are equivalent.

The representation obtained in Theorem 3.1 holds not only when X is finite measure-valued; it is also valid when X is a super-Brownian motion whose values are infinite tempered measures on $\mathbb{R}^d$, and in such context one can apply formula (4.1) to obtain the limit

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t X_s(\psi)ds, \quad \psi \in C_c(\mathbb{R}^d), \tag{1.1}$$

where $C_c(\mathbb{R}^d)$ denotes the space of real-valued continuous functions on $\mathbb{R}^d$ with compact support.

Cox and Griffeath [3], and independently Fleischmann and Gärtner [9] proved that, when $d = 2$ and $X_0 = \lambda$, the limit (1.1) exists and equals $\left(\int_{\mathbb{R}^2} \psi(x)dx\right)\xi$, where ξ is a non-degenerated infinitely divisible random variable. In Theorem 4.2 we give a brief proof of this result and, moreover, identify the random variable ξ as the super-Brownian local time L_1^0 at time $t = 1$, a fact that was discovered by Fleischmann [8] (for $\mathfrak{L}_1^0$ instead of L_1^0) using a different method.

We conclude this section introducing notations and recalling some basic definitions. The set of all finite measures defined on the Borel σ-algebra $\mathcal{B}(\mathbb{R}^d)$ will be denoted by $M_f(\mathbb{R}^d)$. We denote by $\mu(\varphi)$ the integral with respect to the measure μ of the function $\varphi \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ also denotes the space of Borel measurable functions on $\mathbb{R}^d$. Let $B \equiv \{B_t : t \geq 0\}$ be the d-dimensional Brownian motion, and

$$S_t\varphi(x) := E_x[\varphi(B_t)] = \int_{\mathbb{R}^d} \varphi(y)q_t(x,y)dy, \quad \varphi \in B_b(\mathbb{R}^d), \tag{1.2}$$

where

$$q_t(x,y) \;:=\; q_t(x-y) = \frac{1}{(4\pi t)^{d/2}}\exp(-|x-y|^2/4t), \quad t > 0,\ x,y \in \mathbb{R}^d, \tag{1.3}$$

and $B_b(\mathbb{R}^d)$ is the space of bounded Borel measurable functions on $\mathbb{R}^d$. The family of operators $\{S_t,\ t \geq 0\}$ forms a semigroup of contractions with infinitesimal generator the Laplacian Δ in $\mathbb{R}^d$.

A *càdlàg* Markov process $X = \{X_t : t \geq 0\}$ with state space $M_f(\mathbb{R}^d)$ will be called super-Brownian motion if its Laplace transition functional is given by

$$E_\mu\left[e^{-X_t(\varphi)}\right] = E[e^{-X_t(\varphi)}|X_0 = \mu] = e^{-\mu(u_t)}, \quad \mu \in M_f(\mathbb{R}^d), \quad t \geq 0,$$

where $\varphi \in \mathcal{B}(\mathbb{R}^d)$ is non-negative and $u = u_t$ is the unique non-negative solution of the integral equation

$$u_t = S_t\varphi - \int_0^t S_s(u_{t-s})^2 ds, \quad t \geq 0. \tag{1.4}$$

We refer to [4], [5] or [13] for background on super-Brownian motion and Markov measure-valued processes. A basic fact is the martingale problem for super-Brownian motion (e.g. [2]), which says that for each function $f \in C_b^2$ and $t \geq 0$ the random variable $X_t(f)$ can be expressed as

$$X_t(f) = \mu(f) + M_t(f) + \int_0^t X_s(\Delta f) ds, \tag{1.5}$$

where $M(f)$ is a continuous martingale with increasing process determined by

$$\langle M(f), M(g)\rangle_t = 2\int_0^t X_s(fg) ds, \quad t \geq 0. \tag{1.6}$$

2 Some properties of the Green's function

For each $a > 0$ and $\epsilon \geq 0$ we define

$$G_\epsilon^a(x, y) \; := \; \int_0^\infty e^{-at} q_{t+\epsilon}(x, y)\, dt, \quad x, y \in \mathbb{R}^d. \tag{2.1}$$

$G^a := G_0^a$ is termed the Green's function of Brownian motion B; it is obvious that $G_\epsilon^a(x, y) = G_\epsilon^a(|x - y|)$.

Lemma 2.1 *The function G_ϵ^a is in $L^1(\mathbb{R}^d, dx)$ and its characteristic function $\widehat{G_\epsilon^a}(z)$ is $(a + |z|^2)^{-1} \exp(-\epsilon|z|^2)$. Moreover G_ϵ^a is in $L^2(\mathbb{R}^d, dx)$ for $d \leq 3$.*

Proof. It is obvious that $G_\epsilon^a \in L^1(\mathbb{R}^d, dx)$. The assertion regarding $\widehat{G_\epsilon^a}$ follows clearly from the identity $\widehat{q_t}(x) = e^{-t|x|^2}$. By the Plancherel's theorem

$$\begin{aligned}
||G_\epsilon^a||_2^2 &= ||\widehat{G_\epsilon^a}||_2^2 = \int_{\mathbb{R}^d} \left(\frac{e^{-\epsilon|z|^2}}{a + |z|^2}\right)^2 dz \\
&= c\int_0^\infty r^{d-1} \left(\frac{e^{-\epsilon r^2}}{a + r^2}\right)^2 dr \leq c\int_0^\infty r^{d-1} \left(\frac{1}{a + r^2}\right)^2 dr \\
&\leq c\left\{\int_0^1 r^{d-1} \left(\frac{1}{a + r^2}\right)^2 dr + \int_1^\infty r^{d-5} dr\right\} = c(I + II),
\end{aligned}$$

for some positive constant c. The integral I is finite due to the continuity of $r \mapsto r^{d-1}(a + r^2)^{-2}$ on $[0, 1]$, whereas II is convergent if $d \leq 3$. This shows that G_ϵ^a is in $L^2(\mathbb{R}^d, dx)$ for $d \leq 3$. ∎

Lemma 2.2 *$G^a_\epsilon \to G^a$ in $L^1(\mathbb{R}^d, dx)$ as $\epsilon \to 0$, and the convergence is in $L^2(\mathbb{R}^d, dx)$ if $d \le 3$.*

Proof. We will use the estimation

$$\begin{aligned}
|G^a(x) - G^a_\epsilon(x)| &= \left| \int_0^\infty e^{-at} q_t(x)\, dt - \int_\epsilon^\infty e^{-a(t-\epsilon)} q_t(x)\, dt \right| \\
&\le \left| \int_0^\infty e^{-at} q_t(x)\, dt - e^{a\epsilon} \int_0^\infty e^{-at} q_t(x)\, dt \right| \\
&\quad + \left| e^{a\epsilon} \int_0^\infty e^{-at} q_t(x)\, dt - e^{a\epsilon} \int_\epsilon^\infty e^{-at} q_t(x)\, dt \right| \\
&= |1 - e^{a\epsilon}| \int_0^\infty e^{-at} q_t(x)\, dt + e^{a\epsilon} \int_0^\epsilon e^{-at} q_t(x)\, dt.
\end{aligned}$$

The L^1 convergence follows from

$$\begin{aligned}
||G^a - G^a_\epsilon||_{L^1} &= \int_{\mathbb{R}^d} |G^a(x) - G^a_\epsilon(x)| dx \\
&\le |1 - e^{a\epsilon}| \int_0^\infty e^{-at} \int_{\mathbb{R}^d} q_t(x) dx dt \\
&\quad + e^{a\epsilon} \int_0^\epsilon e^{-at} \int_{\mathbb{R}^d} q_t(x) dx dt \\
&= |1 - e^{a\epsilon}|\, a^{-1} + e^{a\epsilon}(1 - e^{-a\epsilon}) a^{-1} = 2a^{-1}(e^{a\epsilon} - 1).
\end{aligned}$$

Notice that $G^a_\epsilon \in L^2$ for $d \le 3$ due to Lemma 2.1. Using Plancherel's theorem it follows, as in the proof of Lemma 2.1, that

$$\begin{aligned}
||G^a - G^a_\epsilon||^2_{L^2} &= ||(G^a - G^a_\epsilon)\widehat{\ }||^2_{L^2} \\
&\le c \left\{ (1 - e^{-\epsilon})^2 \int_0^1 r^{d-1} \left(\frac{1}{a + r^2} \right)^2 dr \right. \\
&\quad \left. + \int_1^\infty r^{d-5} \left(1 - e^{-\epsilon r^2} \right)^2 dr \right\},
\end{aligned}$$

for some $c > 0$. Using the elementary inequality $1 - e^{-x} \le x^{1/8}$, $x \ge 0$, we obtain

$$\begin{aligned}
||G^a - G^a_\epsilon||^2_{L^2} &\le c \left\{ (1 - e^{-\epsilon})^2 \int_0^1 r^{d-1} \left(\frac{1}{a + r^2} \right)^2 dr + \epsilon^{1/4} \int_1^\infty r^{d-9/2} dr \right\} \\
&= c\{(1 - e^{-\epsilon})^2 I + \epsilon^{1/4} II\}.
\end{aligned}$$

The integral I has already been considered in the proof of Lemma 2.1, whereas the integral II is finite for $d \le 3$. Letting $\epsilon \to 0$ yields convergence in L^2 if $d \le 3$. ■

Lemma 2.3 *$\Delta G^a_\epsilon = a G^a_\epsilon - q_\epsilon$ for each $a > 0$ and $\epsilon > 0$.*

Proof. Noting that

$$-aG^a_\epsilon(x)=\int_0^\infty q_{t+\epsilon}(x)\frac{d}{dt}(e^{-at})=q_{t+\epsilon}(x)e^{-at}\big|_0^\infty-\int_0^\infty e^{-at}\frac{d}{dt}(q_{t+\epsilon}(x)),$$

from the equalities

$$\frac{d}{dt}(q_{t+\epsilon}(x))=q_{t+\epsilon}(x)\left(\frac{|x|^2}{4(t+\epsilon)^2}-\frac{d}{2(t+\epsilon)}\right)=\Delta q_{t+\epsilon}(x),$$

we arrive to the expression

$$-aG^a_\epsilon(x)=-q_\epsilon(x)-\int_0^\infty e^{-at}\Delta q_{t+\epsilon}(x)\,dt.$$

From here the result follows by applying the dominate convergence theorem. ■

3 A representation of the local time of super-Brownian motion

Theorem 3.1 *Let X be the super-Brownian motion with $X_0=\mu\in M_f(\mathbb{R}^d)$, where $\mu\ll dx$ and $d\mu/dx\in B_b(\mathbb{R}^d)$. If $d\le 3$, then the local time L^0_t exist and admits the representation:*

$$L^0_t=\mu(G^a)-X_t(G^a)+a\int_0^t X_s(G^a)\,ds+M_t(G^a),\quad a.s. \tag{3.1}$$

for each $t>0$ and $a>0$, where $M(G^a)$ is a square integrable martingale. Moreover, for each $z\in\mathbb{R}^d$ the local time at z, L^z_t, has the expression.

$$L^z_t=\mu(G^a(\cdot-z))-X_t(G^a(\cdot-z))+a\int_0^t X_s(G^a(\cdot-z))\,ds+M_t(G^a(\cdot-z)).$$

By differentiating in the usual way the Laplace functional of X one obtains

$$E_\mu\left[X_t(\varphi)\right]=\mu(S_t\varphi), \tag{3.2}$$

$$E_\mu\left[X_t(f)X_s(g)\right]=\mu(S_tf)\mu(S_sg)+2\int_0^{s\wedge t}\mu\left(S_r\left((S_{t-r}f)(S_{s-r}g)\right)\right)dr, \tag{3.3}$$

for $\mu\in M_f(\mathbb{R}^d)$, $f,g\in B_b(\mathbb{R}^d)$ and $0\le s,t$. From here it is easy to see that for any non-negative $f\in L^1(\mathbb{R}^d,dx)\cap L^2(\mathbb{R}^d,dx)$ and $t\ge 0$

$$E_\mu\left[X_t(f)\right]\le ||d\mu/dx||_\infty||f||_{L^1}, \tag{3.4}$$

$$E_\mu\left[(X_t(f))^2\right]\le c(t)\left(||f||^2_{L^1}+||f||^2_{L^2}\right), \tag{3.5}$$

for some positive constant $c(t)$.

Proof of Theorem 3.1. Using (1.5) and the equality $\Delta G^a_\epsilon = aG^a_\epsilon - q_\epsilon$ that we proved in Lemma 2.3 we obtain

$$\int_0^t X_s(q_\epsilon)\, ds = \mu(G^a_\epsilon) - X_t(G^a_\epsilon) + a\int_0^t X_s(G^a_\epsilon)\, ds + M_t(G^a_\epsilon). \tag{3.6}$$

We want to show that the sequence of random variables $\{\int_0^t X_s(q_\epsilon)\, ds\}_{\epsilon>0}$ converges in L^2 as $\epsilon \to 0$. To prove this it is enough to show convergence in L^2 of the right hand side of (3.6). Using Lemma 2.1, (3.4), (3.5) and Jensen's inequality it follows that

$$\begin{aligned} E\left[\left(\int_0^t X_s(G^a_\epsilon)ds - \int_0^t X_s(G^a)ds\right)^2\right] &= E\left[\left(\int_0^t X_s(G^a_\epsilon - G^a)ds\right)^2\right] \\ &\leq t\int_0^t E\left[(X_s(G^a_\epsilon - G^a))^2\right]\, ds \\ &\leq c(t)\left(\|G^a_\epsilon - G^a\|^2_{L^1} + \|G^a_\epsilon - G^a\|^2_{L^2}\right). \end{aligned}$$

From (1.6) we obtain

$$\begin{aligned} E[(M_t(G^a_\epsilon) - M_t(G^a))^2] &= E[(M_t(G^a_\epsilon - G^a))^2] \\ &= 2\int_0^t E\left[X_s((G^a_\epsilon - G^a)^2)\right]\, ds \\ &\leq 2t\,\|d\mu/dx\|_\infty\, \|G^a_\epsilon - G^a\|^2_{L^2}, \end{aligned}$$

which together with Lemma 2.2 shows that $M_t(G^a_\epsilon) \to M_t(G^a)$ in L^2 as $\epsilon \to 0$. The L^2-convergence of the remaining terms can be obtained in a similar fashion. The last statement in Theorem 3.1 is a consequence of invariance under translation of the Lebesgue measure. ∎

Remark. Consider the sequence of processes $\{L^{0,\epsilon}\}_{\epsilon>0}$ defined by $L^{0,\epsilon} = \{\int_0^t X_s(q_\epsilon)\, ds,\ 0 \leq t \leq T\}$. Let $\{\epsilon_n\}_n$ be a sequence of non-negative numbers that converges to cero, and $0 \leq t_1, ..., t_k \leq T$. Then from the proof of Theorem 3.1,

$$E\left|(L^{0,\epsilon_n}_{t_1}, ..., L^{0,\epsilon_n}_{t_k}) - (L^0_{t_1}, ..., L^0_{t_k})\right| \leq k\sum_{i=1}^k E\left|L^{0,\epsilon_n}_{t_i} - L^0_{t_i}\right| \to 0$$

as $\epsilon_n \to 0$, i.e., the finite-dimensional distributions of $\{L^{0,\epsilon}\}_{\epsilon>0}$ converge to those of L^0. Following the approach of [12] it is possible to show tightness of $\{L^{0,\epsilon}\}_{\epsilon>0}$, and hence weak convergence of $\{L^{0,\epsilon}\}_{\epsilon>0}$ to L^0 as $\epsilon \to 0$ in the Skorokhod space $D_{[0,T]}(\mathbb{R})$.

4 Two applications of the representation of super-Brownian local time

The following theorem, together with Proposition 2.3 in [7], implies that the three notions of super-Brownian local time referred to in the introductory section yield equivalent concepts.

Theorem 4.1 *Let X be the super-Brownian motion of Theorem 3.1 and $d \le 3$. If $f \in L^1(\mathbb{R}^d, dx) \cap L^2(\mathbb{R}^d, dx)$ then*

$$\int_0^t X_s(f)\, ds = \int_{\mathbb{R}^d} f(x) L_t^x \, dx \quad a.s. \tag{4.1}$$

for all $t \ge 0$. The expression (4.1) *holds true if f is only measurable and bounded. In particular, $L_t = \mathfrak{L}_t$, a.s.*

Proof. We first consider the case in which $f \in L^1(\mathbb{R}^d, dx) \cap L^2(\mathbb{R}^d, dx)$. In order to work with the local time at an arbitrary point $z \in \mathbb{R}^d$, we write (3.6) in the form

$$\begin{aligned}\int_0^t X_s(q_\epsilon(\cdot - z))\, ds &= \mu(G_\epsilon^a(\cdot - z)) - X_t(G_\epsilon^a(\cdot - z)) \\ &\quad + a\int_0^t X_s(G_\epsilon^a(\cdot - z))\, ds + M_t(G_\epsilon^a(\cdot - z)).\end{aligned}$$

Multiplying both sides of the above expression by $f(z)$ and integrating with respect to z we obtain

$$\begin{aligned}\int_0^t X_s\left(\int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz\right) ds &= \int_{\mathbb{R}^d} f(z) \int_0^t X_s(q_\epsilon(\cdot - z))\, ds\, dz \\ &= \int_{\mathbb{R}^d} f(z) \left[\mu(G_\epsilon^a(\cdot - z)) - X_t(G_\epsilon^a(\cdot - z))\right. \\ &\quad \left. + a\int_0^t X_s(G_\epsilon^a(\cdot - z)) ds + M_t(G_\epsilon^a(\cdot - z))\right] dz.\end{aligned}$$

Let us start by showing the convergence

$$\int_0^t X_s\left(\int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz\right) ds \to \int_0^t X_s(f) ds \tag{4.2}$$

in $L^2(P)$ when $\epsilon \to 0$. By Jensen's inequality and (3.5)

$$\begin{aligned}& E\left[\left(\int_0^t X_s\left(\int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz\right) ds - \int_0^t X_s(f) ds\right)^2\right] \\ &= E\left[\left(\int_0^t X_s\left(\int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz - f(\cdot)\right) ds\right)^2\right] \\ &\le t\int_0^t E\left[\left(X_s\left(\int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz - f(\cdot)\right)\right)^2\right] ds \\ &\le c(t)\left(\left\| \int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz - f(\cdot)\right\|_{L^1}^2 + \left\| \int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz - f(\cdot)\right\|_{L^2}^2\right).\end{aligned} \tag{4.3}$$

Using the scaling property of the Gaussian density q_t,

$$\begin{aligned}
&\left\| \int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz - f(\cdot) \right\|_{L^1} \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-z) q_\epsilon(z) dz - f(x) \right| dx \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-z) q_1(\epsilon^{-1/2} z) \epsilon^{-d/2} dz - f(x) \right| dx \\
&= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} f(x-\epsilon^{1/2} z) q_1(z) dz - \int_{\mathbb{R}^d} f(x) q_1(z) dz \right| dx \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q_1(z) \left| f(x-\epsilon^{1/2} z) - f(x) \right| dz dx \\
&= \int_{\mathbb{R}^d} q_1(z) ||f(\cdot - \epsilon^{1/2} z) - f(\cdot)||_{L^1} dz,
\end{aligned}$$

where $\lim_{\epsilon \to 0} ||f(\cdot - \epsilon^{1/2} z) - f(\cdot)||_{L^1} = 0$ because $f \in L^1(\mathbb{R}^d, dx)$. Since $q_1(z)||f(\cdot - \epsilon^{1/2} z) - f(\cdot)||_{L^1} \leq 2||f||_{L^1} q_1(z)$, it follows from the bounded convergence theorem that the first summand in the right hand side of (4.3) converges to 0 as $\epsilon \to 0$.

For the other term in the right hand side of (4.3) we use Jensen's inequality:

$$\begin{aligned}
\left\| \int_{\mathbb{R}^d} f(z) q_\epsilon(\cdot - z) dz - f(\cdot) \right\|_{L^2}^2 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} f(z) q_\epsilon(x - z) dz - f(x) \right)^2 dx \\
&\leq \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left| f(x-\epsilon^{1/2} z) - f(x) \right| q_1(z) dz \right)^2 dx \\
&\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(x-\epsilon^{1/2} z) - f(x)|^2 q_1(z) dz dx \\
&= \int_{\mathbb{R}^d} q_1(z) ||f(\cdot - \epsilon^{1/2} z) - f(\cdot)||_{L^2}^2 dz.
\end{aligned}$$

Using now that $f \in L^2(\mathbb{R}^d, dx)$ we conclude as before that the second term in the right hand side of (4.3) tends to 0 as $\epsilon \to 0$. This finishes the proof of (4.2).

Now we will prove that

$$\int_{\mathbb{R}^d} f(z) \left[\mu(G_\epsilon^a(\cdot - z)) - X_t(G_\epsilon^a(\cdot - z)) + a \int_0^t X_s(G_\epsilon^a(\cdot - z))\, ds + M_t(G_\epsilon^a(\cdot - z)) \right] dz$$

converges to

$$\int_{\mathbb{R}^d} f(z) \left[\mu(G^a(\cdot - z)) - X_t(G^a(\cdot - z)) + a \int_0^t X_s(G^a(\cdot - z)) ds + M_t(G^a(\cdot - z)) \right] dz,$$

in $L^2(P)$ when $\epsilon \to 0$. Indeed, using Jensen's inequality, (1.6) and (3.4),

$$\begin{aligned} & E\left[\left(\int_{\mathbb{R}^d} f(z) M_t(G_\epsilon^a(\cdot - z))dz - \int_{\mathbb{R}^d} f(z) M_t(G^a(\cdot - z))dz\right)^2\right] \\ &= E\left[\left(\int_{\mathbb{R}^d} f(z) M_t(G_\epsilon^a(\cdot - z) - G^a(\cdot - z))dz\right)^2\right] \\ &\le E\left[\left(\int_{\mathbb{R}^d} |f(z)|dz\right)\int_{\mathbb{R}^d} (M_t(G_\epsilon^a(\cdot - z) - G^a(\cdot - z)))^2 |f(z)|dz\right] \\ &= ||f||_{L^1} \int_{\mathbb{R}^d} E[(M_t(G_\epsilon^a(\cdot - z) - G^a(\cdot - z)))^2]|f(z)|dz \\ &= ||f||_{L^1} \int_{\mathbb{R}^d} \int_0^t E[X_s((G_\epsilon^a(\cdot - z) - G^a(\cdot - z))^2)]ds|f(z)|dz \\ &\le ||f||_{L^1} c(t) \int_{\mathbb{R}^d} f(z) ||G_\epsilon^a(\cdot - z) - G^a(\cdot - z)||_{L^2}^2 dz \\ &= ||f||_{L^1}^2 c(t) ||G_\epsilon^a - G^a||_{L^2}^2 \end{aligned}$$

Using again Jensen's inequality and (3.5),

$$\begin{aligned} & E\left[\left(\int_{\mathbb{R}^d} f(z) \int_0^t X_s(G_\epsilon^a(\cdot - z))dsdz - \int_{\mathbb{R}^d} f(z) \int_0^t X_s(G^a(\cdot - z))dsdz\right)^2\right] \\ &= E\left[\left(\int_{\mathbb{R}^d} f(z) \int_0^t X_s(G_\epsilon^a(\cdot - z) - G^a(\cdot - z))dsdz\right)^2\right] \\ &\le ||f||_{L^1} \int_{\mathbb{R}^d} \int_0^t E[X_s(G_\epsilon^a(\cdot - z) - G^a(\cdot - z))^2]ds|f(z)|dz \\ &\le c(t)||f||_{L^1} \int_{\mathbb{R}^d} \int_0^t (||(G_\epsilon^a - G^a)(\cdot - z)||_{L^1}^2 + ||(G_\epsilon^a - G^a)(\cdot - z)||_{L^2}^2)ds|f(z)|dz \\ &= c(t)t||f||_{L^1}^2 (||G_\epsilon^a - G^a||_{L^1}^2 + ||G_\epsilon^a - G^a||_{L^2}^2). \end{aligned}$$

The remaining moments $E\left[\left(\int_{\mathbb{R}^d} f(z)\mu\left(G_\epsilon^a(\cdot - z)\right)\, dz - \int_{\mathbb{R}^d} f(z)\mu\left(G^a(\cdot - z)\right)\, dz\right)^2\right]$ and $E\left[\left(\int_{\mathbb{R}^d} f(z)X_t\left(G_\epsilon^a(\cdot - z)\right)\, dz - \int_{\mathbb{R}^d} f(z)X_t\left(G^a(\cdot - z)\right)\, dz\right)^2\right]$ can be bounded in a similar way. Applying Lemma 2.2 we obtain that

$$\begin{aligned} \int_0^t X_s(f)ds &= \int_{\mathbb{R}^d} f(z)(\mu(G^a(\cdot - z)) - X_t(G^a(\cdot - z)) \\ &\quad + a\int_0^t X_s(G^a(\cdot - z))ds + M_t(G^a(\cdot - z)))dz, \quad \text{a.s.} \end{aligned} \tag{4.4}$$

for all $t \ge 0$, and from Theorem 3.1 we conclude that

$$\int_0^t X_s(f)ds = \int_{\mathbb{R}^d} f(z)L_t^z dz, \quad \text{a.s.}$$

Consider now the case in which f is a bounded Borel-measurable function. Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R}^d, dx) \cap L^2(\mathbb{R}^d, dx)$ with $||f_n||_\infty \leq ||f||_\infty$ for each n, and such that $f_n \to f$ pointwise. Then by (4.4) we have, for each n,

$$\begin{aligned}\int_0^t X_s(f_n)ds &= \int_{\mathbb{R}^d} f_n(z)(\mu(G^a(\cdot - z)) - X_t(G^a(\cdot - z)) \\ &\quad + a\int_0^t X_s(G^a(\cdot - z))ds + M_t(G^a(\cdot - z)))dz. \qquad (4.5)\end{aligned}$$

We will show that

$$\int_0^t X_s(f_n)ds \to \int_0^t X_s(f)ds \quad \text{in } L^2(P) \text{ as } \epsilon \to 0. \qquad (4.6)$$

Indeed, by Jensen's inequality and (3.5),

$$\begin{aligned}&E\left[\left(\int_0^t X_s(f_n)ds - \int_0^t X_s(f)ds\right)^2\right] \\ &= E\left[\left(\int_0^t X_s(f_n - f)ds\right)^2\right] \\ &\leq t\int_0^t E[(X_s(f_n - f))^2]ds \\ &= t\int_0^t ((\mu(S_s(f_n - f)))^2 + 2\int_0^s \mu(S_r(S_{s-r}(f_n - f))^2)dr)ds \\ &\leq t\int_0^t (\mu(1)\int_{\mathbb{R}^d}(S_s(f_n - f))^2(x)\mu(dx) + 2\mu(\int_0^s S_r(S_{s-r}(f_n - f))^2 dr))ds \\ &\leq ct\int_0^t \mu((S_s(f_n - f))^2 + \int_0^s S_r(S_{s-r}(f_n - f))^2 dr)ds.\end{aligned}$$

Since $|(f_n - f)(y)|q_s(x, y) \leq 2||f||_\infty q_s(x, y)$, we deduce that $S_s(f_n - f)(x) \to 0$ as $n \to \infty$. Moreover, $|S_s(f_n - f)(x)| \leq 2||f||_\infty$ implies that

$$\left|(S_s(f_n - f)(x))^2 + \int_0^s S_r(S_{s-r}(f_n - f))^2(x)dr\right| \leq 4||f||_\infty^2(1 + s)$$

uniformly in x, and hence

$$(S_s(f_n - f)(x))^2 + \int_0^s S_r(S_{s-r}(f_n - f))^2(x)dr \to 0 \quad \text{as} \quad n \to \infty.$$

In this way (4.6) follows from the dominated convergence theorem.

We now prove the convergence of the right hand side of (4.5). It suffices to show how to achieve this for two of the terms since the convergence of the remaining ones is proved in a similar way.

From (3.4) we obtain

$$\begin{aligned}
& E\left[\left(\int_{\mathbb{R}^d} f_n(z)M_t(G^a(\cdot - z))dz - \int_{\mathbb{R}^d} f(z)M_t(G^a(\cdot - z))dz\right)^2\right] \\
= \; & E\left[\left(\int_{\mathbb{R}^d}(f_n(z) - f(z))M_t(G^a(\cdot - z))dz\right)^2\right] \\
= \; & E\left[\int_{\mathbb{R}^d}(f_n(z) - f(z))M_t(G^a(\cdot - z))dz \int_{\mathbb{R}^d}(f_n(w) - f(w))M_t(G^a(\cdot - w))dw\right] \\
= \; & \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f_n(z) - f(z))(f_n(w) - f(w))E\left[M_t(G^a(\cdot - z))M_t(G^a(\cdot - w))\right]dzdw \\
= \; & E\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f_n(z) - f(z))(f_n(w) - f(w))\int_0^t X_s(G^a(\cdot - z)G^a(\cdot - w))dsdzdw\right] \\
= \; & E\left[\int_0^t X_s\left(\left(\int_{\mathbb{R}^d}(f_n(z) - f(z))G^a(\cdot - z)dz\right)^2\right)ds\right] \\
= \; & \int_0^t \mu\left(S_s\left(\int_{\mathbb{R}^d}(f_n(z) - f(z))G^a(\cdot - z)dz\right)^2\right)ds.
\end{aligned}$$

Since $|(f_n(z) - f(z))G^a(\cdot - z)| \leq 2\|f\|_\infty G^a(\cdot - z)$ and $G^a \in L^1(\mathbb{R}^d, dx)$, we have

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}(f_n(z) - f(z))G^a(\cdot - z)\,dz = 0$$

and

$$\lim_{n\to\infty} S_s\left(\int_{\mathbb{R}^d}(f_n(z) - f(z))G^a(\cdot - z)dz\right)^2(x) = 0$$

for any $s \geq 0$. Using this and that

$$\left|S_s\left(\int_{\mathbb{R}^d}(f_n(z) - f(z))G^a(\cdot - z)dz\right)^2(x)\right| \leq 4||f||_\infty^2||G^a||_{L^1}^2,$$

we conclude from the dominated convergence theorem that

$$\int_{\mathbb{R}^d} f_n(z)M_t(G^a(\cdot - z))dz \to \int_{\mathbb{R}^d} f(z)M_t(G^a(\cdot - z))dz$$

in $L^2(P)$ as $n \to \infty$. Finally, let us show convergence of the term containing $\int_0^t X_s(G^a(\cdot - z))ds$. From (3.5),

$$
\begin{aligned}
&E\left[\left(\int_{\mathbb{R}^d} f_n(z)\int_0^t X_s(G^a(\cdot - z))dsdz - \int_{\mathbb{R}^d} f(z)\int_0^t X_s(G^a(\cdot - z))dsdz\right)^2\right] \\
&= E\left[\left(\int_{\mathbb{R}^d} (f_n(z) - f(z))\int_0^t X_s(G^a(\cdot - z))dsdz\right)^2\right] \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^t\int_0^t (f_n(z) - f(z))(f_n(w) - f(w)) \\
&\quad \cdot E\left[X_s(G^a(\cdot - z))X_r(G^a(\cdot - w))\right] dr\, ds\, dz\, dw \\
&= \left(\mu\left(\int_0^t (f_n(z) - f(z))S_sG^a(\cdot - z)dzds\right)\right)^2 \\
&\quad +2\mu\left(\int_0^t\int_0^t \left\{\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f_n(z) - f(z))(f_n(w) - f(w))\right.\right. \\
&\quad \left.\left. \cdot \int_0^{r\wedge s} S_v(S_{r-v}G^a(\cdot - z)S_{s-v}G^a(\cdot - w))\, dv\, dw\, dz\right\} dr\, ds\right).
\end{aligned}
$$

It suffices to show that the sequence

$$
\begin{aligned}
I_n := &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (f_n(z) - f(z))(f_n(w) - f(w)) \\
&\cdot \int_0^{r\wedge s} S_v(S_{r-v}G^a(\cdot - z)S_{s-v}G^a(\cdot - w))\, dv\, dw\, dz, \quad n \in \mathbb{N},
\end{aligned}
$$

is uniformly bounded and converges to cero. Indeed,

$$
\begin{aligned}
I_n &\leq 4||f||_\infty^2 \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^{r\wedge s} S_v(S_{r-v}G^a(\cdot - z)S_{s-v}G^a(\cdot - w))(x)\, dv\, dw\, dz \\
&= 4||f||_\infty^2 \int_0^{r\wedge s}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} S_{r-v}G^a(\cdot - z)(y)\, dz \\
&\quad \cdot \int_{\mathbb{R}^d} S_{s-v}G^a(\cdot - w)(y)\, dw\, q_v(y, x)\, dy\, dv,
\end{aligned}
$$

where

$$
\int_{\mathbb{R}^d} S_{s-v}G^a(\cdot - w)(y)dw = ||G^a||_{L^1} = \int_{\mathbb{R}^d} S_{r-v}G^a(\cdot - z)(y)dz.
$$

Hence $I_n \leq 4t||f||_\infty^2||G^a||_{L^1}^2$ and the convergence of I_n to cero follows from the fact that

$$
\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_0^{r\wedge s} S_v(S_{r-v}G^a(\cdot - z)S_{s-v}G^a(\cdot - w))(x)\, dv\, dw\, dz,
$$

is bounded.

We have proved that for any bounded Borel-measurable f,

$$\begin{aligned}\int_0^t X_s(f)ds &= \int_{\mathbb{R}^d} f(z)\{\mu(G^a(\cdot - z)) - X_t(G^a(\cdot - z)) \\ &\quad + a\int_0^t X_s(G^a(\cdot - z))ds + M_t(G^a(\cdot - z))\}dz, \quad \text{a.s.},\end{aligned}$$

for all $t \geq 0$. The expression (4.1) follows from Theorem 3.1. Putting $f = 1_B$ in (4.1), where $B \in \mathcal{B}(\mathbb{R}^d)$, yields

$$Y_t(B) = \int_0^t X_s(1_B)ds = \int_B L_t^z dz \quad \text{a.s.}$$

which means that $Y_t(\cdot) \ll dx$ and that its density is precisely the local time. Therefore $\mathfrak{L}_t^z := dY_t/dx = L_t^z$ a.s. for all $t \geq 0$. ■

Theorem 4.2 *Let X be the super-Brownian motion starting with $X_0 = \lambda$ and $d = 2$. Then for each $y \in \mathbb{R}^2$, $t > 0$ and $\psi \in C_c(\mathbb{R}^2)$,*

$$\frac{1}{r}\int_0^{rt} X_s(\psi(\cdot - r^{1/2}y))ds \to \lambda(\psi)L_t^y$$

in distribution as $r \to \infty$, where L_t^y is the local time of X. Moreover, $\mathrm{Var}(L_t^y) = (t^2 \ln 2)/2\pi$.

Remark. Note that the value of $\mathrm{Var}\,(L_t^y)$ is consistent with the one obtained in [3].

Proof of Theorem 4.2. Using the martingale problem of [14] (Proposition 1.7) it is easy to see that our theorems 3.1 and 4.1 remain valid when $X_0 = \lambda$. Moreover [14], for each $R > 0$, $\{R^{-2}X_t(\varphi(\cdot/R))\}$ has the same distribution as $\{X_{t/R^2}(\varphi)\}$. Hence putting $R = r^{1/2}$ and $t = r^{-1}s$ with $r > 0$, $s > 0$, we conclude that $\{rX_{s/r}(\varphi(r^{1/2}\cdot))\}$ and $\{X_s(\varphi)\}$ have a common distribution, and that the same is true for

$$\frac{1}{r}\int_0^{rt} X_s(\psi(\cdot - r^{1/2}y))ds \quad \text{and} \quad \frac{1}{r}\int_0^{rt} rX_{s/r}(\psi(r^{1/2}\cdot - r^{1/2}y))ds.$$

By Theorem 4.1,

$$\begin{aligned}\frac{1}{r}\int_0^{rt} rX_{s/r}(\psi(r^{1/2}\cdot - r^{1/2}y))ds &= \frac{1}{r}\int_0^t rX_s(\psi(r^{1/2}(\cdot - y)))rds \\ &= \int_{\mathbb{R}^2} r\psi(r^{1/2}(x-y))L_t^x dx \quad \text{a.s.} \\ &= \int_{\mathbb{R}^2} \psi(x)L_t^{y+x/r^{1/2}}dx.\end{aligned}$$

Since L_t^x is a process continuous in x [15] and ψ is continuous with compact support, it follows that a.s.

$$\int_{\mathbb{R}^2} \psi(x) L_t^{y+x/r^{1/2}} dx \to \lambda(\psi) L_t^y \quad \text{as } r \to \infty.$$

In Theorem 3.1 we proved that $E[(L_t^y)^2] = \lim_{\epsilon\to 0} E[(L_t^{y,\epsilon})^2]$, where

$$L_t^{y,\epsilon} := \int_0^t X_s(q_\epsilon(\cdot - y))ds, \quad \epsilon > 0.$$

Due to the fact that Lebesgue measure λ is invariant for the Brownian motion, it follows from (3.3) that

$$\begin{aligned}
E\left[(L_t^{y,\epsilon})^2\right] &= \int_0^t \int_0^t E\left[X_u(q_\epsilon(\cdot - y))X_v(q_\epsilon(\cdot - y))\right] du\, dv \\
&= \left(\int_0^t \lambda\left(S_u(q_\epsilon(\cdot - y))\right) du\right)^2 \\
&\quad +2\int_0^t \int_0^t \int_0^{u\wedge v} \lambda\left(S_w(S_{u-w}(q_\epsilon(\cdot - y))S_{v-w}(q_\epsilon(\cdot - y)))\right) dw\, du\, dv \\
&= \left(\int_0^t \lambda(q_\epsilon(\cdot - y))\, du\right)^2 \\
&\quad +2\int_0^t \int_0^t \int_0^{u\wedge v} \lambda\left(S_{u-w}(q_\epsilon(\cdot - y))S_{v-w}(q_\epsilon(\cdot - y))\right) dw\, du\, dv.
\end{aligned}$$

By the Chapman-Kolmogorov equation

$$\begin{aligned}
\lambda(S_{u-w}(q_\epsilon(\cdot - y))S_{v-w}(q_\epsilon(\cdot - y))) &= \lambda(q_{\epsilon+u-w}(\cdot - y)q_{\epsilon+v-w}(\cdot - y)) \\
&= \frac{1}{4\pi(2\epsilon + u + v - 2w)}.
\end{aligned}$$

Hence

$$\begin{aligned}
E[(L_t^{y,\epsilon})^2] &= t^2 + \frac{1}{2\pi}\int_0^t \int_0^t \int_0^{u\wedge v} \frac{1}{2\epsilon + u + v - 2w}\, dw\, du\, dv \\
&= t^2 + \frac{1}{4\pi}\{2t^2 \ln(\epsilon + t) + 2t^2 \ln 2 - 2t^2 \ln(2\epsilon + t) \\
&\quad +6\epsilon t + 4\epsilon t \ln 2 + 4\epsilon t \ln(\epsilon + t) - 8\epsilon t \ln(2\epsilon + t) \\
&\quad +8\epsilon^2 \ln 2 + 6\epsilon^2 \ln \epsilon + 2\epsilon^2 \ln(\epsilon + t) - 8\epsilon^2 \ln(2\epsilon + t)\}.
\end{aligned}$$

Therefore $E[(L_t^y)^2] = t^2 + (t^2 \ln 2)/2\pi$. The result follows noticing that $E[L_t^{y,\epsilon}] = t$ for all $\epsilon > 0$. ■

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