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Abstract. Our aim in this note is to describe an approach for constructing a geometric analog of symplectic reduction for supermanifolds. We discuss how this allows us to obtain the Kostant cotangent supermanifolds as reduced spaces of the geometric cotangent superbundles, and stress some points where the analogy might break down (or at least require more subtle considerations).

I. Introduction

Symplectic reduction has become an essential tool in geometry, with applications ranging from Hamiltonian Mechanics to Geometric Invariant Theory; it is therefore natural to attempt to translate this process to the realm of supermanifolds. However, the classical description of reduction involves several constructions where this translation is not entirely obvious, essentially due to the fact that the notion of point is much subtler than in the non-graded case. In fact, it turns out that this does depend on several choices, so in principle there might not be a unique way to do it.

On the other hand, there are already some important examples in the literature related to this reduction process; notably for our purposes, in [10] it was shown that the geometric cotangent superbundles admit some natural quotients that are special types of supersymplectic supermanifolds; nevertheless, no mention was made in that work of group actions or reduction. Indeed we can say that our specific goal here is to show that their

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construction can be viewed as a kind of supersymplectic reduction, and hence, in our view, it is the existence of this mechanism that makes these quotients natural.

In this work we will consider only real supermanifolds in the sense of Berezin-Leites-Kostant, or BLK- supermanifolds. Detailed accounts on the required supermanifold theory can be found in [2], [8], or [10], and in the original sources [4] and [7], and for the theory of symplectic reduction, we refer to [1]. But, to make the paper more self-contained, here and in the following section we shall recall some points of supermanifold theory that will be important to us:

In the first place, a BLK-supermanifold is a ringed space $\mathcal{M} = (M, \mathcal{A}_{\mathcal{M}})$, where (M, C^∞_M) is an ordinary manifold, and such that $\mathcal{A}_{\mathcal{M}}$ (the sheaf of *superfunctions*) is a sheaf of supercommutative superalgebras, fitting into an exact sequence of the form

$$0 \longrightarrow \mathcal{N}_M \longrightarrow \mathcal{A}_M \xrightarrow{\Delta^*} C^\infty_M \longrightarrow 0.$$

As said, M is an ordinary C^∞ manifold, of dimension say m, $\mathcal{N}_M$ is the sheaf of nilpotent elements of $\mathcal{A}_M$, and the sheaf $\mathcal{E}_M = \mathcal{N}_M/\mathcal{N}_M^2$ is locally free and of constant rank, say n; in particular, $\mathcal{E}$ is equivalent to a vector bundle, the so-called *Batchelor bundle*, which will be denoted E (so that $\mathcal{E} \cong \Gamma(E)$). The pair (m, n) is the *dimension* of the supermanifold. The main structure theorem for real supermanifolds is a—now classical—theorem of M. Batchelor, stating that for real supermanifolds $\mathcal{A}_M$ is isomorphic to the sheaf of sections of the exterior algebra bundle of E; $\mathcal{A}_M \cong \Gamma\left(\bigwedge^*(E)\right)$, as sheaves of $\mathbb{Z}_2$-graded algebras (the isomorphism being however **non** canonical).

A realization of the isomorphism in Batchelor's theorem is usually called a *splitting* of the supermanifold, and the basic examples of split supermanifolds—and also the local models for general supermanifolds—are the *superdomains*: Given $U \subset \mathbb{R}^m$ an open domain, the superdomains having U as underlying manifold are the supermanifolds

$$U^{m|n} = (U, C^\infty_U \otimes \bigwedge{}^*(\xi_1, \dots, \xi_n)), \quad n \in \mathbb{N},$$

where $\bigwedge^*(\xi_1, \dots, \xi_n)$ denotes the Grassmann algebra in the generators $\xi_1, \dots, \xi_n$. We will in fact make here the simplifying assumption that our supermanifolds are all split.

Secondly, there is a natural correspondence between *supervector spaces* and superdomains (the associated *affine supermanifold*) as follows: To each supervector space one associates the superdomain $(V_0, C^\infty(V_0) \otimes \bigwedge^* V_1^*)$; at the level of coordinates this just amounts to regard the dual of some basis of the supervector space as (*standard* or *linear*) coordinates. We will frequently identify both objects, but if we need to distinguish between them, we will write the supervector space associated to $\mathbb{R}^{m|n}$ as $\mathbb{R}(m|n)$.

Finally, supermanifold morphisms are pairs

$$\Phi = (\phi, \Phi^*) \colon \mathcal{M}_1 \to \mathcal{M}_2,$$

where

$$\phi \colon M_1 \to M_2 \quad ; \quad \Phi^* \colon \mathcal{A}_{M_2} \to \mathcal{A}_{M_1}$$

are respectively a map between the underlying manifolds and a morphism of sheaves of superalgebras. In particular, the sheaf map Δ^* in the definition of a BLK-supermanifold, whose effect on a section f we shall denote by $\widetilde{f}$, corresponds to a supermanifold map between the supermanifold M, of dimension $(m, 0)$, to $\mathcal{M}$ (note that $\widetilde{f}$ is an ordinary function). Moreover, a well known result is that supermanifold morphisms are determined by the induced superalgebra morphisms on the global sections. Thus, in coordinate charts, supermanifold morphisms Φ are determined by a collection of superfunctions $F = (F_{0,1} \ldots, F_{0,m}; F_{1,1} \ldots, F_{1,n})$. In particular, a bijection between superfunctions on a supermanifold, $F = (F_0; F_1) \in \Gamma(\mathcal{A}_M)$, and morphisms $\Phi \colon \mathcal{M} \to \mathbb{R}^{1|1}$, can be given as follows: Given a system of linear coordinates $(x; \xi)$ for $\mathbb{R}^{1|1}$, simply set

$$\Phi^* x = F_0 \quad ; \quad \Phi^* \xi = F_1.$$

We will say that such an F *represents* the morphism Φ in the given coordinates.

II. Some geometric superstructures

The two types of geometric structures essential for our purposes are Lie supergroups and vector superbundles, so we now recall the basic facts about them.

First, Lie supergroups are (pointed) supermanifolds $\mathcal{G} = (G, \mathcal{A}_G)$, together with three morphisms: the (super)multiplication morphism

$$\mu : \mathcal{G} \times \mathcal{G} \to \mathcal{G},$$

the inversion morphism $\sigma \colon \mathcal{G} \to \mathcal{G}$, and the morphism ev_e, of evaluation at the base point $e \in G$, as the identity element of $\mathcal{G}$; these morphisms satisfy the analogues of the group relations:

$$\begin{gathered}
\mu \circ (\mu \circ (\pi_1 \times \pi_2) \times \pi_3)) = \mu \circ (\pi_1 \times \mu \circ (\pi_2 \times \pi_3)) \\
\mu \circ (\mathrm{ev}_e \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \mathrm{ev}_e) = \mathrm{id} \\
\mu \circ (\sigma \times \mathrm{id}) = \mu \circ (\mathrm{id} \times \sigma) = \mathrm{ev}_e,
\end{gathered}$$

(where π_i denotes the projection onto the i-th factor).

These type of conditions can be conveniently written as commutative diagrams: For instance (with a slight abuse in notation, omitting the projections to avoid cluttering the notation), the diagram for associativity is:

$$\begin{array}{ccc} \mathcal{G}\times(\mathcal{G}\times\mathcal{G}) & \xrightarrow{\mathrm{id}\times\mu} & \mathcal{G}\times\mathcal{G} \\ {\scriptstyle \mu\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \mu} \\ \mathcal{G}\times\mathcal{G} & \xrightarrow[\mu]{} & \mathcal{G} \end{array} ,$$

where $\mu = (m, \mu^*)$. Furthermore, it is well known that in this situation the underlying manifold G inherits the structure of a Lie group, with product m.

Similarly, one can define supergroup actions as morphisms $\Gamma\colon \mathcal{G}\times\mathcal{M}\to\mathcal{M}$ satisfying:

$$\begin{array}{ccc} \mathcal{G}\times(\mathcal{G}\times\mathcal{M}) & \xrightarrow{\mathrm{id}\times\Gamma} & \mathcal{G}\times\mathcal{M} \\ {\scriptstyle \mu\times\mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma} \\ \mathcal{G}\times\mathcal{M} & \xrightarrow[\Gamma]{} & \mathcal{M} \end{array} ,$$

and so on.

Example. Perhaps the simplest example is given by the superdomain $\mathbb{R}^{1|1}$, which actually admits three non-isomorphic supergroup structures (see e.g. [5] for details). Here we will mostly deal with the additive structure, given by $\mu^* x = x_1 + x_2$; $\mu^*\xi = \xi_1 + \xi_2$.

Second, vector superbundles over supermanifolds are locally free sheaves of $\mathcal{A}_M$-supermodules. However, for any superbundle, $\mathcal{S}$, there exists, in a functorial way, a supermanifold $\mathcal{B}$ and a supermanifold submersion

$$\pi : \mathcal{B} = (B, \mathcal{A}_B) \to \mathcal{M},$$

such that the sections of the superbundle—in the sheaf theoretic sense—correspond to sections—in the geometric sense—of the submersion π (see [10]). These supermanifolds are the *geometric* superbundles, and in the remainder of this paper superbundles will be understood in this sense.

In particular, for any supermanifold $\mathcal{M}$ there is a *cotangent superbundle*. In the split case, the corresponding geometric cotangent superbundle is explicitly given by:

$$\mathcal{S}\mathcal{T}^*\mathcal{M} = \left(T^*M \oplus E, \Gamma\left(\bigwedge\nolimits^{*}(E \oplus TM \oplus E^*)\right)\right),$$

where E is the Batchelor bundle of $\mathcal{M}$, so that its underlying manifold is the Whitney sum of the cotangent and Batchelor bundles, together with the obvious projection. (Of course, in the above expression it is understood that we have pulled back TM and E to $T^*M \oplus E$, but to simplify the notation, we will omit any reference to this pullback, and we will also occasionally write $\mathcal{M}^*$ for the cotangent superbundle.)

We also recall that to any (m, n)-dimensional supermanifold $\mathcal{M}$, one can associate a *cotangent supermanifold*, having the cotangent bundle as underlying manifold. In the split case, it can be explicitely realized as the supermanifold

$$\mathcal{T}^*\mathcal{M} = \left(T^*M, \Gamma\left(\bigwedge^*(E \oplus E^*)\right)\right),$$

where as before E is the Batchelor bundle. The cotangent supermanifold is of dimension $(2m, 2n)$, and is naturally endowed with an even supersymplectic form, for which one can find appropriate local coordinates on the supermanifold, say $(q_i, p_i; \theta_\alpha, \zeta_\alpha)$, $i = 1, \dots, m$, $\alpha = 1, \dots, n$, where it has the expression

$$\omega = \sum_i dq_i dp_i + \sum_\alpha d\theta_\alpha d\zeta_\alpha.$$

This cotangent supermanifolds have been widely used as basis for "super" versions of classical mechanics (see *e.g.* [6]).

$\mathcal{T}^*\mathcal{M}$ is however not a geometric superbundle, and so is not a true analog (for supermanifolds) of the cotangent bundle (of the C^∞ theory). On the other hand, while $\mathcal{M}^*$ is a $(2m + n, 2n + m)$-dimensional supermanifold (so in a definite sense "larger" than the cotangent supermanifold), not only is it a superbundle, but the most important point is that it admits a canonical 1-form Θ on $\mathcal{M}^*$, uniquely determined by the condition:

$$\psi^*\Theta = \psi; \quad \forall\ 1 - \text{superform } \psi \text{ on } \mathcal{M},$$

as in the classical case. In this sense, the geometric cotangent superbundle plays a role closer to that of the cotangent bundle than the cotangent supermanifold. The analogy with the smooth theory is however not perfect, because $d\Theta$ has a kernel, and therefore $\mathcal{M}^*$ is *not* a supersymplectic supermanifold. Moreover, Θ is not homogeneous: in the appropriate system of supercoordinates, written as $(q_i, p_i, z_\alpha; \theta_\alpha, \zeta_\alpha, \eta_i)$, $i = 1, \dots, m$, $\alpha = 1, \dots, n$, it splits into even and odd parts as

$$\Theta_0 = \sum_i dq_i p_i + \sum_\alpha d\theta_\alpha \zeta_\alpha \ ; \quad \Theta_1 = \sum_i dq_i \eta_i + \sum_\alpha d\theta_\alpha z_\alpha.$$

III. Regular values of supermaps

Classical symplectic reduction is performed on inverse images of regular values of momentum maps; hence, we need to define what we mean by the inverse image of a regular value of a supermap.

For this, we recall first that a k-point in a supermanifold $\mathcal{M}$ is, by definition, a morphism Γ from $\mathbb{R}^{0|k}$ to $\mathcal{M}$. In dimension (m,n) we have roughly speaking $n+1$ types of k-points, corresponding to $0 \le k \le n$. But, by considering the standard inclusions, it is clear that if $i \le j$, an i-point can be regarded as a particular type of j-point, so in the end we shall primarily deal with the case $k = n$.

Indeed, the main reason for this is that points in $\mathbb{R}(m|n)$ determine n-points in $\mathbb{R}^{m|n}$ as follows: Denoting the coordinates of $\mathbb{R}^{m|n}$ by y_i, $i = 1,\ldots,m$; θ_α, $\alpha = 1,\ldots,n$, and considering the standard basis of $\mathbb{R}(m|n)$, $\{e_1,\ldots,e_m;\epsilon_1\ldots,\epsilon_n\}$, the point

$$\sum r_i e_i + \sum \rho_\alpha \epsilon_\alpha \in \mathbb{R}(m|n)$$

naturally determines the n-point Γ:

$$\Gamma^* y_i = r_i \quad ; \quad \Gamma^* \theta_\alpha = \rho_\alpha \epsilon_\alpha,$$

where the ϵ_α are regarded as the generators of the global superfunctions on $\mathbb{R}^{0|n}$; this clearly makes sense, since the latter is just a Grassmann algebra in n generators. In this situation we will write $\Gamma = (r;\rho)$, and since the maps that are relevant to the reduction process (momentum supermaps) have as their target a superdomain $\mathbb{R}^{m|n}$, this is the notion of point that will be relevant to us.

Next, if $F = F_0 + F_1 \colon \mathbb{R}^{p|q} \to \mathbb{R}^{m|n}$ is a superfunction between superdomains, we denote by

$$\partial_x F_i = (\partial_{x_1} F_i, \ldots, \partial_{x_p} F_i) \ ; \quad i = 1,2,$$

the matrices of partial dervatives with respect to the even coordinates; similarly, for the odd ones one has $\partial_\xi F_i$. Then, the super Jacobian of F is the $(m+n)\times(p+q)$ matrix of superfunctions

$$\operatorname{SJac} F(x) = \begin{pmatrix} \partial_x F_0 & -\partial_x F_1 \\ \partial_\xi F_0 & \partial_\xi F_1 \end{pmatrix}.$$

Note that $\operatorname{SJac} F$ is *even* as a graded linear map, but the entries of the off-diagonal blocks are *odd* superfunctions. Also note that the odd part of F has the form

$$F_1(x,\xi) = \widetilde{\partial_\xi F_1}(x)\xi + \text{ h.o.t.}$$

The *rank* of the super Jacobian at $x \in \mathbb{R}^m$ is then the usual rank of the matrix

$$\begin{pmatrix} \widetilde{\partial_x F_0}(x) & 0 \\ 0 & \widetilde{\partial_\xi F_1}(x) \end{pmatrix}.$$

More generally, if $\Phi = (\phi, \Phi^*)$ is a morphism, its super Jacobian in a coordinate system is that of any superfunction F representing Φ in those coordinates. Explicitly, if the representation occurs in the coordinates $(x;\xi)$ in $\mathcal{M}$ and $(y;\theta)$ in $\mathbb{R}^{m|n}$; this means that in these coordinates we have

$$\Phi^* y = F_0(x,\xi) = \widetilde{F_0}(x) + \text{h.o.t.} \;\; \Phi^*\theta = F_1(x,\xi) = \widetilde{\partial_\xi F_1}(x)\xi + \text{h.o.t.},$$

and from this follows that, although the definition of the super Jacobian of a morphism is coordinate dependent, its rank is not, so we have the following well known result:

Lemma 1. *The rank of the super Jacobian is independent of the choice of the graded map F representing the morphism.*

We omit the rather easy proof (see the proof of the next proposition); the point here is that this allows us to define regular values:

Definition 1. *A point $\Gamma = (r_0, \rho_0)$ is a regular value of the graded map $\Phi = (\phi, \Phi^*)\colon \mathcal{M} \to \mathbb{R}^{m|n}$ if its super Jacobian has maximal rank everywhere on $\phi^{-1}(r_0)$*

The definition of a regular value given here does not depend on the odd part of the point, so it applies to arbitrary points. Obviously it implies that $V = \phi^{-1}(r_0)$ is a submanifold of M; but in fact, we have the following nice geometrical characterization of regular values (remember that we are assuming that our supermanifolds are all spli):

Proposition 1. *Let $\Phi = (\phi, \Phi^*)\colon \mathcal{M} \to \mathbb{R}^{m|n}$ be a morphism, and F a coordinate represention of Φ. A point $\Gamma = (r_0, \rho_0)$ is regular if and only if r_0 is a regular value of the map ϕ, and the pointwise defined map $\widetilde{\partial_\xi F_1}(x)$ defines a vector bundle map from the Batchelor bundle E of $\mathcal{M}$ restricted to $V = \phi^{-1}(r_0)$, to the Batchelor bundle of $\mathbb{R}^{m|n}$.*

The bundles corresponding to different representations of Φ are isomorphic.

Proof: The first assertion is simply the previous remark; and since locally the Batchelor bundles are generated by the odd coordinates, and the rank of $\widetilde{\partial_\xi F_1}(x)$ is assumed constant on V, this means that it is a vector bundle map.

For the last assertion, observe that changes of coordinates, $X = \Psi^* x$; $\Xi = \Psi^*\xi$ in $\mathcal{M}$, and $Y = \chi^* y$; $\Theta = \chi^*\theta$ in $\mathbb{R}^{m|n}$, are given by expressions of the type

$$X = \widetilde{H_0}(x) + \text{ h.o.t.; } \Xi = \widetilde{\partial_\xi H_1}(x)\xi + \text{ h.o.t.;}$$
$$Y = \widetilde{G_0}(y) + \text{ h.o.t.;} \Theta = \widetilde{\partial_\theta G_1}(y)\theta + \text{ h.o.t.,}$$

where H_0 and G_0 are diffeomorphisms, and the matrix valued functions $\widetilde{\partial_\xi H_1}$ and $\widetilde{\partial_\theta G_1}$ are invertible. Then the function $\widetilde{\partial_\xi F_1}$ changes as

$$\widetilde{\partial_\xi F_1}(X) = \widetilde{\partial_\theta G_1}(\widetilde{F_0}(x))\widetilde{\partial_\xi F_1}(\widetilde{H_0}(x))\widetilde{\partial_\xi H_1}(x),$$

showing that different representations of the bundle are equivalent; this completes the proof. □

Let us now look somewhat more closely—from a categorical point of view—at the notion of inverse image of a point: Namely, given a morphism $\Phi\colon \mathcal{M} \to \mathbb{R}^{m|n}$, the inverse image of a k-point in $\mathbb{R}^{m|n}$ should be defined by a commutative diagram of the type:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\Upsilon} & \mathcal{M} \\ {\scriptstyle\Psi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ \mathbb{R}^{0|n} & \xrightarrow[\Gamma]{} & \mathbb{R}^{m|n} \end{array}; \tag{1}$$

where Υ embeds the supermanifold $\mathcal{V}$ as a subsupermanifold of $\mathbb{R}^{m|n}$, and this supermanifold would be the inverse image of the point.

Now, if $\Phi = (\phi, \Phi^*)$, $\Psi = (\psi, \Psi^*)$, $\Upsilon = (\iota, \Upsilon^*)$, and $\Gamma = (\gamma, \Gamma^*)$, this corresponds to a pair of diagrams; one between classical manifolds

$$\begin{array}{ccc} V & \xrightarrow{\iota} & M \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\ \{*\} & \xrightarrow[\gamma]{} & \mathbb{R}^m \end{array}, \tag{2}$$

where γ picks up a point $p \in \mathbb{R}^m$, underlying the k-point, ι is an embedding, and ψ the unique map $V \to \{*\}$; and another of sheaf morphisms

$$\begin{array}{ccc} \mathcal{A}_V & \xleftarrow{\Upsilon^*} & \mathcal{A}_M \\ {\scriptstyle\Psi^*}\big\uparrow & & \big\uparrow{\scriptstyle\Phi^*} \\ \bigwedge^* \mathbb{R}^k & \xleftarrow[\Gamma^*]{} & C^\infty_{\mathbb{R}^m} \otimes \bigwedge^* \mathbb{R}^n \end{array}. \tag{3}$$

where the assertion that $\mathcal{V}$ is a subsupermanifold basically means that the sheaf map Υ^* is onto (see [2]). At any rate, for a given point the construction of (2) is clear; but for (1) to actually be in the category of supermanifolds, we still need to define the morphisms Υ and Ψ so as to make (3) commutative; sufficient conditions for this are given in the next proposition:

Proposition 2. *Let $\Phi\colon \mathcal{M} \to \mathbb{R}^{1|1}$ be a graded map represented by F, and assume $\Gamma = (r_0; 0)$ is a regular value. Let $V = \phi^{-1}(0)$, $\hat{B} = E|_V$ and define $B = \ker(\widetilde{\partial_\xi F_1}|_{\hat{B}})$ $\mathcal{A}_V = \Gamma^*(\bigwedge(B))$. Then, there exist morphisms Ψ and Υ, such that (3) is commutative. Therefore, the inverse image of the regular point is the graded submanifold $\mathcal{V} = (V, \mathcal{A}_V)$ of $\mathcal{M}$.*

Proof: Taking a local representation F of Φ the inverse image of a point $\Gamma = (r_0; \rho_0)$ is given by

$$\Phi^* r = F_0(x;\xi)\ ; \quad \Phi^*\rho = F_1(x;\xi).$$

The equations for commutativity of the diagram are then

$$\Upsilon^* F_0(x;\xi) = \Psi^* r\ ; \quad \Upsilon^* F_1(x;\xi) = \Psi^*\rho,$$

but since the morphism Ψ must be defined by $\Psi^* r = r_0$ and $\Psi^*\rho = \rho_0$, to lowest order they reduce to

$$\Upsilon^* \tilde{F}(x) = r_0\ ; \quad \Upsilon^* \sum F_\alpha(x)\xi_\alpha = \rho_\alpha \Psi^* e_\alpha.$$

Solutions to the first equation are obtained in the standard way, while, for $\rho_0 = 0$, solutions to the second just amount to a local representation of the kernel B of the vector bundle map $\widetilde{\partial_\xi F_1}$, and thus follow from proposition 1. This allows us to define $\mathcal{A}_V$ as the sheaf of sections of $\bigwedge^* B$, and then the diagrams obviously are commutative by construction.

That Υ is an embedding follows from the fact that B is a subbundle of $E|_V$. $\square$

IV. Reduction of the cotangent superbundles

Let $\mathcal{M}$ be a supermanifold of dimension $(m|n)$. We wish to describe the even supersymplectic quotient supermanifold of $\mathcal{M}^*$ discussed in [10] as a symplectic reduction.

Let us begin by defining an action of the additive group $\mathcal{G} = \mathbb{R}^{n|m}$ on $\mathcal{M}^*$ as follows. Consider a local system of supercoordinates $(q, p, z; \theta, \eta, \zeta)$, where z and ζ are linear (throughout this section we let α run through $1, \ldots, n$, and i through $1, \ldots, m$). If we choose linear coordinates for $\mathcal{G}$, say $(x;\xi)$, then we have a natural action

$$\Psi\colon \mathcal{G} \times \mathcal{M}^* \to \mathcal{M}^*$$

given by

$$\Psi^* z_\alpha = z_\alpha + x_\alpha;\ \Psi^*\eta_i = \eta_i + \xi_i;\ \Psi^* q_i = q_i;\ \Psi^* p_i = p_i;\ \Psi^*\theta_\alpha = \theta_\alpha;\ \Psi^*\zeta_\alpha = \zeta_\alpha.$$

This action is well-defined, since only the coordinates q, p can not in general be taken as linear, but the action on them is trivial.

Lemma 2. *The action above of $\mathcal{G}$ on $\mathcal{M}^*$ is "supersymplectic" in the sense that $\iota^*\Psi^* d\Theta = \pi^* d\Theta$, where $\pi\colon \mathcal{G}\times\mathcal{M}^* \to \mathcal{M}^*$ is the projection, and $\iota\colon \mathcal{M}^* \to \mathcal{G}\times\mathcal{M}^*$ is the natural injection determined by the condition $\iota^* x_\alpha = 0$, $\iota^*\xi_i = 0$.*

Proof: The proof is again quite straightforward: since exterior differentiation commutes with pullbacks, it suffices to check that $\iota^*\Psi^*\Theta = \pi^*\Theta$. But in fact it suffices to check that $\iota^*\Psi^* = \pi^*$ for each of the supercoordinates of $\mathcal{M}^*$, which is obvious from the definitions. $\square$

The key point is now the following:

Proposition 3. *The action Ψ admits an* **odd** *momentum supermap. Moreover 0 is a regular value for this map*

We will define explicitly a momentum supermap; but, before proving the proposition, let us indicate what we mean here by an odd supermap:

To wit, there is a functor, Π, called the *change of parity functor*, that to a vector superspace $V = V_0 \oplus V_1$, associates the vector superspace $W = \Pi V = W_0 \oplus W_1$, where $W_0 = V_1$ and $W_1 = V_0$. Hence, identifying supervector spaces with their corresponding superdomains, an odd map into V is just a map into W. (These construction is in fact essential to the definition of the cotangent superbundle, since the "new" coordinates z and ζ might be locally defined as $\Pi\theta$ and $\Pi\partial_q$; see [10].)

Proof of proposition 3: Now, recall that a momentum map is an application that allows to view the infinitesimal generators of the action as Hamiltonian fields. In our context this means that the momentum supermap is a morphism $\mathbf{J}\colon \mathcal{M} \to \mathfrak{g}^*$, defined by the relation:

$$i_{X_\xi}\, d\Theta = dJ(\xi),$$

for each infinitesimal generator of the action X_ξ (corresponding to the point $\xi \in \mathfrak{g}$), and where $d\Theta$ is the supersymplectic form and J is a superfunction representing $\mathbf{J}$.

Thus, as a first step, we must determine the infinitesimal generators of the action. But, since the action is linear, in the coordinates chosen the non trivial infinitesimal generators are necessarily the constant superfields ∂_{z_α} and ∂_{η_i}.

We next compute the contractions of these non-trivial generators with $d\Theta$ to get:

$$i_{\partial_{z_\alpha}} d\Theta = d\theta_\alpha \quad ; \quad i_{\partial_{\eta_i}} d\Theta = dq_i.$$

These conditions completely determine dJ, and so, in order for $\mathbf{J}$ to be a momentum supermap, we simply have to set

$$J^* r_\alpha = \Pi\theta_\alpha \quad ; \quad J^* \rho_i = \Pi q_i.$$

Finally, it is clear from these expressions that the rank of SJac J is $m+n$, so 0 is indeed a regular value for $\mathbf{J}$. $\square$

Remark: We may also describe these momentum supermaps as Lie superalgebra morphisms: Indeed, suppose $e_\alpha; \epsilon_i$ is a basis for $\mathfrak{g}$; then, in order for both constructions to be compatible, at the level of Hamiltonian fields we have to set $e_\alpha \mapsto \partial_{z_\alpha}$ and $\epsilon_i \mapsto \partial_{\eta_i}$, so that the map

$$\mathcal{J} : \mathfrak{g} \to \mathcal{A}_{M^*}$$

corresponding to $\mathbf{J}$ is given by

$$e_\alpha \mapsto \theta_\alpha \quad ; \quad \epsilon_i \mapsto q_i.$$

At any rate, we have now all the ingredients for our main result:

Theorem 1. *For the regular value* 0*, the reduced space of the above construction is $\mathcal{T}^*\mathcal{M}$, with its standard even supersymplectic structure.*

Proof: In fact, the computations above show that in the chosen coordinates, the equations of the regular value become

$$z_\alpha = 0 \; ; \; \eta_i = 0.$$

Thus, the underlying manifold of the inverse image of 0 is the manifold T^*M, corresponding to the first equation. But then, the subbundle of the Batchelor bundle determined by the second equation is isomorphic to $E \oplus E^*$.

To make the reduction, we have now to compute the action of the stabilizer subsupergroup on the inverse image of the regular value 0. A feature of this case that greatly simplifies the discussion is that the action is a simple linear one: Indeed, as easily follows from the coordinate expression of the action, the isotropy supergroup corresponds to

$x_\alpha = 0$ and $\xi_i = 0$, and is thus necessarily the trivial supergroup $\mathbb{R}^{0|0}$. Hence, the quotient supermanifold is isomorphic to $\mathcal{T}^*\mathcal{M}$, as stated.

It only remains to show that the induced superform is in fact the supersymplectic structure of $\mathcal{T}^*\mathcal{M}$, but this is immediate from the constructions, since we now have $dz_\alpha = 0$, $d\eta_i = 0$. $\square$

Remarks: Thus, *a posteriori*, it is not entirely surprising that the momentum supermap for this action turns out to be odd, since in the quotient we intend to kill the odd superform $d\Theta_1$. Also, we point out that the momentum supermap so constructed is not equivariant, since, the supergroup $\mathbb{R}^{n|m}$ being abelian, the adjoint action is trivial, and hence so is the coadjoint action; thus, the isotropy subsupergroup of 0 should be the whole supergroup $\mathbb{R}^{n|m}$. This certainly requires further investigation.

On the other hand, a very similar construction, given this time by an additive action of $\mathbb{R}^{m|n}$ on the coordinates p and ζ, and with associated (even) momentum supermap

$$L^*s = -q \; ; \; L^*\zeta = \theta,$$

would give the odd supersymplectic supermanifold also described in [10], namely the supermanifold

$$(E, \Gamma(\bigwedge^*(E \oplus E^*)),$$

with the odd supersymplectic structure obtained by pulling back $d\Theta_1$.

Finally, although the scheme presented here is not a general theory of reduction, it does raise several interesting questions. Chiefly, we believe that our construction suggests that, associated to any supermanifold $\mathcal{M}$, one should look for a true supersymplectic supermanifold—which would certainly be larger than the cotangent superbundle—such that, again by a process of reduction, would give both, the cotangent superbundle and supermanifold. The exact nature of this supersymplectic supermanifold is unclear to us at this point, and its study will be delayed to a future work.

V. Another example: Liftings of the additive $\mathbb{R}^{1|1}$ action

Let us now discuss in some detail a related, but different construction; namely, the reduction of arbitrary liftings of the additive action of $\mathbb{R}^{1|1}$ on itself.

In this section, to distinguish the supergroup $G = \mathbb{R}^{1|1}$ from the supermanifold $\mathcal{M} = \mathbb{R}^{1|1}$ on which the supergroup acts, we write the coordinates of the former as (x, ξ), and those of the latter as (q, θ). The coordinates for the dual of the Lie algebra of G, $\mathfrak{g}^* = \mathbb{R}^{1|1}$ will be denoted (r, ρ).

Lemma 3. *The additive action of* $\mathbb{R}^{1|1}$ *on itself (i.e., the type I product,* μ*) lifts to actions on* $\mathcal{ST}^*\mathbb{R}^{1|1} \cong \mathbb{R}^{3|3}$*. These liftings,* Φ*, are again "supersymplectic" in the sense that* $\iota^*\Phi^* d\Theta = \pi^* d\Theta$*, where* $\pi\colon \mathbb{R}^{1|1}\times\mathbb{R}^{3|3}\to\mathbb{R}^{3|3}$ *is the projection, and* $\iota\colon \mathbb{R}^{3|3}\to\mathbb{R}^{1|1}\times\mathbb{R}^{3|3}$ *is the natural injection* $\iota^*x=0$*,* $\iota^*\xi=0$*.*

Proof: The morphism Φ will be a lift if: (i) it is an action, and, (ii) it makes the following diagram commute:

$$\begin{array}{ccc} \mathbb{R}^{1|1}\times\mathbb{R}^{3|3} & \xrightarrow{\Phi} & \mathbb{R}^{3|3} \\ {\scriptstyle \mathrm{id}\times\pi}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \mathbb{R}^{1|1}\times\mathbb{R}^{1|1} & \xrightarrow{\mu} & \mathbb{R}^{1|1}. \end{array}$$

By simple and standard computations, we then get the following four parameter family of liftings for the type I action:

$$\begin{array}{lcl} \Phi^*q = x+q & ; & \Phi^*\theta = \xi+\theta \\ \Phi^*p = bx+p & ; & \Phi^*\eta = e\xi+\eta \\ \Phi^*z = cx+z & ; & \Phi^*\zeta = f\xi+\zeta \end{array}$$

where b, c, e, f are real parameters; here, the two equations in the top row essentially come frome condition (ii) and the remaining ones from (i).

Moreover, again since exterior differentiation commutes with pullbacks, it is immediate that the liftings are supersymplectic in the sense described above. $\square$

Now, the point is that for $(1,1)$-dimensional actions, the infinitesimal generators of the action Φ may be unambiguously defined by the integration of an ordinary differential equation on the supermanifold (see [9]). This leads therefore to the flow of a vector superfield X, determined by a condition of the form:

$$(\partial_x + \partial_\xi)\circ\Phi^* = \Phi^*\circ X,$$

unique up to a linear change of the supercoordinates. Since this action is again linear, we can immediately solve this equation to find that the infinitesimal generators are just

$$X = \partial_q + b\partial_p + c\partial_z + \partial_\theta + e\partial_\eta + f\partial_\zeta$$

To obtain the momentum supermap, $\mathbf{J}\colon \mathbb{R}^{3|3}\to\mathfrak{g}^*$, again we need to solve the equation $i_X\, d\Theta = dJ$, where J represents $\mathbf{J}$, and X is as above. This explicitly reads

$$\partial_q J = -(b+e)\ ;\quad \partial_p J = 1\ ;\quad \partial_z J = 1\ ;\quad \partial_\theta J = f-c\ ;\quad \partial_\eta J = 1\ ;\quad \partial_\zeta = 1,$$

and so one finally gets as possible momentum supermaps the four parameter familiy of superfunctions

$$J = -(b+e)q + p + z + (f-c)\theta + \eta + \zeta.$$

or

$$J^*r = -(b+e)q + p + z\ ; \quad J^*\rho = (f-c)\theta + \eta + \zeta.$$

The previous computations then lead to our final result:

Proposition 4. *For the lifting of the type I action, there is a 2-parameter family of equivariant momentum maps. This family is parametrized by $b, c \in \mathbb{R}$, and is given by*

$$J^*r = -(b+c)q + p + z\ ; \quad J^*\rho = -c\theta + \eta + \zeta.$$

Consequently, all the 1-points in $\mathfrak{g}^*$ are regular.

By reduction, the supermanifold corresponding to the regular value $0 \in \mathfrak{g}^*$ carries a supersymplectic form, whose even part makes it isomorphic to the cotangent supermanifold $\mathcal{T}^*\mathbb{R}^{1|1}$.

Proof: Again, since the type I structure is abelian, the coadjoint action is the trivial action. Thus, we find that the equivariant lifts satisfy the relations

$$\Phi^*J^*r = -(b+e)(x+q) + bx + p + cx + z = J^*r = -(b+e)q + p + z\ ;$$
$$\Phi^*J^*\rho = (f-c)(\xi+\theta) + e\xi + \eta + f\xi + \zeta = J^*\rho = (f-c)\theta + \eta + \zeta,$$

which together imply

$$c = e\ ; \quad f = 0,$$

with b and c free parameters. It is also clear from these formulas that all 1-points are regular values for J.

Only the assertions in the last paragraph remain now to be proved, so we find next the inverse image of the regular value 0: It has as underlying manifold the plane $z = (b+c)q+p$, so we can take as coordinates q, p, and we can take as generators of the corresponding Batchelor bundle θ and ζ, with η subject to $\eta = c\theta - \zeta$.

Once more we are confronted with the problem that this $(2,2)$-dimensional domain is not invariant under the action, its stabilizer being again trivial. So, it is in this space that we perform the reduction, and it is clear that the restriction of $d\Theta$ to this subsupermanifold is

$$dq\,dp + d\theta\,d\zeta + (d\zeta + b\,d\theta)\,dq - d\theta\,dp.$$

Thus, taking the even part of this superform, we get again the supersymplectic form of Kostant's cotangent supermanifold to $\mathbb{R}^{1|1}$. $\square$

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