R-MATRICES FOR LEIBNIZ ALGEBRAS

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Abstract. Leibniz agebras are a generalization of Lie algebras, where no symmetry properties of the bracket are required. In this note we introduce a notion of R-matrices for this structure and the related Yang-Baxter equations, and discuss some of their basic properties.

Introduction.

Largely because of their importance to String Theory, Quantum Field Theory, and other branches of fundamental research in Mathematical Physics, non commutative analogs of many classical constructions have received a considerable deal of attention in the past several years. To name just one that is close in spirit to our work here (but see also *e.g.* [3] and the references therein), in [2] the non-commutative (and non associative) structures of di- and tri-algebras are analyzed in relation to the so-called Rota-Baxter relation.

In this brief note we would like to introduce yet another non commutative variation on a classical theme: namely, R-matrices for Leibniz algebras. We believe that this construction, which to the best of our knowledge is new to the literature, is relevant for a number of problems; in particular, we believe it can be used as basis for a non commutative extension of some integrable systems, such as the Toda lattice or the discrete KP equations.

I. Leibniz Algebras and Dialgebras.

Leibniz algebras generalize Lie algebras, but with no symmetry requirements. Their definition, given by Loday almost ten years ago (see [5]), goes as follows:

Definition 1. *Let V denote a vector space. A Leibniz bracket on V is a bilinear operation $[\cdot,\cdot]\colon V\times V\to V$, satisfying the following form of the Jacobi identity.*

$$[X,[Y,Z]]-[[X,Y],Z]+[[X,Z],Y]=0. \tag{1}$$

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A Leibniz algebra is a vector space provided with a Leibniz bracket

Obviously, in finitely many dimensions, Leibniz algebras can be defined in terms of structure constants: If $E_1, \ldots, E_n$ is a basis of V, then $[E_i, E_j] = c_{ij}^k E_k$ (summation convention), where the constants satisfy $c_{jk}^l c_{il}^m - c_{ij}^l c_{lk}^m + c_{ik}^l c_{lj}^m = 0$; $m = 1, \ldots, n$, as in the usual Lie algebra case, but no symmetry with respect to the lower indices is required. For instance, if $\dim V = 2$, with basis E_1, E_2, it is not hard to show that up to isomorphism there are four 2- dimensional Leibniz algebras, whose brackets might be given as:

$$\begin{array}{llll}
[E_1, E_1] = 0 \quad ; & [E_1, E_2] = 0 \quad ; & [E_2, E_1] = 0 \quad ; & [E_2, E_2] = 0 \\
[E_1, E_1] = 0 \quad ; & [E_1, E_2] = E_2 \ ; & [E_2, E_1] = -E_2 \ ; & [E_2, E_2] = 0 \\
[E_1, E_1] = E_2 \ ; & [E_1, E_2] = 0 \quad ; & [E_2, E_1] = 0 \quad ; & [E_2, E_2] = 0 \\
[E_1, E_1] = E_2 \ ; & [E_1, E_2] = 0 \quad ; & [E_2, E_1] = E_2 \quad ; & [E_2, E_2] = 0;
\end{array}$$

the first two are Lie algebras, but the remaining two are not.

Leibniz algebras appear in several contexts, but to explicitly construct more general examples, we recall that they are related to dialgebras (see [6]), in a similar way as Lie algebras are related to associative algebras:

Definition 2. *A dialgebra is a vector space V together with two bilinear operators, $\vdash$, $\dashv$, satisfying the following relations*

$$\begin{aligned}
X \dashv (Y \dashv Z) &= X \dashv (Y \vdash Z) \\
(X \vdash Y) \dashv Z &= X \vdash (Y \dashv Z) \\
(X \dashv Y) \vdash Z &= (X \vdash Y) \vdash Z,
\end{aligned}$$

called respectively right and left products.

Given a dialgebra, the associated Leibniz algebra is obtained by defining the bracket as

$$[X, Y] = X \dashv Y - Y \vdash X.$$

With this idea, a non trivial class of (in general irreducible) Leibniz algebras can be constructed as follows: Consider a vector space V with basis $E_1, E_2, \ldots, E_n$, and a partition of $\{1, \ldots, n\}$ into the disjoint subsets $\{1, \ldots, m\}$ and $\{m+1, \ldots, n\}$; then

- Set the right product $\vdash$ equal to zero
- Define the left product $\dashv$ so that, for some arbitrary constants c_{ij}^k, with $k > m$,

$$E_i \dashv E_j = c_{ij}^k E_k \quad \text{for } j \leq m,$$

while

$$E_i \dashv E_j = 0 \quad \text{for } j > m.$$

The c_{ij}^k then become the structure constants of the Leibniz algebra. In general, *none* of these Leibniz algebras is a Lie algebra.

On the other hand, we remark that suppressing the antisymmetry of the bracket does not necessarily allow more freedom in the construction of algebras, as the following (infinite dimensional) examples show:

Example 1. (Virasoro-type algebras.) A non trivial Leibniz analog of the Virasoro algebra does not exist: That is, given an infinite family of generators $\{L_n\}_{n\in\mathbb{Z}}$, required to satisfy relations of the form

$$[L_n, L_m] = c(n,m)L_{n+m},$$

the only solution is the standard Virasoro algebra.

Indeed, imposing the Leibniz condition gives the functional relation among the coefficients $c(m,n)$:

$$c(m,k)c(n,m+k) - c(n,m)c(n+m,k) + c(n,k)c(n+k,m) = 0.$$

Setting $k = 0$ in this equation gives

$$c(m,0) + c(n,0) - c(m+n,0) = 0,$$

which shows that the coefficients are of the form $c(m,n) = am + f(m,n)$, with $f(m,0) = 0$, while setting $m = n = 0$ gives the antisymmetry relation

$$c(0,k)\,(c(0,k) + c(k,0)) = 0.$$

With a little more work it follows that $c(m,n) = a(m-n)$, and so the algebra is just the standard Virasoro algebra.

Example 2. (Hamiltonian systems.) Also, we can consider $2n$-dimensional space, and assume a Poisson-type bracket for functions is given by a matrix M, so that

$$\{F,G\} = \nabla F^t M \nabla G$$

(where the superscript t denotes transpose). If we wish to verify the Leibniz identity, in the form given in definition 1 and without assuming any symmetry property of M, then, from the relation

$$\nabla\{F,G\} = (\mathrm{Hes}\,F)M\nabla G + \mathrm{Hes}\,G M^t \nabla F,$$

where Hes denotes the Hessian matrix, we readily get the following condition for the Leibniz identity:

$$(\nabla F)^t M^t\,\mathrm{Hes}\,H\,M\nabla G + (\nabla F)^t M^t\,\mathrm{Hes}\,H\,M^t\nabla G = 0,$$

for all F, G, H. From this in turn follows that $M^t = -M$, and so the Poisson bracket comes from a closed 2-form, again as in the standard case.

II. R-matrices for Leibniz algebras.

We now come to the key definition:

Definition 3. *Let $(V, [\cdot,\cdot])$ be a Leibniz algebra and $R\colon V \to V$ a linear mapping. Define two new bilinear operators $[\cdot,\cdot]_{R_-}, [\cdot,\cdot]_{R_+}\colon V \to V$ by*

$$[X, Y]_{R_+} = [RX, Y] + [X, RY].$$

$$[X, Y]_{R_-} = [RX, Y] - [RY, X];$$

Then, we will say that R is an $R_\pm$-matrix if $(V, [\cdot,\cdot]_{R_\pm})$ satisfies the Jacobi-Leibniz identity.

Plainly, both are generalizations of the way classical R-matrices are defined, but again, because the bracket $[\cdot,\cdot]$ is not antisymmetric, they will not in general be equivalent, so one should consider both. Also, to simplify, we will use the following notation: $[\cdot,\cdot]_{R_\pm} = [\cdot,\cdot]_\pm$.

Remarks. $[\cdot,\cdot]_-$ is always antisymmetric, so the resulting algebra is a Lie algebra. In the case $R = I$ (the identity operator), this is just the antisymmetrization of the bracket, and this is a Lie algebra naturally attached to the Leibniz algebra. On the other hand, given a Leibniz algebra, another Lie algebra naturally associated is obtained by quotienting out the ideal generated by $[x, y] - [y, x]$. Obviously, the two Lie algebras so obtained are in general different.

Now, we want to rewrite the Leibniz condition for these new brackets. Arguing with the bracket $[\cdot,\cdot]_+$, from the bilinearity of the original bracket $[\cdot,\cdot]$, we see that the Leibniz condition becomes

$$\begin{aligned}
&[X, R([Y, Z]_+)] - [[X, RY], RZ] + [[X, RZ], RY] \\
&+[R([X, Z]_+), Y] + [RX, [Y, RZ]] - [[RX, Y], RZ] \\
&-[R([X, Y]_+), Z] + [RX, [RY, Z]] + [[RX, Z], RY] = 0.
\end{aligned}$$

Now, if we notice that, say, the last three terms contain Z but not RZ, the Leibniz identity for the original bracket allows us to collect these three terms as $-[B_R(X, Y), Z]$, where

$$B_R(X, Y) = R([X, Y]_+) - [RX, RY].$$

We can restate this as:

Proposition 1. *Let $(V, [\cdot,\cdot])$ be a Leibniz algebra. A linear mapping $R\colon V \to V$ is an R_+-matrix if and only if it satisfies:*

$$[X, B_R(X, Z)] + [B_R(X, Z), Y] - [B_R(X, Y), Z] = 0,$$

where

$$B_R(X, Y) = R([X, Y]_+) - [RX, RY]. \tag{2}$$

And similar computations for the $[\cdot,\cdot]_-$ bracket give:

Proposition 2. *The linear mapping $R\colon V \to V$ is an R_--matrix if and only if it satisfies:*

$$C_R(X,Y;Z) + C_R(Y,Z;X) - C_R(X,Z;Y) = 0,$$

where

$$C_R(X,Y;Z) = [R([X,Y]_-),Z] + [RY,[RX,Z]] - [RX,[RY,Z]]. \tag{3}$$

Although the two results have the same aspect, they are actually quite different. Indeed, notice that the tensor C has the antisymmetry property

$$C_R(X,Y;Z) = -C_R(Y,X;Z)$$

so it is a V-valued 2-cocycle. Moreover, for the proof one does not require $[\cdot,\cdot]$ to be a Leibniz bracket, so this is in fact *the* obstruction to get a Lie bracket by this procedure from *any* bilinear operation on a vector space. On the other hand, we stress the fact that B is not antisymmetric *unless* the Leibniz algebra is already a Lie algebra.

III. Yang-Baxter type equations.

Even for Lie brackets, the classification of R-matrices is extremely difficult. Thus, one is led to the question of finding sufficient conditions. To this end, if V is a Leibniz algebra, besides the tensors B and C, we define

$$D_R(X,Y,Z) = [R[RX,Y],Z] - [RX,[RY,Z]]$$

so that the tensor C further decomposes as

$$C_R(X,Y;Z) = D_R(X,Y,Z) - D_R(Y,X,Z).$$

Then we propose the following equations:

Definition 4. *The Yang-Baxter-type equations for Leibniz algebras are*

$$B_R(X,Y) = 0. \tag{YB$_+$}$$

$$D_R(X,Y,Z) = 0. \tag{YB$_-$}$$

Remark. In spite of the non-commutativity in this context, (YB$_+$) is actually the same condition as for Lie algebras, so this is the natural definition.

On the other hand, (YB$_-$) could seem artificial at first sight, so one certainly might try to choose different conditions, and a natural choice could be $C_R = 0$. However, since as pointed out this has *nothing* to do with a Leibniz condition on the original algebra, the condition chosen seems preferable to us.

From the above considerations we immediately get the following result:

Proposition 3. *If the operator R satisfies ($YB_\pm$), then it is an $R_\pm$-matrix.*

For completeness, we point out that in terms of bases, if $RE_i = r_i^j E_j$, then the Yang-Baxter equatons take on the form:

$$r_m^n \left(r_i^l c_{lj}^m + r_j^l c_{il}^m \right) - r_i^l r_j^m c_{lm}^n = 0. \tag{YB$_+$}$$

$$r_i^l \left(r_m^n c_{lj}^m c_{nk}^s - r_j^m c_{mk}^n c_{ln}^s \right) = 0, \tag{YB$_-$}$$

Let us now describe some non trivial special solutions to the proposed Yang-Baxter equations.

First, for $[\cdot,\cdot]_+$, consider a decomposition $V = V_0 \oplus V_1$, where the V_i are subalgebras satisfying

$$[V_i, V_j] = \{0\} \quad \text{if } i \neq j.$$

Then, following [7], we can introduce a modified Yang-Baxter equation

$$B_R(X,Y) = [X,Y], \tag{MYB$_+$}$$

Straightforward calculations then show that both, the identity I and the operator $R = P_0 - P_1$, where P_i is projection onto V_i, satisfy this equation; thus, their difference, $I - R$, satisfies (YB$_+$) as well.

The above construction does not work for $[\cdot,\cdot]_-$, but we can still find some non trivial solutions to the corresponding Yang-Baxter equation in the following way:

Let $\langle\cdot,\cdot\rangle$, be an interior product on V, $H \in V$; and R the projection onto the span of H:

$$R(X) = \langle X, H\rangle H.$$

The new bracket is readily found to be

$$[X,Y]_- = \langle X,H\rangle [H,Y] - \langle Y,H\rangle [H,X],$$

so that a sufficient condition for $C_R = 0$ is, for instance,

$$\langle [H,Y], H\rangle = 0$$

(*i.e.*, that the image of H under the "adjoint mapping," $H \mapsto \operatorname{Ad} H = [H,\cdot]$ be orthogonal to H). The resulting Lie algebra is not abelian, so we have a nontrivial solution to our problem.

(We point out that, "conversely," this construction does not work for $[\cdot,\cdot]_+$, so the two cases are indeed very different.)

IV. Concluding Remarks.

Let us outline here the possible relation of these constructins to integrable systems, that we mentioned in the introduction.

First of all, one would like to think of a Leibniz algebra as the "tangent space at the identity" to some manifold possessing some algebraic structure (Loday's "coquecigrue," [5]), in a way similar to the relation between Lie algebras and Lie groups.

This is an elusive task, but certainly there are good reasons to believe this is possible, since—as shown in [6]—there is a sort of universal envelopping algebra for a Leibniz algebra, so there is a candidate for the space of functions on the coquecigrue. In fact, Kinyon and Weinstein ([4]) have quite recently given a more concrete partial answer to the problem, by considering reductive homogeneous spaces with an invariant connection, which in a sense integrate the Leibniz algebra.

Our construction then shows that, using Leibniz brackets, equations of evolution on these spaces in Lax pair form can be written down. In fact, the construction of special solutions to (YB_+) given above shows that R_+-matrices for Leibniz algebras should relate to some kind of "Riemann-Hilbert decomposition problem" (again see [7]), and thus suggests that at least some of the standard methods of integrable systems theory could be applied to this case.

On the other hand, we must point out that Kinyon and Weinstein's construction uses the antisymmetrization of the Leibniz bracket, and thus seems to relate instead to $[\cdot,\cdot]_-$. Obviously all of this requires further investigation, which will be the subject matter of a forthcoming work.

We would like to end this note by stating another interesting question (of a completely different nature) suggested by our computations, which is whether or not *every* Lie algebra can be obtained from a Leibniz algebra by a suitable $[\cdot,\cdot]_-$ deformation; and if so, in how many ways. The problem is certainly not trivial, for one can easily show that the two non trivial 2-dimensional Leibniz algebras—mentioned in section I—might be deformed into the non abelian Lie algebra by solving an equation of the form

$$[E_1, E_2]_- = [r_1^1 E_1 + r_1^2 E_2, E_2] - [r_2^1 E_1 + r_2^2 E_2, E_1] = E_2,$$

whose solution in both cases is easily seen to involve free parameters: for the first case listed we have:

$$[E_1, E_2]_- = -r_2^1 E_2,$$

while for tthe other we have:

$$[E_1, E_2]_- = -\left(r_2^1 + r_2^2\right) E_2.$$

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