HARMONIC MAPS OF C TO GL_nC IN LOW DIMENSIONS

Raúl Felipe

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Raúl Felipe
CIMAT
Jalisco S/N, Valenciana,
36240 Guanajuato, Gto.
APDO. POSTAL 402,
36000 Guanajuato, Gto.
ICIMAF
Calle E No 309 esquina a 15,
Vedado. La Habana, Cuba.

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Abstract

In this paper an ample class of non-trivial harmonic maps of $D \subset \mathbb{C}$ to $GL_2\mathbb{C}$ are presented. Several of them are related to some quasi-conformal mappings. These can be applied, in turn, to construct other harmonic maps from $D \subset \mathbb{C}$ into $GL_n\mathbb{C}$ for $n > 2$.

1 Introduction

Recently the harmonic maps of a compact domain D in the complex plane into a Lie group have been extensively studied [2], [3] and [6] and their references, in particular in [6] is given a comprehensive treatment of harmonic maps into $U(n)$. On the other hand, harmonic maps arise in physics in a natural manner as solutions of the Chiral model (see [7], where only the case $n = 2$ was considered).

One of the techniques used in this area for almost 30 years has consisted of reformulating the harmonic map equation as an integrable system (for instance as a Zakharov-Shabat type equation) and lately, this development has been particularly strong in Great Britain. In the case, of a matrix Lie group, as is well-known, the two-dimensional Toda lattices have a relevant role. In this paper, non-trivial new harmonic maps into $U(2)^{\mathbb{C}} = GL_2\mathbb{C}$ are presented. But, our approach instead of starting with some integrable system consists of studying the harmonic map equation directly. The idea behind our method is very simple: the solutions are expressed in the form of full Kostant-Toda matrices, which linearize the expression of the harmonic map equation by components. Hence, we make the results more explicit and their proofs very clear. On the other hand, from our point of view harmonic maps $\phi : \mathbb{C} \to GL_2\mathbb{C}$ turn out to be related to quasi-conformal mappings.

Let G be a matrix Lie group; the equation for a harmonic map $\phi : \mathbb{C} \to G$ is the following partial differential equation

$$\left(\phi^{-1}\phi_{\overline{z}}\right)_z + \left(\phi^{-1}\phi_z\right)_{\overline{z}} = 0, \tag{1}$$

where $z = x + iy$ and the overbar means complex conjugation. It is well known that the equation (1) is the Euler-Lagrange equation for the fuctional $\phi \to \int_D \|d\phi\|^2$ where D is a compact domain of $\mathbb{C}$, i.e., $\phi_{|D}$ satisfies the harmonic map equation if and only if it is a critical point of this functional.

On the other hand, the harmonic map equation is equivalent to the system

$$\begin{aligned} U_{\overline{z}} - V_z &= [U, V] \\ U_{\overline{z}} + V_z &= 0 \end{aligned} \tag{2}$$

where $U, V : \mathbb{C} \to g$. Here g is the Lie algebra of G, see [2] and [3] for more details. It is well-known that the first equation of (2) belongs to a distinguished class of PDEs known as integrable systems (it is the Zakharov-Shabat type equation).

One comment is in order. The equations in (2) can be written as only one equation using a complex parameter establishing relations between harmonic maps and loop groups, Riemann-Hilbert problems, etc.

2 Full Kostant-Toda harmonic maps

The aim of this section is to present several non-trivial families of harmonic maps of D into $GL_2\mathbb{C}$ and starting from there obtain a large set of solutions of the matrix Zakharov-Shabat type equation. The results of this section can be considered as analogous to the "Weierstrass formula in immersion theory", by means of which harmonic maps $D \to GL_2\mathbb{C}$ are described explicitly in terms of two functions (in some cases of two quasi-conformal mappings) on D.

Let D be a compact domain of $\mathbb{C}$ and $\mu(z) = \frac{dp}{d\overline{z}}(\overline{z}) / \frac{dq}{dz}(z)$ a smooth function defined in D. We will assume that, $\frac{dq}{dz}(z) \neq 0$ for all $z \in D$, then we have

Lemma 1 *The Beltrami type equation*

$$\frac{\partial u}{\partial \overline{z}} = \mu(z) \frac{\partial u}{\partial z} \qquad\qquad z \in D \tag{3}$$

has a solution of the form

$$u(z, \overline{z}) = q(z) + p(\overline{z}) + u_0$$

where $u_0 \in \mathbb{C}$.

Proof. The Lemma can be proved directly. ■

Note that when $|\mu| < k < 1$, the solution $u(z,\overline{z})$ of (3) will be a quasi-conformal mapping (see [1]).

Thanks to this simple Lemma we can find some non-trivial solutions of (1), that throughout this paper we will call full Kostant-Toda solutions for the harmonic map equation. In what follows ϕ is an invertible matrix of the form

$$\phi = \begin{pmatrix} a & 1 \\ b & c \end{pmatrix} \qquad \text{with} \qquad b - ac = 1. \tag{4}$$

It turns out that matrices of the form (4) are generally not invertible (for instance, when $c = -a$ and $b = -a^2$), hence the condition $b - ac = 1$; in this case, we have

$$\phi^{-1} = \begin{pmatrix} -c & 1 \\ b & -a \end{pmatrix}.$$

A matrix of the form (4) with $b - ac = 1$ will be called a full Kostant-Toda matrix and a full Kostant harmonic map if this is a solution of the harmonic map equation.

Let us suppose that a, b and c depend on variables z and $\overline{z}$, then

$$\phi^{-1}\phi_{\overline{z}} = \begin{pmatrix} b_{\overline{z}} - ca_{\overline{z}} & c_{\overline{z}} \\ ba_{\overline{z}} - ab_{\overline{z}} & -ac_{\overline{z}} \end{pmatrix}, \qquad \phi^{-1}\phi_z = \begin{pmatrix} b_z - ca_z & c_z \\ ba_z - ab_z & -ac_z \end{pmatrix}$$

therefore, we have

$$\left(\phi^{-1}\phi_{\overline{z}}\right)_z = \begin{pmatrix} b_{\overline{z}z} - c_{\overline{z}}a_z - ca_{\overline{z}z} & c_{\overline{z}z} \\ b_z a_{\overline{z}} - a_z b_{\overline{z}} + ba_{\overline{z}z} - ab_{\overline{z}z} & -a_z c_{\overline{z}} - ac_{\overline{z}z} \end{pmatrix}$$

and

$$\left(\phi^{-1}\phi_z\right)_{\overline{z}} = \begin{pmatrix} b_{\overline{z}z} - c_z a_{\overline{z}} - ca_{\overline{z}z} & c_{\overline{z}z} \\ b_{\overline{z}}a_z - a_{\overline{z}}b_z + ba_{\overline{z}z} - ab_{\overline{z}z} & -a_{\overline{z}}c_z - ac_{\overline{z}z} \end{pmatrix}.$$

Now, in order for the full Kostant-Toda matrix ϕ to be a solution of (1) we must do

$$\begin{pmatrix} 2b_{\overline{z}z} - c_{\overline{z}}a_z - c_z a_{\overline{z}} - 2ca_{\overline{z}z} & 2c_{\overline{z}z} \\ 2ba_{\overline{z}z} - 2ab_{\overline{z}z} & -a_z c_{\overline{z}} - a_{\overline{z}}c_z - 2ac_{\overline{z}z} \end{pmatrix} = 0, \tag{5}$$

thus, we have $2c_{\overline{z}z} = 0$. Below, we choose

$$c(z,\overline{z}) = g(z) + f(\overline{z}) + c_0 \tag{6}$$

where $c_0 \in \mathbb{C}$ and $g(z)$, $f(\overline{z})$ are smooth functions of each argument in a domain D of the complex plane. Thus, the function $a(z,\overline{z})$ is seen to be a solution of the following equation

$$\frac{\partial a}{\partial \overline{z}} = -\frac{\frac{df}{d\overline{z}}(\overline{z})}{\frac{dg}{dz}(z)}\frac{\partial a}{\partial z} \qquad z \in D \tag{7}$$

if $\frac{dg}{dz}(z) \neq 0$ in D; in the rest of this section we assume that such a condition is satisfied. On the other hand,

$$2b_{\overline{z}z} - 2ca_{\overline{z}z} = 0, \qquad 2ba_{\overline{z}z} - 2ab_{\overline{z}z} = 0 \tag{8}$$

multiplying the first equation of (8) by a we have $ab_{\overline{z}z} = aca_{\overline{z}z}$, now having in mind that $b - ac = 1$ and the last relation, from the second equation of (8) we have that, $a_{\overline{z}z} = 0$; also it is easy to see that $b_{\overline{z}z} = 0$. Then, a and b can be functions of the form $a(z,\overline{z}) = m(z) + n(\overline{z}) + a_0$, $b(z,\overline{z}) = h(z) + i(\overline{z}) + b_0$ where a_0, $b_0 \in \mathbb{C}$. By Lemma 1, one can readily see that the following functions are solutions of (7)

$$\begin{aligned} a)\quad a(z,\overline{z}) &= g(z) - f(\overline{z}) + a_0 \\ b)\quad a(z,\overline{z}) &= -g(z) + f(\overline{z}) + a_0 \end{aligned} \tag{9}$$

Recall that, we have established a relationship between a, b and c: $b - ac = 1$. In the case a) the function $b(z,\overline{z})$ is given explicitly by

$$\begin{aligned} a)\quad b(z,\overline{z}) &= g^2(z) + (a_0 + c_0)\,g(z) - f^2(\overline{z}) + (a_0 - c_0)\,f(\overline{z}) \\ &\quad +1 + a_0 c_0 \end{aligned} \tag{10}$$

in fact,

$$\begin{aligned} 1 &= (h(z) + i(\overline{z}) + b_0) - (g(z) - f(\overline{z}) + a_0)(g(z) + f(\overline{z}) + c_0) \\ &= (h(z) + i(\overline{z}) + b_0) - g^2(z) - g(z)f(\overline{z}) - c_0 g(z) + g(z)f(\overline{z}) + f^2(\overline{z}) \\ &\quad + c_0 f(\overline{z}) - a_0 g(z) - a_0 f(\overline{z}) - a_0 c_0. \end{aligned}$$

become (10). With similar arguments we can calculate $b(z,\overline{z})$ in the case b),

$$\begin{aligned} b)\quad b(z,\overline{z}) &= \left(-g^2(z) + (a_0 - c_0)g(z)\right) + \left(f^2(\overline{z}) + (a_0 + c_0)f(\overline{z})\right) \\ &\quad +1 + a_0 c_0 \end{aligned} \tag{11}$$

In summary, we have the following result:.

Theorem 2 *Let g and f be two functions defined on D (in particular they can be rational functions). Then, the full Kostant-Toda matrices*

$$\phi_1 = \begin{pmatrix} g(z) - f(\overline{z}) + a_0 & 1 \\ g^2(z) - f^2(\overline{z}) + (a_0 + c_0)g(z) + & g(z) + f(\overline{z}) \\ (a_0 - c_0)f(\overline{z}) + 1 + a_0 c_0 & +c_0 \end{pmatrix} \tag{12}$$

and

$$\phi_2 = \begin{pmatrix} -g(z) + f(\overline{z}) + a_0 & 1 \\ -g^2(z) + (a_0 - c_0)g(z) + f^2(\overline{z}) + & g(z) + f(\overline{z}) \\ (a_0 + c_0)f(\overline{z}) + 1 + a_0 c_0 & +c_0 \end{pmatrix} \tag{13}$$

are full Kostant-Toda harmonic maps of D to $GL_2\mathbb{C}$.

The following Theorem establishes a very interesting property of the full Kostant-Toda harmonic maps.

Theorem 3 *If $\phi = \begin{pmatrix} a & 1 \\ b & c \end{pmatrix}$ is a full Kostant-Toda harmonic map, then*

$$\phi^a = \begin{pmatrix} c & 1 \\ b & a \end{pmatrix}$$

is also a full Kostant-Toda harmonic harmonic map of D into $GL_2\mathbb{C}$.

Proof. The role of the functions a and c in (5) as also in the relation $b - ac = 1$, can be interchanged. In fact, both cases imply the following equations

$$c_{\overline{z}z} = 0, \quad a_{\overline{z}z} = 0, \quad b_{\overline{z}z} = 0, \quad a_z c_{\overline{z}} + a_{\overline{z}} c_z = 0, \quad b - ac = 1.$$

■

Corollary 4 *Associated to ϕ_1 and ϕ_2 of the Theorem 2 are the following full Kostant-Toda harmonic maps*

$$\phi_1^a = \begin{pmatrix} g(z) + f(\overline{z}) + c_0 & 1 \\ g^2(z) - f^2(\overline{z}) + (a_0 + c_0)g(z) + & g(z) - f(\overline{z}) \\ (a_0 - c_0)f(\overline{z}) + 1 + a_0 c_0 & +a_0 \end{pmatrix}$$

and

$$\phi_2^a = \begin{pmatrix} g(z) + f(\overline{z}) + c_0 & 1 \\ -g^2(z) + (a_0 - c_0)g(z) + f^2(\overline{z}) + & -g(z) + f(\overline{z}) \\ (a_0 + c_0)f(\overline{z}) + 1 + a_0 c_0 & +a_0 \end{pmatrix}$$

As a consequence of Theorem 2 we have the following solutions of the matrix Zakharov-Shabat type equation.

Example 5 *Let* $H = \phi_1$ *for* $g(z) = z$ *and* $f(\overline{z}) = \overline{z}$*; then*

$$H_z = \begin{pmatrix} 1 & 0 \\ 2z + (a_0 + c_0) & 1 \end{pmatrix} \qquad H_{\overline{z}} = \begin{pmatrix} 1 & 0 \\ -2\overline{z} + (a_0 - c_0) & 1 \end{pmatrix}$$

and

$$H^{-1} = \begin{pmatrix} -z - \overline{z} - c_0 & 1 \\ z^2 - \overline{z}^2 + (a_0 + c_0)z + (a_0 - c_0)\overline{z} + 1 + a_0 c_0 & -z + \overline{z} - a_0 \end{pmatrix}.$$

So, defining

$$U = H^{-1} H_z = \begin{pmatrix} z - \overline{z} + a_0 & 1 \\ -z^2 - \overline{z}^2 + 2z\overline{z} + 2a_0 (\overline{z} - z) + 1 - a_0^2 & -z + \overline{z} - a_0 \end{pmatrix},$$

$$V = H^{-1} H_{\overline{z}} = \begin{pmatrix} z - \overline{z} + a_0 & 1 \\ -z^2 - \overline{z}^2 + 2z\overline{z} + 2a_0 (\overline{z} - z) - 1 - a_0^2 & -z + \overline{z} - a_0 \end{pmatrix}$$

we compute now $U_{\overline{z}} - V_z$, $U_{\overline{z}} + V_z$, $[U, V]$ *with* $a_0 = 0$. *A direct calculation gives* $U_{\overline{z}} + V_z = 0$, *on the other hand we have*

$$U_{\overline{z}} - V_z = \begin{pmatrix} -2 & 0 \\ 4(z - \overline{z}) & 2 \end{pmatrix} \tag{14}$$

finally

$$UV = \begin{pmatrix} -1 & 0 \\ 2(z - \overline{z}) & 1 \end{pmatrix}, \qquad VU = \begin{pmatrix} 1 & 0 \\ 2(\overline{z} - z) & -1 \end{pmatrix}$$

then

$$[U, V] = \begin{pmatrix} -2 & 0 \\ 4(z - \overline{z}) & 2 \end{pmatrix} \tag{15}$$

clearly (14) *and* (15) *imply that* $U_{\overline{z}} - V_z = [U, V]$.

Example 6 *Beginning from the family* ϕ_2 *for* $g(z) = z$ *and* $f(\overline{z}) = \overline{z}$ *we also have that the matrices*

$$U=\begin{pmatrix} -z+\overline{z}+a_0 & 1 \\ -z^2-\overline{z}^2+2z\overline{z}+2a_0\left(z-\overline{z}\right)-1-a_0^2 & z-\overline{z}-a_0 \end{pmatrix},$$

and

$$V=\begin{pmatrix} -z+\overline{z}+a_0 & 1 \\ -z^2-\overline{z}^2+2z\overline{z}+2a_0\left(z-\overline{z}\right)+1-a_0^2 & z-\overline{z}-a_0 \end{pmatrix}$$

are solutions of (2)*.*

We consider now full Kostant-Toda harmonic maps induced by the shift matrix

$$\Lambda=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and its transpose Λ^T.

Theorem 7 *There exist full Kostant-Toda harmonic maps ϕ_Λ and ϕ_{Λ^T}, such that*

$$\phi_\Lambda^{-1}\left(\phi_\Lambda\right)_z=\Lambda=\phi_\Lambda^{-1}\left(\phi_\Lambda\right)_{\overline{z}} \tag{16}$$

and

$$\phi_{\Lambda^T}^{-1}\left(\phi_{\Lambda^T}\right)_z=\Lambda^T=\phi_{\Lambda^T}^{-1}\left(\phi_{\Lambda^T}\right)_{\overline{z}} \tag{17}$$

Proof. The proof of the Theorem reduces to determine ϕ_Λ and ϕ_{Λ^T} such that (16) and (17) are satisfied. Let $\phi_\Lambda=\begin{pmatrix} a & 1 \\ b & c \end{pmatrix}$ be a Kostant matrix, such that, $\left(\phi_\Lambda\right)_z=\phi_\Lambda\Lambda$. In this case we have

$$\begin{pmatrix} a_z & 0 \\ b_z & c_z \end{pmatrix}=\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \tag{18}$$

from (18) it follows that $a=0$, $b_z=0$ and $c_z=b$, we now take $b=1$; it implies that $c\left(z,\overline{z}\right)=z+h(\overline{z})$ where $h(\overline{z})$ is a smooth function. On the other hand, the condition $\left(\phi_\Lambda\right)_{\overline{z}}=\phi_\Lambda\Lambda$ asserts that

$$\begin{pmatrix} 0 & 0 \\ 0 & c_{\overline{z}} \end{pmatrix}=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

therefore $h(\overline{z}) = \overline{z} + h_0$ where $h_0 \in \mathbb{C}$, thus

$$\phi_\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & z + \overline{z} + h_0 \end{pmatrix}.$$

Now, in the same way it is easy to see that

$$\phi_{\Lambda^T} = \begin{pmatrix} z + \overline{z} + h_1 & 1 \\ z + \overline{z} + h_1 + 1 & 1 \end{pmatrix}$$

■

Notice that the full Kostant-Toda harmonic maps ϕ_Λ and ϕ_{Λ^T} are not included in the family of solutions of the Theorem 2 and the Corollary 4. Its easy to show that the other element of the base of $\mathfrak{sl}_2\mathbb{C}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ does not generate a full Kostant-Toda harmonic map, since $|H| \neq 0$.

Corollary 8 *The following full Kostant-Toda matrices*

$$\phi^a_\Lambda = \begin{pmatrix} z + \overline{z} + h_0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \phi^a_{\Lambda^T} = \begin{pmatrix} 1 & 1 \\ z + \overline{z} + h_1 + 1 & z + \overline{z} + h_1 \end{pmatrix}$$

are full Kostant-Toda harmonic maps of D to $GL_2\mathbb{C}$.

Proof. It follows from the Theorem 3. ■

Remark 9 *Consider the matrix $S = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$, then is obvious that $S^{-1} = \begin{pmatrix} 1 & 0 \\ -s & 1 \end{pmatrix}$. Le us denote $L = S\Lambda S^{-1}$, from (16) follows that*

$$\left(\phi_\Lambda S^{-1}\right)^{-1}\left(\phi_\Lambda S^{-1}\right)_z = L - S_z S^{-1} \tag{19}$$

and

$$\left(\phi_\Lambda S^{-1}\right)^{-1}\left(\phi_\Lambda S^{-1}\right)_{\overline{z}} = L - S_{\overline{z}} S^{-1} \tag{20}$$

thus, (19) and (20) imply that if

$$L_z + L_{\overline{z}} - \left(\left(S_{\overline{z}} S^{-1}\right)_z + \left(S_z S^{-1}\right)_{\overline{z}}\right) = 0 \tag{21}$$

then $\phi_\Lambda S^{-1}$ will be a harmonic map. After a straightforward calculation we can see that the equation (21) is equivalent to the following equation

$$\begin{pmatrix} 0 & 0 \\ 2s_{z\overline{z}} & 0 \end{pmatrix} = \begin{pmatrix} -\left(s_z + s_{\overline{z}}\right) & 0 \\ -2s\left(s_z + s_{\overline{z}}\right) & s_z + s_{\overline{z}} \end{pmatrix}$$

hence, $s = z - \overline{z} + s_0$ *or* $s = \overline{z} - z + s_1$. *In this case*

$$L_1 = \begin{pmatrix} z - \overline{z} + s_0 & 1 \\ -\left(z - \overline{z} + s_0\right)^2 & -\left(z - \overline{z} + s_0\right) \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} \overline{z} - z + s_1 & 1 \\ -\left(\overline{z} - z + s_1\right)^2 & -\left(\overline{z} - z + s_1\right) \end{pmatrix}.$$

The corresponding full Kostant-Toda harmonic maps are

$$\phi_{L_1} = \begin{pmatrix} z - \overline{z} + s_0 & 1 \\ z^2 - \overline{z}^2 + (s_0 + h_0)\, z + (s_0 - h_0)\, \overline{z} + 1 + s_0 h_0 & z + \overline{z} + h_0 \end{pmatrix},$$

$$\phi_{L_2} = \begin{pmatrix} -z + \overline{z} + s_1 & 1 \\ -z^2 + \overline{z}^2 + (s_1 - h_0)\, z + (s_1 + h_0)\, \overline{z} + 1 + s_1 h_0 & z + \overline{z} + h_0 \end{pmatrix}.$$

The solutions ϕ_{L_1} and ϕ_{L_2} belong to the family ϕ_1 and ϕ_2 of the Theorem 2, respectively. Therefore, dressing Λ we do not obtain new solutions of the hamonic map equation.

Remark 10 *Choosing two* 2×2 *constant matrices* A *and* B *such that* $[A, B] = 0$, *we can construct a harmonic map that satisfies*

$$\phi^{-1}\left(\phi\right)_z = A, \qquad \phi^{-1}\left(\phi\right)_{\overline{z}} = B; \tag{22}$$

clearly if ϕ *satisfies* (22) *then* ϕ *is a harmonic map. Notice that, before we considered the cases* $A = B = \Lambda$ *and* $A = B = \Lambda^T$, *and full Kostant-Toda harmonic maps were constructed. But, in general for* A *and* B *arbitrary constant matrices then the corresponding* ϕ *are not full Kostant-Toda harmonic maps.*

For instance, let us consider the Pauli matrix

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The matrix

$$\phi_\sigma = \begin{pmatrix} a_0 e^{z+\overline{z}} & b_0 e^{-(z+\overline{z})} \\ c_0 e^{z+\overline{z}} & d_0 e^{-(z+\overline{z})} \end{pmatrix}.$$

is a harmonic map if $\left| \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \right| \neq 0$, because

$$\phi_\sigma^{-1} (\phi_\sigma)_z = \sigma = \phi_\sigma^{-1} (\phi)_{\overline{z}},$$

but ϕ_σ is not a full Kostant-Toda harmonic map.

On the other hand we want to remark that the full Kostant-Toda harmonic maps determined by the family ϕ_1 and ϕ_2 of Theorem 2 and Corollary 4 do not satisfy (21) unless g and f are both constant functions.

3 More full Kostant-Toda harmonic maps

In the previous section, we have seen that using harmonic functions of the form

$$u(x, y) = c_0 f(x + iy) + c_1 g(x - iy) + c_2 \tag{23}$$

where c_0, c_1 and c_2 are complex numbers, it is possible to construct harmonic maps of D into $GL_2\mathbb{C}$. But we do not have to restrict ourselves to this case, in fact, as we shall now show the functions a, b and c can also have a very different appearance to (23), i.e, we can also consider functions a, b and c of a more general nature. These harmonic maps are connected with more analytic methods. We recall that the equation (5) shows that if

$$c_{\overline{z}z} = 0, \quad a_{\overline{z}z} = 0, \quad b_{\overline{z}z} = 0, \quad a_z c_{\overline{z}} + a_{\overline{z}} c_z = 0, \quad b - ac = 1. \tag{24}$$

then ϕ given by (4) will be a harmonic map of D into $GL_2\mathbb{C}$.

Let D be a domain in $\mathbb{C}$, with its boundary $\Gamma = \partial D$ compact in $\mathbb{C}$. The domain D is said to be regular if the Dirichlet problem has a solution for every $k \in C(\Gamma)$, i.e, if given $k \in C(\Gamma)$ it is possible to find a function F that is harmonic in D, continuous in the closure $\overline{D}$ and satisfies $F(z) = k(z)$, $z \in D$ (see [5])

Theorem 11 *Let D be a regular domain and $c \in C^2(D)$ a harmonic function in D, i.e, $c_{\overline{z}z} = 0$, then for every $z_0 \in D$ there is a ball $B(z_0) = |z - z_0| < \epsilon$ and two functions a and b not unique, such that*

$$a_{\overline{z}z} = 0, \quad b_{\overline{z}z} = 0, \quad a_z c_{\overline{z}} + a_{\overline{z}} c_z = 0, \quad b - ac = 1 \tag{25}$$

in $\overline{B(z_0)}$.

Proof. The purpose is to build solutions of (25) through a known harmonic function c in D. The system of partial differential equations in (25) has an abundance of solutions having the desired property $b - ac = 1$. In fact, let c be a given harmonic function in D (we recall that D is regular). The solution of the equation

$$a_z c_{\overline{z}} + a_{\overline{z}} c_z = 0. \tag{26}$$

will be required to have one of the following two properties:

$$\begin{aligned} 1)\ a_z &= -c_z, \qquad a_{\overline{z}} = c_{\overline{z}}, \\ 2)\ a_z &= c_z, \qquad a_{\overline{z}} = -c_{\overline{z}}, \end{aligned} \tag{27}$$

in some closed ball of D with center z_0. It is clear that those a for which $(27-1)$ or $(27-2)$ is true are solutions of (26). Let us write $(27-1)$ as a system of four equations for $\widehat{u}$ and $\widehat{v}$ $(a = \widehat{u} + i\widehat{v})$:

$$\widehat{u}_x = -v_y, \qquad \widehat{u}_y = v_x, \qquad \widehat{v}_x = u_y, \qquad \widehat{v}_y = -u_x, \tag{28}$$

where $c = u + iv$. Let $z_0 \in D$, let $B(z_0)$ be a circular neighborhood of z_0 such that $B(z_0) \subset D$. Hence, $d\widehat{u} = -v_y dx + v_x dy$ and $d\widehat{v} = u_y dx - u_x dy$ are exact differentials in $B(z_0)$, $\widehat{u}$ and $\widehat{v}$ can be found by evaluating the following linear integrals

$$\int_{(x_0,y_0)}^{(x,y)} \left[-v_t(s,t)\, ds + v_s(s,t)\, dt\right] = \widehat{u}(x,y) - \widehat{u}(x_0,y_0), \tag{29}$$

and

$$\int_{(x_0,y_0)}^{(x,y)} \left[u_t(s,t)\, ds - u_s(s,t)\, dt\right] = \widehat{v}(x,y) - \widehat{v}(x_0,y_0), \tag{30}$$

along a line segment in $B(z_0)$, where $\widehat{u}(x_0,y_0)$ and $\widehat{v}(x_0,y_0)$ can be chosen arbitrarily. But then $\widehat{u}(x,y)$ and $\widehat{v}(x,y)$ are uniquely determined in $B(z_0)$. Now, since c is a harmonic function also in $B(z_0)$,

$$c_z = \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right), \tag{31}$$

is an analytic function in $B(z_0)$, then the Cauchy-Riemann equations for the function c_z are:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \ \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \tag{32}$$

Hence, from (28) and (32) we have

$$\begin{aligned}\frac{\partial}{\partial x}\left(\frac{\partial \widehat{u}}{\partial y}-\frac{\partial \widehat{v}}{\partial x}\right) &= -\frac{\partial}{\partial x}\left(\frac{\partial \widehat{v}}{\partial x}-\frac{\partial \widehat{u}}{\partial y}\right) \\ &= -\frac{\partial}{\partial y}\left(-\frac{\partial \widehat{v}}{\partial y}-\frac{\partial \widehat{u}}{\partial x}\right) \\ &= \frac{\partial}{\partial y}\left(\frac{\partial \widehat{u}}{\partial x}+\frac{\partial \widehat{v}}{\partial y}\right)\end{aligned} \tag{33}$$

and also

$$\begin{aligned}\frac{\partial}{\partial y}\left(\frac{\partial \widehat{u}}{\partial y}-\frac{\partial \widehat{v}}{\partial x}\right) &= -\frac{\partial}{\partial y}\left(\frac{\partial \widehat{v}}{\partial x}-\frac{\partial \widehat{u}}{\partial y}\right) \\ &= \frac{\partial}{\partial x}\left(-\frac{\partial \widehat{v}}{\partial y}-\frac{\partial \widehat{u}}{\partial x}\right) \\ &= -\frac{\partial}{\partial x}\left(\frac{\partial \widehat{u}}{\partial x}+\frac{\partial \widehat{v}}{\partial y}\right).\end{aligned} \tag{34}$$

In view of (33) and (34) the function a_z will be an analytic function in $B(z_0)$. Thus, $a_{z\overline{z}}=0$ in $B(z_0)$. Let us define $b=1+ac$, then

$$b_{z\overline{z}}=a_z c_{\overline{z}}+a_{\overline{z}}c_z+a_{z\overline{z}}+c_{z\overline{z}}=0.$$

It remains to take ϵ such that $\overline{B(z_0,\epsilon)}\subset B(z_0)$. The construction of a and b in the case $(27-2)$ is reduced to the case $(27-1)$, replacing a by $-a$. ■

Theorem 12 *Let D be a regular domain in $\mathbb{C}$, with its boundary $\Gamma=\partial D$ compact in $\mathbb{C}$. Then for every $z_0\in D$ there is $\varepsilon>0$ for which one can find at least two families of full Kostant-Toda harmonic maps of $B(z_0,\epsilon)$ into $GL_2\mathbb{C}$.*

Proof. This is a consequence of the Theorem 8. ■

4 Full Kostant-Toda harmonic maps into $GL_3\mathbb{C}$

For $n>2$ one cannot hope to construct full Kostant-Toda harmonic maps even in the case $n=3$ by means of some simple algorithm, however the results of the sections 2 and 3 allow us to also construct full Kostant-Toda harmonic maps of D into $GL_3\mathbb{C}$ starting with a full Kostant-Toda harmonic map for $n=2$. We have

Proposition 13 *Let $\phi=\begin{pmatrix} a & 1 \\ b & c\end{pmatrix}$ be a full Kostant-Toda harmonic map, where $b-ac=1$, then*

$$\widehat{\phi} = \begin{pmatrix} a & 1 & 0 \\ b & c & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

is a full Kostant-Toda harmonic map of D into $GL_3\mathbb{C}$.

Proof. It is easy to show that,

$$\left(\widehat{\phi}\right)^{-1} = \begin{pmatrix} -c & 1 & -1 \\ b & -a & a \\ 0 & 0 & 1 \end{pmatrix},$$

then, with a straightforward calculation we obtain

$$\left(\widehat{\phi}\right)^{-1} \widehat{\phi}_z = \begin{pmatrix} b_z - ca_z & c_z & 0 \\ ba_z - ab_z & -ac_z & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad . \tag{35}$$

and

$$\left(\widehat{\phi}\right)^{-1} \widehat{\phi}_{\overline{z}} = \begin{pmatrix} b_{\overline{z}} - ca_{\overline{z}} & c_{\overline{z}} & 0 \\ ba_{\overline{z}} - ab_{\overline{z}} & -ac_{\overline{z}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{36}$$

Since ϕ is a harmonic map, by (35) and (36) $\widehat{\phi}$ will be a harmonic map. ∎

Now, a similar method enables one to obtain the following less evident result

Proposition 14 *Let $\phi = \begin{pmatrix} a & 1 \\ b & c \end{pmatrix}$ be a full Kostant-Toda harmonic map, where as always let us assume that $b - ac = 1$. Then, the matrix*

$$\widetilde{\phi} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & a & 1 \\ 0 & b & c \end{pmatrix}$$

itself is a full Kostant-Toda harmonic map of $\mathbb{C}$ into $GL_3\mathbb{C}$.

Proof. Recall first that ϕ is a harmonic map, then

$$c_{\overline{z}z} = 0, \quad a_{\overline{z}z} = 0, \quad b_{\overline{z}z} = 0, \quad a_z c_{\overline{z}} + a_{\overline{z}} c_z = 0, \quad b - ac = 1.$$

On the other hand, we have

$$\left(\widetilde{\phi}\right)^{-1} = \begin{pmatrix} 1 & c & -1 \\ 0 & -c & 1 \\ 0 & 1+ac & -a \end{pmatrix},$$

thus we see that

$$\left(\widetilde{\phi}\right)^{-1} \widetilde{\phi}_z = \begin{pmatrix} 0 & ca_z - b_z & -c_z \\ 0 & -ca_z + b_z & c_z \\ 0 & ba_z - ab_z & -ac_z \end{pmatrix}. \tag{37}$$

But then it follows from (37) that

$$\begin{aligned} \left(\left(\widetilde{\phi}\right)^{-1} \widetilde{\phi}_z\right)_{\overline{z}} &= \begin{pmatrix} 0 & c_{\overline{z}}a_z + ca_{z\overline{z}} - b_{z\overline{z}} & -c_{z\overline{z}} \\ 0 & -c_{\overline{z}}a_z - ca_{z\overline{z}} + b_{z\overline{z}} & c_{z\overline{z}} \\ 0 & b_{\overline{z}}a_z + ba_{z\overline{z}} - a_{\overline{z}}b_z - ab_{z\overline{z}} & -a_{\overline{z}}c_z - ac_{z\overline{z}} \end{pmatrix} \quad (38) \\ &= \begin{pmatrix} 0 & c_{\overline{z}}a_z & 0 \\ 0 & -c_{\overline{z}}a_z & 0 \\ 0 & b_{\overline{z}}a_z - a_{\overline{z}}b_z & -a_{\overline{z}}c_z \end{pmatrix}, \end{aligned}$$

also, in a similar manner it follows at once that

$$\begin{aligned} \left(\left(\widetilde{\phi}\right)^{-1} \widetilde{\phi}_{\overline{z}}\right)_z &= \begin{pmatrix} 0 & c_z a_{\overline{z}} + ca_{z\overline{z}} - b_{z\overline{z}} & -c_{z\overline{z}} \\ 0 & -c_z a_{\overline{z}} - ca_{z\overline{z}} + b_{z\overline{z}} & c_{z\overline{z}} \\ 0 & b_z a_{\overline{z}} + ba_{z\overline{z}} - a_z b_{\overline{z}} - ab_{z\overline{z}} & -a_z c_{\overline{z}} - ac_{z\overline{z}} \end{pmatrix} \quad (39) \\ &= \begin{pmatrix} 0 & c_z a_{\overline{z}} & 0 \\ 0 & -c_z a_{\overline{z}} & 0 \\ 0 & b_z a_{\overline{z}} - a_z b_{\overline{z}} & -a_z c_{\overline{z}} \end{pmatrix} \end{aligned}$$

now, the conclusion of the Theorem is clear from (38) and (39) taking into account the equations at the begining of the proof. ■

In this part we complete the study of full Kostant-Toda harmonic maps from a compact D from $\mathbb{C}$ into $GL_3\mathbb{C}$. The following Lemma is immediately verified:

Lemma 15 *The equation*

$$uu_{z\overline{z}} - u_z u_{\overline{z}} = 0 \qquad z \in D \tag{40}$$

admits the following two solutions $u_+(z,\overline{z}) = \mu_1(z)\, e^{k\overline{z}}$ *and* $u_-(z,\overline{z}) = \mu_2(\overline{z})\, e^{kz}$, *where* $\mu_1(z)$ *and* $\mu_2(\overline{z})$ *are functions defined in* D *different from zero and* $k \in \mathbb{C}$.

We now construct more full Kostant harmonic maps

Proposition 16 *Let c be a harmonic function in D. Then, the following matrices*

$$\phi_+ = \begin{pmatrix} 1 & 1 & 0 \\ 1+c & c & 1 \\ 0 & 0 & u_+(z,\overline{z}) \end{pmatrix} \quad \phi_- = \begin{pmatrix} 1 & 1 & 0 \\ 1+c & c & 1 \\ 0 & 0 & u_-(z,\overline{z}) \end{pmatrix},$$

where $u_+(z,\overline{z})$ and $u_-(z,\overline{z})$ are the solutions of (40) described in the Lemma 15, are full Kostant-Toda harmonic maps from $\mathbb{C}$ into $GL_3\mathbb{C}$.

Proof. In fact, given a matrix of the form

$$\phi = \begin{pmatrix} a & 1 & 0 \\ b & c & 1 \\ 0 & 0 & f \end{pmatrix},$$

for which $b - ac = 1$ and $f \neq 0$, we have

$$\phi^{-1} = \begin{pmatrix} -c & 1 & -\frac{1}{f} \\ b & -a & \frac{a}{f} \\ 0 & 0 & \frac{1}{f} \end{pmatrix},$$

then, it should be noted that

$$\phi^{-1}\phi_z = \begin{pmatrix} b_z - ca_z & c_z & -\frac{f_z}{f} \\ ba_z - ab_z & -ac_z & a\frac{f_z}{f} \\ 0 & 0 & \frac{f_z}{f} \end{pmatrix} \tag{41}$$

and

$$\phi^{-1}\phi_{\overline{z}} = \begin{pmatrix} b_{\overline{z}} - ca_{\overline{z}} & c_{\overline{z}} & -\frac{f_{\overline{z}}}{f} \\ ba_{\overline{z}} - ab_{\overline{z}} & -ac_{\overline{z}} & a\frac{f_{\overline{z}}}{f} \\ 0 & 0 & \frac{f_{\overline{z}}}{f} \end{pmatrix} \tag{42}$$

from (41) and (42) it follows that in the particular case: $a = 1$, then $b = c + 1$ and obviously b is a harmonic function in D. On the other hand, in order for ϕ to be a full Kostant-Toda harmonic map, is necessary that

$$ff_{z\overline{z}} - f_z f_{\overline{z}} = 0 \qquad z \in D.$$

Now the Proposition is an inmediate consequence of the Lemma 15. ∎

5 Other harmonic maps into $GL_3\mathbb{C}$

In this section, we will construct other harmonic maps by a procedure that we call "dimensional extension". The central idea is already found in section 4. We have the following

Lemma 17 *Let* $\phi = \begin{pmatrix} a & 1 \\ b & c \end{pmatrix}$ $D \longrightarrow GL_2\mathbb{C}$ *be a harmonic map, then the following matrices*

$$\Phi_\alpha = \begin{pmatrix} \alpha e^{G(z,\overline{z})} & 0 \quad 0 \\ 0 & \\ 0 & \phi \end{pmatrix}, \quad \Phi_\beta = \begin{pmatrix} & 0 \\ \phi & 0 \\ 0 \quad 0 & \beta e^{F(z,\overline{z})} \end{pmatrix} \tag{43}$$

and

$$\Phi_\sigma = \begin{pmatrix} a & 0 & 1 \\ 0 & \sigma e^{E(z,\overline{z})} & 0 \\ b & 0 & c \end{pmatrix} \tag{44}$$

will be harmonic maps of D into $GL_3\mathbb{C}$, whenever α, β and σ are complex numbers different from zero and G, F and E are harmonic functions in D.

Proof. Let $\alpha \neq 0$, and let

$$\Phi_\alpha = \begin{pmatrix} \alpha T(z,\overline{z}) & 0 \quad 0 \\ 0 & \\ 0 & \phi \end{pmatrix}$$

where ϕ $D \longrightarrow GL_2\mathbb{C}$ is a harmonic map, we shall choose $T(z,\overline{z})$ such that Φ_α is a harmonic map. Now,

$$\Phi_\alpha^{-1} = \begin{pmatrix} \frac{1}{\alpha} T^{-1}(z,\overline{z}) & 0 \quad 0 \\ 0 & \\ 0 & \phi^{-1} \end{pmatrix}$$

and

$$(\Phi_\alpha)_z = \begin{pmatrix} \alpha \left(T(z,\overline{z})\right)_z & 0 \quad 0 \\ 0 & \\ 0 & \phi_z \end{pmatrix}, \qquad (\Phi_\alpha)_{\overline{z}} = \begin{pmatrix} \alpha \left(T(z,\overline{z})\right)_{\overline{z}} & 0 \quad 0 \\ 0 & \\ 0 & \phi_{\overline{z}} \end{pmatrix}$$

then

$$\Phi_\alpha^{-1}\left(\Phi_\alpha\right)_z=\begin{pmatrix} T^{-1}\left(z,\overline{z}\right)\left(T\left(z,\overline{z}\right)\right)_z & 0 \quad 0 \\ 0 & \\ 0 & \phi^{-1}\phi_z \end{pmatrix} \tag{45}$$

and similarly

$$\Phi_\alpha^{-1}\left(\Phi_\alpha\right)_{\overline{z}}=\begin{pmatrix} T^{-1}\left(z,\overline{z}\right)\left(T\left(z,\overline{z}\right)\right)_{\overline{z}} & 0 \quad 0 \\ 0 & \\ 0 & \phi^{-1}\phi_{\overline{z}} \end{pmatrix} \tag{46}$$

from $(45)-(46)$ we see that if

$$\left(\ln T\left(z,\overline{z}\right)\right)_{z\overline{z}}=0 \tag{47}$$

then Φ_α is a harmonic map. Thus, equation (47) gives $T\left(z,\overline{z}\right)=e^{G(z,\overline{z})}$ where $G\left(z,\overline{z}\right)$ is a harmonic function in D. Similar arguments can be used to prove that the matrix Φ_β and Φ_σ are harmonic maps. ■

The previous results of the present paper allow us to construct many more harmonic maps from D into $GL_3\mathbb{C}$. For instance, from (43) and (44) it follows that

$$\Phi_\alpha=\begin{pmatrix} \alpha e^{G(z,\overline{z})} & 0 & 0 \\ 0 & z+\overline{z}+h_1 & 1 \\ 0 & z+\overline{z}+h_1+1 & 1 \end{pmatrix}$$

and

$$\Phi_\beta=\begin{pmatrix} 0 & 1 & 0 \\ 1 & z+\overline{z}+h_0 & 0 \\ 0 & 0 & \beta e^{F(z,\overline{z})} \end{pmatrix}, \quad \Phi_\sigma=\begin{pmatrix} z+\overline{z}+h_1 & 0 & 1 \\ 0 & \sigma e^{E(z,\overline{z})} & 0 \\ z+\overline{z}+h_1+1 & 0 & 1 \end{pmatrix}$$

are harmonic maps of D into $GL_3\mathbb{C}$.

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