TWO COMPARISON CRITERIA FOR THE FUJITA EQUATION IN THE GLOBAL REGIME

Matthias Brikner, José A. López – Mimbela
and Anton Wakolbinger

Comunicación Técnica No I-02-24/30-10-2002
(PE/CIMAT)

TWO COMPARISON CRITERIA FOR THE FUJITA EQUATION IN THE GLOBAL REGIME*

MATTHIAS BIRKNER, JOSÉ A. LÓPEZ-MIMBELA
and ANTON WAKOLBINGER

Abstract

Using the Feynman-Kac representation of the Fujita equation $\frac{\partial}{\partial t}u(t,x) = -(-\Delta)^{\alpha/2}u(t,x) + G(u(t,x))$, $x \in \mathbf{R}^d$, where $0 < \alpha \leq 2$, $\alpha < d$ and $G : \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function obeying certain growth conditions, it is shown that if v is a given global solution and $0 < \varepsilon < 1$, then any solution u satisfying $0 \leq u(0,\cdot) \leq (1-\varepsilon)v(0,\cdot)$ vanishes asymptotically as $t \to \infty$. The case $0 \leq v(0,\cdot) \leq (1+\varepsilon)u(0,\cdot)$ is also studied. For $G(z) = z^{(d+\alpha)/(d-\alpha)}$ a two-parameter family of radially symmetric stationary solutions of the above equation is obtained.

1 Introduction

In this paper we continue our study initiated in [2] on the Fujita equation

$$\begin{aligned} \frac{\partial}{\partial t}u(t,x) &= Lu(t,x) + G(u(t,x)), \quad t \geq 0,\ x \in \mathbf{R}^d, \\ u(0,x) &= \varphi(x) \geq 0, \end{aligned} \tag{1.1}$$

where $L = \Delta_\alpha$ is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \leq 2$, $G : \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function satisfying conditions (2.1) and (2.2) below, and φ is a non-negative bounded measurable function on $\mathbf{R}^d$.

A well-known fact is that for any non-trivial initial value φ there exists a number $T_\varphi \in (0,\infty]$ such that (1.1) has a unique solution u on $\mathbf{R}^d \times [0,T_\varphi)$ which is bounded on $\mathbf{R}^d \times [0,T]$ for any $0 < T < T_\varphi$, and if $T_\varphi < \infty$, then $\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^d)} \to \infty$ as $t \uparrow T_\varphi$.

When $T_\varphi = \infty$ we say that u is a global solution, and when $T_\varphi < \infty$ we say that u blows up in finite time or that u is non-global.

The study of blow up properties of (1.1) goes back to the fundamental work of Fujita [6], who studied Eq. (1.1) with $\alpha = 2$ and $G(z) = z^{1+\beta}$, $\beta > 0$. The investigation of (1.1) with a general α was initiated by Sugitani [13], who showed that if $d \leq \alpha/\beta$, then for any non-vanishing initial condition the solution blows up

*Research partially supported by CONACyT (Mexico) Grant 32401E.

in finite time. Using a Feynman-Kac representation for semilinear problems of the form (1.1), this conclusion was re-derived in [2] and the corresponding behaviors of equations with time-dependent nonlinearities and of various systems of semilinear pde's were studied.

It is known (e.g. [11], [12]) that in supercritical dimensions $d > \alpha/\beta$, Eq. (1.1) admits global as well non-global positive solutions, and that the "size" of the initial condition determines which of these two behaviors is exhibited by the solutions. For short we address this parameter constellation as the *global regime.*

In this note we prove two comparison criteria in the global regime:

(i) Assume the initial value $\varphi \geq 0$ leads to a globally bounded solution. Then any initial value ψ with $0 \leq \psi \leq (1-\varepsilon)\varphi$, $\varepsilon > 0$, gives rise to a solution converging to zero.

(ii) Assume the initial value $\varphi \geq 0$ leads to a solution which is uniformly bounded away from 0 for all $t > 0$ and all x in some ball $\subset \mathbf{R}^d$. Then any initial value ψ with $\psi \geq (1+\varepsilon)\varphi$, $\varepsilon > 0$, gives rise to a solution which blows up in finite time.

The essential tool in proving (i) and (ii) is the probabilistic representation of the solution of (1.1) provided by the Feynman-Kac formula, that was obtained in [2] (see (2.3) below).

Natural candidates for the comparison in (i) and (ii) are (time-)stationary solutions of (1.1), i.e. solutions of the "elliptic" equation

$$\Delta_\alpha u(x) + G(u(x)) = 0, \quad x \in \mathbf{R}^d. \tag{1.2}$$

In the case $\alpha = 2$ and $G(z) = z^{1+\beta}$, it is known that (see [8], [4], [7], [14])

- for $d > 2$, $1+\beta < (d+2)/(d-2)$, apart from $u \equiv 0$, no bounded non-negative solution of (1.2) exists

- for $d > 2$, $\beta = (d+2)/(d-2) - 1$, all bounded stationary solutions of (1.2) are given by the family

$$u_{c,A}(x) = \frac{A(d(d-2)^{(d-2)/2}}{\left(d(d-2) + (A^{2/(d-2)}\|x-c\|)^2\right)^{(d-2)/2}}, \quad c, x \in \mathbf{R}^d, \quad A \in \mathbf{R}_+. \tag{1.3}$$

(Note that the two parameters of the family are the symmetry center c of u and its value A at c.)

In the case $\alpha < 2$, much less is known. For $d > \alpha$ and $\beta = (d+\alpha)/(d-\alpha) - 1$ we specify in Proposition 3.1 a two-parameter family $u_{c,A}$, $c \in \mathbf{R}^d$, $0 < A < \infty$, of radially symmetric solutions of

$$\Delta_\alpha u(x) + u^{1+\beta}(x) = 0, \quad x \in \mathbf{R}^d \tag{1.4}$$

with the property

$$u_{c,A}(c) = A^{(\alpha-d)/2}, \quad \|x\|^{d-\alpha} u_{c,A}(\|x\|) \to K(d,\alpha,\beta) \text{ as } \|x\| \to \infty$$

where $K(d,\alpha,\beta)$ is a positive constant.

We also show that the solution set of (1.4) is invariant under Kelvin transformations, and exhibit an example of a radially symmetric singular solution of (1.4).

2 Two comparison criteria

In this section we assume that the function G in Eq. (1.1) satisfies the conditions

$$\lim_{z\to 0+} \frac{G(z)}{z^{1+\beta}} = c \in (0,\infty) \tag{2.1}$$

and

$$\int_\theta^\infty \frac{dz}{G(z)} < \infty \tag{2.2}$$

for certain positive numbers β and θ.

Lemma 2.1 *Let G be a convex function satisfying (2.1), and $\varepsilon > 0$. For any $M > 0$ there exists $\varepsilon' > 0$ such that*

$$\frac{G((1+\varepsilon)z)}{(1+\varepsilon)z} > (1+\varepsilon')\frac{G(z)}{z} \quad \textit{for } 0 < z \le M.$$

Proof By considering G/c instead of G we can assume that $c = 1$. Given $\tilde\varepsilon > 0$ there exists $\delta > 0$ such that

$$(1-\tilde\varepsilon)z^\beta < \frac{G(z)}{z} < (1+\tilde\varepsilon)z^\beta$$

for $z \in (0,\delta)$. Take $\tilde\varepsilon < ((1+\varepsilon)^\beta - 1)/((1+\varepsilon)^\beta + 1)$. Then for $z < \delta/(1+\varepsilon) := x_0$,

$$\frac{G((1+\varepsilon)z)}{(1+\varepsilon)z} > (1-\tilde\varepsilon)(1+\varepsilon)^\beta z^\beta > \frac{(1-\tilde\varepsilon)(1+\varepsilon)^\beta}{(1+\tilde\varepsilon)}\frac{G(z)}{z} = (1+c_1)\frac{G(z)}{z}$$

where $c_1 > 0$. Since $z \mapsto G(z)/z$ is continuous and strictly increasing in $(0,\infty)$ it follows that $\inf_{z\in[x_0,M]}\left(\frac{G((1+\varepsilon)z)}{(1+\varepsilon)z}/\frac{G(z)}{z}\right) > 1+c_2$ with $c_2 > 0$. Taking $\varepsilon' = c_1 \wedge c_2$ yields the assertion.

□

Lemma 2.2 *If v is a globally bounded solution of (1.1) then, for all x,*

$$P_t v(0,x) \to 0 \quad \textit{as} \quad t\to\infty,$$

where (P_t) is the semigroup with generator L.

Proof Indeed, from the integral form of (1.1)

$$\begin{aligned} v(t,x) &= P_t v(0,x) + \int_0^t P_{t-s} G(v(s,x))\,ds \\ &\ge P_t v(0,x) + \int_0^t P_{t-s} G(P_s v(0,x))\,ds \\ &\ge P_t v(0,x) + \int_0^t G(P_t v(0,x))\,ds \\ &\ge tG(P_t v(0,x)), \end{aligned}$$

where we used in the first inequality that $P_t v(0,\cdot) \le v(t,\cdot)$, and Jensen's inequality after the second line. It follows from the global boundedness of v that

$$\lim_{t\to\infty} P_t v(0,x) \le \lim_{t\to\infty} G^{-1}(\text{Const.}t^{-1}) = 0.$$

□

Proposition 2.1 *Let G be a convex, increasing function satisfying (2.1) and (2.2). Assume the initial value $\varphi \ge 0$ leads to a globally bounded solution of (1.1). Then any initial value ψ with $0 \le \psi \le (1-\varepsilon)\varphi$, $\varepsilon > 0$, gives rise to a solution converging to zero.*

Proof Recall that the Feynman-Kac representation of solutions of (1.1) is given by (see [2])

$$u(t,x) = \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{G(u(s,X_s))}{u(s,X_s)}\,ds\,\middle|\, X_t = x\right] dy, \tag{2.3}$$

where (X_t) is the Lévy process with generator L, and $p_t(x,y)$, $t>0$, $x,y \in \mathbf{R}^d$, are its transition densities.

Suppose that v is a globally bounded solution of (1.1) and that $0 \le u(0,\cdot) \le (1-\varepsilon)v(0,\cdot)$ where $0<\varepsilon<1$. As (1.1) preserves ordering we have $u(t,x) \le v(t,x)$ for all $t \ge 0$ and $x \in \mathbf{R}^d$, which together with (2.3) improves to

$$\begin{aligned} u(t,x) &\le \int_{\mathbf{R}^d}(1-\varepsilon)v(0,y)p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{G(v(s,X_s))}{v(s,X_s)}\,ds\,\middle|\, X_t = x\right] dy \\ &= (1-\varepsilon)v(t,x) \end{aligned}$$

uniformly in t and x. Inserting this bound again into the Feynman-Kac representation of u yields

$$u(t,x) \le \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{G((1-\varepsilon)v(s,X_s))}{(1-\varepsilon)v(s,X_s)}\,ds\,\middle|\, X_t = x\right] dy.$$

Putting $z(t,x) := (1-\varepsilon)v(t,x)$ in the above inequality and using Lemma 1 (with ε in Lemma 1 substituted by $\tilde\varepsilon := \varepsilon/(1-\varepsilon)$) we get

$$\begin{aligned} u(t,x) &\le \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{1}{1+\varepsilon'}\frac{G((1+\tilde\varepsilon)z(s,X_s))}{(1+\tilde\varepsilon)z(s,X_s)}\,ds\,\middle|\, X_t = x\right] dy \\ &= \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[e^{(1-\varepsilon'')A_t}\,\middle|\, X_t = x\right] dy, \end{aligned}$$

where $\varepsilon' > 0$ is given by Lemma 1, $\varepsilon'' := \varepsilon'/(1+\varepsilon')$ and

$$A_t := \int_0^t \frac{G(v(s,X_s))}{v(s,X_s)}\,ds. \tag{2.4}$$

Thus,

$$\begin{aligned} u(t,x) &\le \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[e^{(1-\varepsilon'')A_t}1(e^{A_t}<(P_tv(0,x))^{-1/2})\Big|\, X_t=x\right]dy \\ &\quad + \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[e^{(1-\varepsilon'')A_t}1(e^{A_t}\ge(P_tv(0,x))^{-1/2})\Big|\, X_t=x\right]dy. \end{aligned}$$

Since $e^{A_t}\ge(P_tv(0,x))^{-1/2}$ implies $e^{(1-\varepsilon'')A_t}\le e^{A_t}\,(P_tv(0,x))^{\varepsilon''/2}$, we obtain

$$\begin{aligned} u(t,x) &\le \int_{\mathbf{R}^d} u(0,y)p_t(y,x)\,(P_tv(0,x))^{-1/2} \\ &\quad + (P_tv(0,x))^{\varepsilon''/2}\int_{\mathbf{R}^d} u(0,y)p_t(y,x)\mathbf{E}_y\left[e^{A_t}\big|\, X_t=x\right]dy \\ &\le (1-\varepsilon)\left((P_tv(0,x))^{1/2}+v(t,x)(P_tv(0,x))^{\varepsilon''/2}\right) \end{aligned}$$

which tends to 0 uniformly as $t\to\infty$ due to Lemma 2.2. □

Proposition 2.2 *Let G be a convex, increasing function satisfying (2.1) and (2.2). Assume the initial value $\varphi=v(0,\cdot)\ge 0$ leads to a solution of (1.1) which for some open ball $B\subset\mathbf{R}^d$ and some $\kappa>0$ obeys*

$$\inf_{x\in B} v(t,x)\ge\kappa \quad \textit{for all sufficiently large } t>0. \tag{2.5}$$

Then for any $\varepsilon>0$, the initial condition $(1+\varepsilon)\varphi$ leads to blow-up in finite time.

Proof By the Feynman-Kac formula,

$$v(t,x)=\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y\left[e^{A_t}\big|\, X_t=x\right]dy,$$

where A_t is given by (2.4). If $K>0$ then

$$\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y\left[e^{A_t};A_t\le K\big|\, X_t=x\right]dy\le e^K\mathbf{E}_xv(0,X_t)\to 0$$

as $t\to\infty$ uniformly in x due to Lemma 2.2. Therefore, for all $K>0$ there exists $T_0=T_0(K,\gamma)>0$ such that for $t>T_0$

$$\frac{v(t,x)}{\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y\left[e^{A_t};A_t\ge K|\, X_t=x\right]dy}\le 2. \tag{2.6}$$

Without loss of generality we can assume that $\inf_{x\in B_1(0)} v(t,x)\ge\kappa$ for all t large enough, where $B_r(x)$ denotes the ball in $\mathbf{R}^d$ of radius r centered at x. Arguing as above we check via the Feynman-Kac representation that for all $t\ge 0$ and $x\in\mathbf{R}^d$, $u(t,x)\ge(1+\varepsilon)v(t,x)$. Plugging this again into the Feynman-Kac representation for u yields

$$\begin{aligned} u(t,x) &\ge (1+\varepsilon)\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t\frac{G((1+\varepsilon)v(s,X_s))}{(1+\varepsilon)v(s,X_s)}ds\Big|\, X_t=x\right]dy \\ &\ge (1+\varepsilon)\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y\left[e^{(1+\varepsilon')A_t}\Big|\, X_t=x\right]dy \end{aligned}$$

for some $\varepsilon' > 0$ by Lemma 2.1. Using this and (2.6) we obtain for given $K > 0$ nd t big enough that

$$\begin{aligned} u(t,x) &\geq (1+\varepsilon)e^{K\varepsilon'} \int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y \left[e^{A_t}; A_t \geq K \middle| X_t = x \right] dy \\ &\geq (1+\varepsilon)e^{K\varepsilon'} \frac{v(t,x)}{2}. \end{aligned}$$

Hence, for any $K > 0$ we find $\inf_{x \in B_1(0)} u(t,x) \geq \kappa(1+\varepsilon)e^{K\varepsilon'}/2$ for all sufficiently large t. As is well known (see e.g. [10]), this inequality together with (2.2) are sufficient for finite-time blowup of u. □

Corollary 2.1 *Let G be a convex, increasing function satisfying (2.1) and (2.2), and $\varphi \geq 0$ a non-trivial positive bounded solution of*

$$L\varphi(x) + G(\varphi(x)) = 0.$$

a) For each $\varepsilon > 0$ the solution u of (1.1) with initial value $u(0,x) = (1+\varepsilon)\varphi(x)$, $x \in \mathbf{R}^d$, blows up in finite time.

b) For each $\varepsilon \in (0,1)$ the solution u of (1.1) with initial value $u(0,x) = (1-\varepsilon)\varphi(x)$, $x \in \mathbf{R}^d$ converges to 0 as $t \to \infty$.

Proof This is immediate from Proposition 2.2. □

3 A class of radially symmetric stationary solutions

We now set out to specify a family of positive stationary solutions of (1.1) in the particular case of $G(z) = z^p$, where $p = (d+\alpha)/(d-\alpha)$. Before doing this, we still consider the case of a general p, and note that the "elliptic" equation

$$\Delta_\alpha u(x) + u^p(x) = 0, \quad x \in \mathbf{R}^d \tag{3.1}$$

can be rewritten in integral form as

$$u(x) = \int_0^\infty \mathbf{E}_x \left[u^p(X_t) \right] dt, \quad x \in \mathbf{R}^d, \tag{3.2}$$

where (X_t) denotes the (symmetric) α-stable process in $\mathbf{R}^d$. Hence, for $d \leq \alpha$, due to recurrence of (X_t), the only non-negative solutions of (3.1) are $u \equiv 0$ and $u \equiv \infty$. Therefore, we henceforth assume that $d > \alpha$, in which case (3.2) rewrites as

$$u(x) = \int_{\mathbf{R}^d} \frac{\mathcal{A}(d,\alpha)u^p(y)}{\|y-x\|^{d-\alpha}} \, dy, \quad x \in \mathbf{R}^d, \tag{3.3}$$

where $\mathcal{A}(d,\alpha) := \Gamma(\frac{1}{2}(d-\alpha))/[\Gamma(\frac{1}{2}\alpha)2^\alpha \pi^{d/2}]$.

Proposition 3.1 *If $p = (d+\alpha)/(d-\alpha)$, then for any $A \in (0,\infty)$ and $c \in \mathbf{R}^d$ the function*

$$u_{c,A}(x) = \frac{A}{\left[1 + \left(A^{2/(d-\alpha)} 2^{-1} \left(\Gamma\left(\frac{d+\alpha}{2}\right)/\Gamma\left(\frac{d-\alpha}{2}\right)\right)^{-1/\alpha} \|x-c\|\right)^2\right]^{(d-\alpha)/2}}$$

solves (3.1).

Proof Without loss of generality we assume that c is the origin. Due to (3.3) it suffices to show that

$$u_{0,A}(x) = \int_{\mathbf{R}^d} \frac{\mathcal{A}(d,\alpha) u_{0,A}^p(y)}{\|y-x\|^{d-\alpha}}\, dy, \quad x \in \mathbf{R}^d. \tag{3.4}$$

Let us write $a := A^{-2/(d-\alpha)} 2 \left(\Gamma\left(\frac{d+\alpha}{2}\right)/\Gamma\left(\frac{d-\alpha}{2}\right)\right)^{1/\alpha}$. We first note that

$$u_{0,A}(x) = \frac{A}{(1+4\pi^2\|\frac{x}{2\pi a}\|^2)^{(d-\alpha)/2}} = A\widehat{B_{d-\alpha}}\left(\frac{x}{2\pi a}\right)$$

(see [5], p. 155), where for any $f \in L^1(\mathbf{R}^n)$, $\widehat{f}(x) := \int_{\mathbf{R}^d} e^{-2\pi i y\cdot x} f(y)\, dy$ is the Fourier transform of f, and for any complex w with $\mathrm{Re}(w) > 0$

$$B_w(x) = \frac{1}{\Gamma(\frac{w}{2})(4\pi)^{d/2}} \int_0^\infty r^{(w-d)/2-1} e^{-r-\|x\|^2/4r}\, dr$$

is the Bessel potential of order w. Hence

$$\widehat{u_{0,A}}(x) = A\left[\widehat{B_{d-\alpha}}\left(\frac{\cdot}{2\pi a}\right)\right]\widehat{\ }(x) = A(2\pi a)^d \widehat{\widehat{B_{d-\alpha}}}(2\pi a x) = A(2\pi a)^d B_{d-\alpha}(-2\pi a x).$$

Since

$$B_w(x) = \frac{2^{(d-w)/2+1}}{\Gamma\left(\frac{w}{2}\right)(4\pi)^{d/2}} \|x\|^{(w-d)/2} K_{(d-w)/2}(\|x\|), \quad x \in \mathbf{R}^d, \tag{3.5}$$

where for any complex ν, K_ν is the Macdonald's function ([15], §6·22)

$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^\nu \int_0^\infty r^{-\nu-1} e^{-r-z^2/4r} dr, \quad \mathrm{Re}(z^2) > 0,$$

it follows that

$$\widehat{u_{0,A}}(x) = A a^{d-\alpha/2} \pi^{(d-\alpha)/2} \frac{2}{\Gamma(\frac{d-\alpha}{2})} \|x\|^{-\alpha/2} K_{\alpha/2}(\|2\pi a x\|). \tag{3.6}$$

To compute the Fourier transform of the other side of (3.4) we use the convolution theorem, (3.5) and

$$\left[\mathcal{A}(d,\alpha)\|\cdot\|^{-(d-\alpha)}\right]\widehat{\ }(x) = (2\pi\|x\|)^{-\alpha}, \quad x \in \mathbf{R}^d, \quad 0 < \mathrm{Re}(\alpha) < d,$$

(e.g. [5], p. 154) to obtain

$$\begin{aligned}
\left[\int_{\mathbf{R}^d} \frac{\mathcal{A}(d,\alpha) u_{0,A}^p(y)}{\|y-\cdot\|^{d-\alpha}}\, dy\right]^{\widehat{}}(x) &= (2\pi)^{-\alpha}\|x\|^{-\alpha} A^p \left[\frac{1}{\left(1+4\pi^2\|\frac{\cdot}{2\pi a}\|^2\right)^{(d+\alpha)/2}}\right]^{\widehat{}}(x) \\
&= (2\pi)^{-\alpha}\|x\|^{-\alpha} A^p \left[\widehat{B_{d+\alpha}}\left(\frac{\cdot}{2\pi a}\right)\right]^{\widehat{}}(x) \\
&= (2\pi)^{-\alpha}\|x\|^{-\alpha} A^p (2\pi a)^d B_{d+\alpha}(-2\pi a x) \\
&= 2^{-\alpha}\pi^{(d-\alpha)/2}\|x\|^{-\alpha/2} A^p a^{d+\alpha/2} \frac{2}{\Gamma\left(\frac{d+\alpha}{2}\right)} K_{-\alpha/2}(\|2\pi a x\|)
\end{aligned}$$

Since $K_\nu = K_{-\nu}$ ([1], Formula 9.6.6), by comparing the RHS of the last equality with that of (3.6) we see that they are equal for the value of a stated at the beginning of the proof. The result follows from uniqueness of Fourier transforms. □

Remarks 1. Recall [3, 9] that for $0 < \alpha \le 2$ and $d > \alpha$ the Kelvin transform of u is defined by

$$v(x) := \frac{1}{\|x\|^{d-\alpha}} u\left(\frac{x}{\|x\|^2}\right), \quad x \in \mathbf{R}^d, \quad x \neq 0. \tag{3.7}$$

A simple calculation shows that for any fixed $c \in \mathbf{R}^d$ the family of solutions $\{u_{c,A}\}_{A\ge 0}$ rendered by Proposition 3.1 is invariant under the Kelvin transform with center at c, given by

$$v_c(x) := \frac{1}{\|x-c\|^{d-\alpha}} u_{c,A}\left(c + \frac{x-c}{\|x-c\|^2}\right).$$

2. Moreover, if u is *any* given regular positive solution of (3.1) (where $0 < \alpha \le 2$, $p > 1$ and $d > \alpha$) and $x \neq 0$, then its Kelvin transform $v(x)$ satisfies

$$\Delta_\alpha v(x) + \frac{v^p(x)}{\|x\|^{(d+\alpha)-p(d-\alpha)}} = 0. \tag{3.8}$$

Indeed, let G_α denote the Green's operator corresponding to Δ_α, and $x \neq 0$. Then,

$$\begin{aligned}
-G_\alpha\left(-\frac{v^p(x)}{\|x\|^{(d+\alpha)-p(d-\alpha)}}\right) &= \int \frac{\mathcal{A}(d,\alpha)}{\|x-y\|^{d-\alpha}} \cdot \frac{1}{\|y\|^{(d+\alpha)-p(d-\alpha)}} \\
&\quad \cdot \frac{1}{\|y\|^{(d-\alpha)p}} \cdot u^p\left(\frac{y}{\|y\|^2}\right) dy \\
&= \mathcal{A}(d,\alpha) \int \frac{1}{\left\|x - \frac{z}{\|z\|^2}\right\|^{d-\alpha}} \cdot \frac{1}{\|z\|^{d-\alpha}} \cdot u^p(z)\, dz \\
&= \mathcal{A}(d,\alpha) \int \frac{1}{\left\|x\cdot\|z\| - \frac{z}{\|z\|}\right\|^{d-\alpha}} \cdot u^p(z)\, dz \\
&= \mathcal{A}(d,\alpha) \int_{\mathbf{R}^d} \frac{u^p(z)\, dz}{\left\|\frac{x}{\|x\|} - \|x\|\cdot z\right\|^{d-\alpha}} \\
&= \frac{1}{\|x\|^{d-\alpha}} u\left(\frac{x}{\|x\|^2}\right) = v(x),
\end{aligned}$$

where we used the elementary identity $\| x \cdot \|z\| - \frac{z}{\|z\|} \| = \| \frac{x}{\|x\|} - \|x\| \cdot z\|$ in the fourth equality.

3. Proceeding as in the proof of Proposition 3.1 one can verify that a singular explicit solution to (3.1) is given by

$$u_{\text{sing}}(x) = \left[2^{\alpha} \left(\Gamma\left(\frac{d+\alpha}{4}\right) \Big/ \Gamma\left(\frac{d-\alpha}{4}\right) \right)^2 \right]^{1/(p-1)} \cdot \frac{1}{\|x\|^{(d-\alpha)/2}}, \quad x \neq 0,$$

and that u_{sing} is a fixed point of the Kelvin transform (3.7).

ACKNOWLEDGEMENT

J.A. López-Mimbela appreciates the kind hospitality of Frankfurt University during his visit in summer 2002.

References

[1] M. Abramowitz and I.A. Stegun (1972). Handbook of Mathematical Functions, 9th Edition. Dover, New York.

[2] M. Birkner, J.A. López-Mimbela and A. Wakolbinger (2002). Blow-up of semilinear PDE's at the critical dimension. A probabilistic approach. *Proc. Am. Math. Society* **130**, No. 8, 2431-2442.

[3] K. Bogdan, T. Byczkowski, Potential theory of Schrödinger operator based on fractional Laplacian. *Probab. Math. Statist.* **20** (2000), no. 2, 293–335.

[4] W.X. Chen and C. Li (1991). Classification of solutions of some nonlinear elliptic equations. *Duke Math. J.* **63** , no. 3, 615–622.

[5] G.B. Folland (1995). Introduction to Partial Differential Equations 2nd Edition. Princeton University Press. Princeton, New Jersey.

[6] H. Fujita (1966). On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Univ. Tokyo Sect. I* **13**, 109-124.

[7] B. Gidas, W.M. Ni, and L. Nirenberg (1981). Symmetry of positive solutions of nonlinear elliptic equations in $\mathbf{R}^n$, *Mathematical analysis and applications, Part A, Adv. in Math. Suppl. Stud.* **7**, 369-402.

[8] B. Gidas and J. Spruck (1981). Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34**, no. 4, 525–598.

[9] D. Gilbarg and N.S. Trudinger (1998). Elliptic Partial Differential Equations of Second Order. Springer Verlag, New York.

[10] K. Kobayashi, T. Sirao and H. Tanaka (1977). On the growing up problem for semilinear heat equations. *J. Math. Soc. Japan* **29**, 407-424.

[11] J.A. López-Mimbela and A. Wakolbinger (2000). A probabilistic proof of non-explosion of a non-linear PDE system. *J. Appl. Probab.* **37**, no. 3, 635–641.

[12] M. Nagasawa and T. Sirao (1969). Probabilistic treatment of the blowing up of solutions for a nonlinear integral equation. *Trans. Amer. Math. Soc.* **139**, 301–310.

[13] S. Sugitani (1975). On nonexistence of global solutions for some nonlinear integral equations. *Osaka J. Math.* **12**, 45-51.

[14] X. Wang (1993). On the Cauchy problem for reaction-diffusion equations. *Trans. Amer. Math. Soc.* **337**, 549–590.

[15] G.N. Watson (1944). A Treatise on the Theory of Bessel Functions 2nd Edition. Cambridge University Press.

Matthias Birkner
FB Mathematik, J.W. Goethe Universität
D-60054 Frankfurt am Main, Germany.

J. Alfredo López-Mimbela
Centro de Investigación en Matemáticas
Apartado Postal 402
36000 Guanajuato, México.

Anton Wakolbinger
FB Mathematik, J.W. Goethe Universität
D-60054 Frankfurt am Main, Germany.