LEVY PROCESSES ON BANACH SPACES : DISTRIBUTIONAL PROPERTIES AND SUBORDINATION

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Abstract

The paper deals with subordination of Banach space valued Lévy processes via a random time change by a one dimensional subordinator. Distributional properties of Banach valued Lévy processes that are needed for this purpose are first derived.

1 Introduction

Lévy processes have received increasing attention in recent years, both from the applied and the theoretical point of view; see for example the books by Bertoin (1996) and Sato (1999) as well as the Proceedings of the two recent MaPhySto Conferences on Lévy Processes: Theory and Applications.

Lévy processes taking values in separable Banach spaces have been studied by several authors. Gihman and Skorohod (1975) derived the corresponding Lévy-Khintchine formula for the characteristic functional of such processes, while Dettweiler (1982) and Albeverio and Rudiger (2002) studied their associated stochastic integrals and the corresponding Lévy-Ito decomposition. The purpose of the present paper is to study distributional properties of two kind of transformations of Banach-valued Lévy processes that are still Lévy processes: linear transformations and subordination via a random time change by a one dimensional increasing Lévy process. In both cases we identified the generating triplet in an explicit manner. As a key tool for the subordination case, we first derive distributional properties that are of interest in their on, such as conditions for the existence of g-moments as well as tail and moment inequalities for Banach space valued Lévy processes.

Section 2 of this paper recalls basic facts about Banach space valued Lévy processes that are used in the sequel. In Section 3 we prove an equivalence between finiteness of g-moments of the processes and g-moments of the corresponding Lévy measures. We also derive some useful tail and moment inequalities for Banach space Lévy processes which are used in Section 4 to obtain the characteristic triplet of the Banach space valued Lévy process subordinated to a one dimensional subordinator.

2 Preliminaries and notation

Let B be a real separable Banach space with norm $\|\cdot\|$ and strong dual B^*. A *Lévy process* $\{X_t : t \geq 0\}$ with values in B is a stochastic process defined on a probability space $(\Omega, \mathcal{F}, P)$ satisfying the following properties:
i) For any $0 \leq t_1 < t_2 < ... < t_n$, the random variables $X_{t_2} - X_{t_1}, ..., X_{t_n} - X_{t_{n-1}}$ are independent (*independent increments*).
ii) For any $s \geq 0, t \geq 0$, $X_{t+s} - X_s$ and X_t have the same law (*stationary increments*).
iii) Almost surely, $X_0 = 0$.
iv) It is *stochastically continuous*, i.e., for every $\varepsilon > 0$, $P(\|X_t - X_s\| > \varepsilon) \to 0$ as $s \to t$.
v) Almost surely, it is *right-continuous* in $t \geq 0$ and has *left limits* in $t > 0$.

The structure of these processes is similar as for the finite dimensional case. Gihman and Skorohod (1975) decompose the process into a sum of a nonrandom, discrete and stochastically continuous parts. As usual, denote $X_{s-} := \lim_{s\uparrow t} X_s$. Let $\mathcal{B}_\varepsilon$ be the ring of Borel sets of B whose distance from at point 0 is at least ε. Define for any A in $\mathcal{B}_\varepsilon$,

$$N_t(A) = \textstyle\sum_{s<t} 1_A (X_s - X_{s-}) \quad \text{and}$$

$$X_t^A = \textstyle\sum_{s<t} (X_s - X_{s-}) 1_A (X_s - X_{s-}),$$

that is, the number of jumps and the sum of jumps of the process which have occurred up to time t and took place in A, respectively. They are independent processes.

For any continuous linear functional f on B, the one dimensional process $\{f(X_t)\}$ is stochastically continuous and has independent increments. This fact supplies a useful non trivial information of the Banach-valued Lévy process $\{X_t\}$ to obtain the following Lévy-Khintchine representation from Gihman and Skorohod (1975, Th. IV.5).

Theorem 1 *Let $\{X_t : t \geq 0\}$ be a Lévy process on a separable Banach space B. Then, its characteristic functional $Ee^{if(X_t)}$ has the form*

$$exp\left\{ t\left(-\frac{1}{2}A(f,f) + if(\gamma) + \int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x) \right] \nu(dx) \right) \right\}, \tag{1}$$

for $f \in B^*$, where γ is in B, $A(f,f)$ is a nonnegative quadratic functional in f, $\nu(A)$ is a finite measure in $A \in \mathcal{B}_\varepsilon$ for all $\varepsilon > 0$, such that for any continuous linear functional f

$$\int_{\|x\|\leq 1} f^2(x)\,\nu(dx) < \infty. \tag{2}$$

Remark 2 *a) For a fix A in $\mathcal{B}_\varepsilon$ with $\varepsilon > 0$, $N_t(A)$ is a Poisson process. For a fix $t \geq 0$, $A \longmapsto N_t(A)$ is a Poisson random measure in $A \in \mathcal{B}_\varepsilon$, for all $\varepsilon > 0$, with intensity measure given by the product of Lebesgue measure on $[0, \infty)$ with ν, i.e., $EN_t(A) = t\nu(A)$. Moreover, ν has a unique σ-finite extension to the σ-algebra $\mathcal{B}(B\backslash\{0\})$ containing the ring $\mathcal{B}_\varepsilon$ for all $\varepsilon > 0$ (see Albeverio and Rudiger (2002)).*

b) Let $\triangle_\varepsilon = \{\|x\| > \varepsilon\}$. The process $X_t^\varepsilon = X_t - X_t^{\triangle_\varepsilon}$ has moments of any order. The centering process $X_t^\varepsilon - EX_t^\varepsilon$ converges as $\varepsilon \downarrow 0$ to a continuous process with stationary e independent increments X_t^G. The process X_t^G is Gaussian, that is, $f(X_t^G)$ is a one dimensional Gaussian process for any continuous linear functional.

c) (Af, f) is a quadratic functional, i.e., A is a symmetric nonnegative operator from B^* into B and (Af, f) denotes a dual pair, that is, $(Af, f) = f(Af)$.

d) The triplet of parameters (A, ν, γ) in (1) is called the generating triplet of the Lévy process $\{X_t\}$.

e) This triplet is unique, recalling that for an infinite dimensional Banach space B, a σ-finite measure ν on B with $\nu(\{0\}) = 0$ is a Lévy measure if the function

$$f \longmapsto \exp\{\int \left[e^{if(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x) \right] \nu(dx)\} \qquad f \in B^*, \tag{3}$$

is characteristic functional of a probability measure on B (see Araujo and Giné (1980) and Linde (1986)).

From Theorem IV.8 of Gihman and Skorohod (1975) we obtain the next result for bounded variation Banach space valued Lévy processes.

Proposition 3 *Let $\{Z_t : t \geq 0\}$ be a B-valued Lévy process. Then, $\{Z_t\}$ has bounded variation on each interval $[0, t]$, with probability 1, if and only if, it has characteristic functional given by*

$$Ee^{if(Z_t)} = exp\left\{ t\left(\int_B \left(e^{if(x)} - 1 \right) \nu(dx) + if(\gamma) \right) \right\} \qquad f \in B^*,$$

where $\gamma \in B$ and the Lévy measure ν satisfies

$$\int_{0<\|x\|\leq 1} \|x\| \, \nu\,(dx) < \infty. \tag{4}$$

Moreover, $s(t) = var_{0\leq s\leq t}\, Z(s)$ is a Lévy process in $\mathbb{R}$ and

$$Ee^{irs(t)} = exp\left\{ t\left(\int_B \left(e^{ir\|x\|} - 1 \right) \nu(dx) + ir\,\|\gamma\|) \right) \right\} \qquad r \in \mathbb{R}.$$

3 Distributional Properties

In this section we prove some useful distributional properties of Lévy processes. We begin by proving that general linear transformations of Lévy processes are still Lévy processes being possible to describe their generating triplets. The case of linear transformations of Lévy processes on $\mathbb{R}^d$ is in Sato (1999, Th. 11.10).

Proposition 4 *Let $\{X_t : t \geq 0\}$ be a B-valued Lévy process with (A, ν, γ). Let B_1 be a Banach space such that the map $V \longmapsto f_V$ is an isomorphism of B_1 onto B^*. Let $T : B \to B$ and let $T^{'} : B_1 \to B_1$ be continuous linear transformations with the property*

$$f_V(TS) = f_{T'V}(S), \tag{5}$$

for every $V \in B_1$ and $S \in B$. Then $\{T(X_t) : t \geq 0\}$ is a B-valued Lévy process with generating triplet (A_T, ν_T, γ_T) given by

$$\begin{aligned} A_T &= TAT^{'}, \\ \nu_T &= (\nu T^{-1})\mid_{B\setminus\{0\}}, \\ \gamma_T &= T\gamma + \textstyle\int Tx[1_{\{\|Tx\|\leq 1\}}(Tx) - 1_{\{\|x\|\leq 1\}}(x)]\nu(dx). \end{aligned} \tag{6}$$

Here $\nu T^{-1}(C) = \nu(\{x : Tx \in C\})$ and $(\nu T^{-1})\mid_{B\setminus\{0\}}$ denotes the restriction of the measure νT^{-1} to $B\setminus\{0\}$ and the last integral is a Bochner integral.

Proof. From the continuity of T and the fact that $\{X_t\}$ is a Lévy process it is clear that $\{T(X_t)\}$ is also a Lévy process. We now obtain its generating triplet. From (5)

$$\begin{aligned} Ee^{if_V(TX_1)} &= Ee^{if_{T'V}(X_1)} = \exp\left\{ -\frac{1}{2}A(f_{T'V}\,,\, f_{T'V}) + if_{T'V}(\gamma) \right. \\ &\quad + \int \left(e^{if_{T'V}(x)} - 1 - if_{T'V}(x)1_{\{\|x\|\leq 1\}}\,(x) \right) \nu(dx), \end{aligned}$$

where $A(f_{T'V}, f_{T'V}) = f_{T'V}(AT'V)$ (see Remark 2(c)). Using (5) the above expression becomes

$$\exp\left\{-\frac{1}{2}f_V(TAT'V) + if_V(T\gamma) \right. \tag{7}$$
$$+i\int f_V(Tx)\left[1_{\{\|Tx\|\le 1\}}(Tx) - 1_{\{\|x\|\le 1\}}(x)\right]\nu(dx)$$
$$\left. +\int\left[e^{if_V(y)} - 1 - if_V(y)1_{\{\|y\|\le 1\}}(y)\right]\nu T^{-1}(dy)\right\}.$$

Next, for any $f_V \in B^*$,

$$\int \left(f_V^2(y)\wedge 1\right)\nu T^{-1}(dy) = \int\left(f_{T'V}^2(x)\wedge 1\right)\nu(dx) < \infty$$

and therefore the last integral in (7) is well defined. Since the infinitely divisible random variable TX_1 has the characteristic functional (7) in which the last integral is a characteristic functional, then by Remark 2(e) we get (6). ■

As in the finite dimensional case (Sato (1999, Th. 25.3), for certain class of functions g defined on a separable Banach space, g-moments of Lévy processes are intimately connected with g-moments of their Lévy measures. Kruglov ((1970), (1972)) proved that these are equivalent for Lévy processes on $\mathbb{R}$ and even on Hilbert spaces. For Poisson measures on Banach spaces the equivalence is due to de Acosta (1980). In Proposition 5 below we prove this equivalence for a general Lévy process on a separable Banach space. Let $g : B \to \mathbb{R}^+$ be a locally bounded function. We say that g is a *submultiplicative* function if there exists a constant $a > 0$ such that $g(x + y) \le ag(x)g(y)$ for every $x \in B$ and $y \in B$.

Proposition 5 *Let $\{X_t : t \ge 0\}$ be Lévy process on a separable Banach space B with triplet (A, ν, γ). Then*

$$Eg(X_t) < \infty \quad \text{for every } t, \qquad \text{if and only if,} \qquad \int_{\|x\|>1} g(x)\nu(dx) < \infty.$$

Proof. Let $b > 1$ such that $sup_{\|x\|\le 1}|g(x)| \le b$ and let us choose n such that $n - 1 < \|x\| \le n$, then

$$\begin{aligned} g(x) &= g(\frac{x}{n} + ... + \frac{x}{n}) \le a^{n-1}g(\frac{x}{n})^n \\ &\le a^{n-1}b^n \le b(ab)^{n-1} = be^{c\|x\|}, \end{aligned} \tag{8}$$

where $c = \log(ab) > 0$ choosing $a > 1$.

Let $\{X_t^0\}$ and $\{X_t^1\}$ be independent Lévy processes with generating triplets (A, ν_0, γ) and $(0, \nu_1, 0)$ respectively, where $\nu_0 = [\nu]_{\|x\|\le 1}$ and $\nu_1 = [\nu]_{\|x\|>1}$ denote the restrictions of ν to the set $\{\|x\| \le 1\}$ and $\{\|x\| > 1\}$, respectively.

Denote by μ, μ_0 and μ_1 the distributions of X_1, X_1^0 and X_1^1 respectively. Assume that $Eg(X_t) < \infty$ for some $t > 0$. Then

$$Eg(X_t) = \int_B g(x)\mu^t(dx) = \int_B \int_B g(x+y)\mu_0^t(dx))\mu_1^t(dy) < \infty,$$

which implies, for some $x \in B$, that $\int_B g(x+y)\mu_1^t(dy) < \infty$ and hence

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_B g(x+y)\nu_1(dy) < \infty.$$

In view of (8), $g(y) \le ag(-x)g(x+y) \le abe^{c\|x\|}g(x+y)$ and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \int_B g(y)\nu_1(dy) < \infty. \tag{9}$$

Then, for $n = 0$, $\int_B g(y)\nu_1(dy) < \infty$.

Reciprocally, assume that $\int_B g(y)\nu_1(dy) < \infty$. By submultiplicativity, we have for every n,

$$\int_B g(y)\nu_1(dy) \le a^{n-1} \left\{ \int_B g(y)\nu_1(dy) \right\}^n < \infty.$$

This implies (9) and therefore $Eg(X_t^1) < \infty$ for every $t > 0$. It remains to prove that $Eg(X_t^0) < \infty$ for every $t > 0$ since $Eg(X_t) \le aE\left[g(X_t^0)\right] E\left[g(X_t^1)\right]$. Let $\left\{G_t^0\right\}, \left\{P_t^0\right\}$ and $\left\{D_t^0\right\}$ be homogeneous and independent random processes on B, where $\left\{G_t^0\right\}$ is Gaussian , $\left\{P_t^0\right\}$ is generalized Poisson, i.e., P_1^0 has the characteristic functional (3) with Lévy measure ν_0 and $\left\{D_t^0\right\}$ is a non random function, such that $X_t^0 \overset{d}{=} G_t^0 + P_t^0 + D_t^0$. We only need to prove that $E\left[g(G_1^0)\right] E\left[g(P_1^0)\right] E\left[g(D_1^0)\right] < \infty$. That $E\left[g(P_1^0)\right]$ is finite follows from Corollary 3.3 in de Acosta (1980) since ν_0 is concentrated in the unit ball of B. Now, since G_1^0 is a centered Gaussian random variable then $G_1^0 \overset{d}{=} \frac{1}{\sqrt{2}}G_1 + \frac{1}{\sqrt{2}}G_2$ where G_1 and G_2 are independent copies of G_1^0 (see Araujo and Giné (1980, Prop.III.6.4)). Then

$$\begin{aligned}
Eg(G_1^0) &= Eg(\frac{1}{\sqrt{2}}G_1 + \frac{1}{\sqrt{2}}G_2) \\
&\le aE\left[g\left(\frac{1}{\sqrt{2}}G_1^0\right)\right]^2 = a\left[Eg\left(\frac{1}{(\sqrt{2})^2}G_1 + \frac{1}{(\sqrt{2})^2}G_2\right)\right]^2 \\
&\le aa^2\left[\left\{Eg\left(\frac{1}{(\sqrt{2})^2}G_1^0\right)\right\}^2\right]^2 \le a^7\left[Eg\left(\frac{1}{(\sqrt{2})^3}G_1^0\right)\right]^8 \\
\ldots &\le a^{2^n-1}\left[Eg\left(\frac{1}{(\sqrt{2})^n}G_1^0\right)\right]^{2^n} \qquad \text{for } n \ge 1.
\end{aligned}$$

We can choose $\beta > 0$ such that $Ee^{\beta\|G_1^0\|} < \infty$ which is a consequence of Fernique's theorem, see Araujo and Giné (1980, Th. III.6.5), see also Skorohod (1970) who gave an independent proof of this fact. Choose n sufficiently large such that $c\frac{1}{(\sqrt{2})^n} < \beta$, then $Eg(G_1^0) < \infty$, which concludes the proof. ■

Remark 6 *The proof of the "if " part of the above Proposition is on the same lines of the finite dimensional case of Sato (1999, Th. 25.3).*

For exponential and regular moments of Lévy processes we obtain the following conditions for their existence, which are independent of t.

Corollary 7 *a) For every $f \in B^*$, $Ee^{f(X_t)} < \infty$ for all $t > 0$ if and only if $\int_{\|x\|>1} e^{f(x)}\nu(dx) < \infty$.*

b) $Ee^{\alpha\|X_t\|} < \infty$ for all $t > 0$, for some $\alpha > 0$, if and only if $\int_{\|x\|>1} e^{\alpha\|x\|}\nu(dx) < \infty$.

c) Let $\alpha > 0$. Then $E\|X_t\|^\alpha < \infty$ for all $t > 0$ iff $\int_{\|x\|>1} \|x\|^\alpha \nu(dx) < \infty$.

Proof. a) and b) follows sine $g(x) = e^{f(x)}$ and $g(x) = e^{\alpha\|x\|}$ are submultiplicative functions.
b) Assume that $E\|X_t\|^\alpha < \infty$ for every $t > 0$, which is equivalent to the finiteness of $E[1 \vee \|X_t\|^\alpha]$ for every $t > 0$, which in turn is equivalent to $\int_{\|x\|>1} (1 \vee \|x\|^\alpha)\nu(dx) < \infty$ since $g(x) = 1 \vee \|x\|^\alpha$ is a submultiplicative function. Finally, this is equivalent to $\int_{\|x\|>1} \|x\|^\alpha \nu(dx) < \infty$. ■

The following lemma on estimations of moments and tails of Lévy processes generalizes and extends, to the Banach space case, the one presented in Sato (1999, Lemma 30.3) for the Euclidean case.

Lemma 8 *Let $\{X_t : t \geq 0\}$ be a Lévy process on a separable Banach space B with triplet (A, ν, γ). Then*
a) There exist positive constants $C(\varepsilon)$, C_1 and C_2 such that, for every $t > 0$,

$$\begin{aligned} P(\|X_t\| > \varepsilon) &\leq C(\varepsilon)t \qquad \textit{for } \varepsilon > 0, & (10)\\ E\left[\|X_t\|^2; \|X_t\|\right] &\leq C_1 t, & (11)\\ E[\|X_t\|; \|X_t\| \leq 1] &\leq C_2 t^{1/2}. & (12)\end{aligned}$$

b) For any continuous linear functional f on B there exist positive constants $C_1(f)$ and $C_2(f)$ such that, for every $t > 0$,

$$\begin{aligned} E\left[|f(X_t)|^2; \|X_t\| \leq 1\right] &\leq C_1(f)t, & (13)\\ |E[f(X_t); \|X_t\| \leq 1]| &\leq C_2(f)t. & (14)\end{aligned}$$

c)There exists a positive constant C_3 such that, for every $t > 0$,

$$\|E[X_t; \|X_t\| \leq 1]\| \leq C_3 t. \qquad (15)$$

Proof. a) Let $\{X_t^0\}$ and $\{X_t^1\}$ be independent Lévy processes with generating triplets (A, ν_0, γ) and $(0, \nu_1, 0)$ respectively, where $\nu_0 = [\nu]_{\|x\| \leq 1}$ and $\nu_1 = [\nu]_{\|x\|>1}$ denote the restrictions of ν to the set $\{\|x\| \leq 1\}$ and $\{\|x\| > 1\}$, respectively. Then $\{X_t\} \stackrel{d}{=} \{X_t^0\} + \{X_t^1\}$. By Corollary 7 $E\left\|X_t^0\right\|^2 < \infty$ and hence

$$
\begin{aligned}
P(\|X_t\| &> \varepsilon) = P(\left\|X_t^0 + X_t^1\right\| > \varepsilon) \leq P(X_t^1 \neq 0) + P(X_t^1 = 0,\ \left\|X_t^0\right\| > \varepsilon) \\
&\leq 1 - e^{-t\nu_1(\{\|x\|>1\})} + \frac{E\left\|X_t^0\right\|^2}{\varepsilon^2} \leq \\
&\quad \left\{\nu_1\left(\{\|x\| > 1\}\right) + \frac{E\left\|X_1^0\right\|^2}{\varepsilon^2}\right\} t,
\end{aligned}
$$

where we have applied Chevyshev's inequality and considered $t \leq 1$. This proves (10). Next, note that $\{\|X_t\| \leq 1\} = \left\{\left\|X_t^0\right\| \leq 1\right\} \cup \left\{[\|X_t\| \leq 1] \cap \left[X_t^1 \neq 0\right]\right\}$ then

$$
\begin{aligned}
E\left[\|X_t\|^2; \|X_t\| \leq 1\right] &\leq E\left[\left\|X_t^0\right\|^2 1_{\{\|X_t^0\| \leq 1\}}\right] + E\left[\|X_t\|^2 1_{\{\|X_t\| \leq 1\} \cap \{X_t^1 \neq 0\}}\right] \\
&\leq E\left[\left\|X_t^0\right\|^2 1_{\{\|X_t^0\| \leq 1\}}\right] + P\left(X_t^1 \neq 0\right) \\
&\leq E\left\|X_t^0\right\|^2 + P\left(X_t^1 \neq 0\right)
\end{aligned}
$$

since $\|X_t\|^2 1_{\{\|X_t\| \leq 1\} \cap \{X_t^1 \neq 0\}} \leq 1_{\{X_t^1 \neq 0\}}$. This reduces to the above case and hence (11) follows. To obtain (12) apply Hölder's inequality to get $E\left[\|X_t\|; \|X_t\| \leq 1\right] \leq \left(E\left[\|X_t\|^2; \|X_t\| \leq 1\right]\right)^{1/2} \leq C_1^{1/2} t^{1/2}$ where C_1 is given by (11).

b) Let f be a continuous linear functional on B.
Since $E\left[|f(X_t)|^2; \|X_t\| \leq 1\right] \leq |f|^2 E\left[\|X_t\|^2; \|X_t\| \leq 1\right]$, (13) follows from (11). To prove (14) we use the three elementary estimates $\left|e^{it} - 1 - it\right| \leq t^2/2$, $t \in \mathbb{R}$, $|e^z - 1| \leq e^{|z|} - 1$, $z \in \mathbb{C}$, and $|e^z - 1| \leq \frac{7}{4}|z|$, $|z| \leq 1$. Note that

$$
\begin{aligned}
&\left|E\left[f(X_t); \|X_t\| \leq 1\right]\right| \\
= &\left|\begin{array}{c} E\left[e^{if(X_t)} - 1\right] - E\left[e^{if(X_t)} - 1; \|X_t\| > 1\right] \\ -E\left[e^{if(X_t)} - 1 - if(X_t); \|X_t\| \leq 1\right] \end{array}\right| \\
\leq &\left|E\left[e^{if(X_t)} - 1\right]\right| + 2P\left(\|X_t\| > 1\right) + \frac{1}{2}E\left[f^2(X_t); \|X_t\| \leq 1\right].
\end{aligned}
$$

The last two terms of the sum are bounded by constant multiples of t by (10) and (13) respectively. Next, $E\left|e^{if(X_t)} - 1\right| = \left|e^{t\Psi_X(f)} - 1\right|$ where $\Psi_X(f) = if(\gamma) - \frac{1}{2}A(f, f) + \int\left[e^{if(x)} - 1 - if(x)1_{\{\|x\| \leq 1\}}(x)\right]\nu(dx)$ is given by (1). Let $z = \Psi_X(f)$. When $t \geq 1$, $E\left|e^{if(X_t)} - 1\right| \leq 2t$. If $0 \leq t \leq \frac{1}{|z|}$ then $|e^z - 1| \leq \frac{7}{4}|z|t$. Finally, if $\frac{1}{|z|} \leq t \leq 1$, $|e^z - 1| \leq e^{|z|} - 1 \leq e^{|z|}|z|t$.

c) By (12) $E[\|X_t\|;\|X_t\| \leq 1]$ is finite for each t and hence $X_t 1_{\{\|X_t\| \leq 1\}}$ is Bochner integrable (Araujo and Giné [1980] Prop. III.2.2). Let $V = X_t 1_{\{\|X_t\| \leq 1\}}$. Since V is also Pettis integrable, we have

$$f(EV) = Ef(V) \qquad \text{for every } f \in B^*.$$

On the other hand, by the Hahn-Banach Theorem there exists a continuous linear functional $\tilde{f} \in B^*$ satisfying $\|EV\| = \tilde{f}(EV)$ and then $\|EV\| = E\tilde{f}(V)$. This means $\|E[X_t;\|X_t\| \leq 1]\| = E\left[\tilde{f}(X_t);\|X_t\| \leq 1\right]$. The assertion follows from (14) applied to $\tilde{f}$. ■

4 Subordination

Recall that a one dimensional subordinator is an increasing Lévy process $\{Z_t; t \geq 0\}$ in $\mathbb{R}$ with generating triplet (A, ν, γ) satisfying

$$A = 0, \quad \nu((-\infty, 0)) \quad \text{and} \quad \gamma - \int_{(0,1]} x\nu(dx) \geq 0,$$

where $\gamma - \int_{(0,1]} x\nu(dx) = \gamma_0$, the drift of $\{Z_t\}$, and its law is specified by its Laplace transform given by

$$Ee^{-uZ_t} = \exp\{-t\Psi(u)\} \qquad u \geq 0, \tag{16}$$

where for any $w \in \mathbb{C}$ whose $\mathrm{Re}(w) \leq 0$

$$\Psi(w) = \int_{(0,\infty)} (e^{ws} - 1)\,\nu(ds) + \gamma^0 w, \tag{17}$$

and

$$\int_{(0,\infty)} (1 \wedge s)\,\nu(ds) < \infty. \tag{18}$$

In this section we study subordination of a Banach space valued Lévy process by a real valued subordinator. That is, we study the process that is obtained as the random time change of a Banach space valued Lévy process $\{X_t\}$ by a one dimensional subordinator $\{Z_t\}$. The corresponding process $\{Y_t\}$ -called subordinated process- is also a Lévy process.

Theorem 9 *Let $\{Z_t : t \geq 0\}$ be a one dimensional subordinator with Laplace transform (16) and let $\{X_t : t \geq 0\}$ be a Lévy process on a separable Banach space B. Assume that $\{X_t\}$ and $\{Z_t\}$ are independent and define $Y_t = X_{Z_t}$ for all $t \geq 0$. Then the process $\{Y_t : t \geq 0\}$ is a Lévy process.*

The proof of this result follows in a similar manner as the proof of the first part of Theorem 30.1 for the Euclidean case in Sato (1999).

Theorem 10 *Let $\{Z_t : t \geq 0\}$ be a one dimensional subordinator with Lévy measure ν_Z and drift $\gamma_Z^0 \geq 0$. Let $\{X_t : t \geq 0\}$ be a Lévy process on a separable Banach space B with generating triplet (A, ν, γ) and let $\mu = \mathcal{L}(X_1)$. Assume that $\{X_t\}$ and $\{Z_t\}$ are independent and let $\{Y_t : t \geq 0\}$ be the subordinated process obtained from subordination of $\{X_t\}$ by $\{Z_t\}$, that is $Y_t = X_{Z_t}$. Then we have the following:*

a) The characteristic functional of the Lévy process $\{Y_t\}$ is given by $Ee^{if(Y_t)} = e^{t\Psi(\log \hat{\mu}(f))}$ for all real-valued $f \in B^*$.

b) Let $\mu_s = \mathcal{L}(X_s)$. The generating triplet (A_Y, ν_Y, γ_Y) of $\{Y_t\}$ is given by

$$\begin{aligned} A_Y &= \gamma_Z^0 A, && (19) \\ \nu_Y(C) &= \int_{(0,\infty)} \mu_s(C)\nu_Z(ds) + \gamma_Z^0 \nu(C) \qquad C \in \mathcal{B}(B\backslash\{0\}), && (20) \\ \gamma_Y &= \int_{(0,\infty)} \int_{\|x\|\leq 1} x\mu_s(dx)\nu_Z(ds) + \gamma_Z^0 \gamma, && (21) \end{aligned}$$

where the last integral is a Bochner integral.

c) If $\gamma_Z^0 = 0$ and $\int_{(0,1]} s^{1/2}\nu_Z(ds) < \infty$, then $A_Y = 0$, $\int_{\|x\|\leq 1} \|x\| \nu_Y(dx) < \infty$ and $\gamma_Y - \int_{\|x\|\leq 1} x\nu_Y(dx) = 0$. Moreover, the Banach space Lévy process $\{Y_t\}$ has bounded variation on each interval $[0, t]$.

Proof. a) Let f be a real-valued continuous linear functional on B. We observe that $Ee^{if(X(s))} = e^{s \log \hat{\mu}(f)}$ for each $s \geq 0$. Since $\mathrm{Re}(\log \hat{\mu}(f)) \leq 0$ we use (17) to get

$$Ee^{if(Y_t)} = E\left[\left(Ee^{if(X(s))}\right)_{s=Z_t}\right] = E\left[\left(e^{s \log \hat{\mu}(f)}\right)_{s=Z_t}\right] = e^{t\Psi(\log \hat{\mu}(f))}.$$

b) We use Lemma 8 in order to find the generating triplet of $\{Y_t\}$. From (17) we have

$$\begin{aligned} Ee^{if(Y_t)} &= e^{t\Psi(\log \hat{\mu}(f))} = \\ &\quad \exp\left\{t\left(\int_{(0,\infty)} (e^{s \log \hat{\mu}(f)} - 1)\nu_Z(ds) + \gamma_Z^0 \log \hat{\mu}(f)\right)\right\} \\ &= \exp\left\{t\left(\int_{(0,\infty)} (\hat{\mu}_s(f) - 1)\nu_Z(ds) + \gamma_Z^0 \log \hat{\mu}(f)\right)\right\}. \qquad (22) \end{aligned}$$

Let $g(f, x) = e^{f(x)} - 1 - if(x)1_{\{\|x\|\leq 1\}}(x)$. Note that

$$\gamma_Z^0 \log \hat{\mu}(f) = -\frac{1}{2}\gamma_Z^0(Af, f) + i\gamma_Z^0 f(\gamma) + \int_B g(f, x)\, \gamma_Z^0 \nu(dx). \qquad (23)$$

On the other hand, by Lemma 8 $\int_{\|x\|\leq 1} x\mu_s(dx)$ is a well defined Bochner integral. From (15) and (18) we have that $\int_{(0,\infty)} \left\|\int_{\|x\|\leq 1} x\mu_s(dx)\right\| \nu_Z(ds)$ is finite and hence $\int_{(0,\infty)} \int_{\|x\|\leq 1} x\mu_s(dx)\nu_Z(ds)$ is a well defined Bochner integral.

Therefore

$$\int_{(0,\infty)} \int_{\|x\|\le 1} f(x)\mu_s(dx)\nu_Z(ds) = f\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_Z(ds)\right).$$

Then we have

$$\int_{(0,\infty)} (\hat{\mu}_s(f)-1)\nu_Z(ds) = \int_{(0,\infty)} \int_B (e^{if(x)}-1)\mu_s(dx)\nu_Z(ds)$$

$$= \int_{(0,\infty)} \int_B g(f,x)\mu_s(dx)\nu_Z(ds) + if\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_Z(ds)\right). \quad (24)$$

From (22), (23) and (24) we have

$$\begin{aligned} Ee^{if(Y_t)} &= \exp\left\{t\left[-\frac{1}{2}\gamma_Z^0 A(f,f) + if\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_Z(ds) + \gamma_Z^0\gamma\right)\right]\right. \\ &\quad + \int_B g(f,x)\left(\int_{(0,\infty)} \mu_s(\cdot)\nu_Z(ds) + \gamma_Z^0\nu(\cdot)\right)(dx). \end{aligned} \quad (25)$$

Define $\nu_1(C) = \int_{(0,\infty)} \mu_s(C)\nu_Z(ds)$ for $C \in \mathcal{B}(B\backslash\{0\})$. It remains to prove that ν_1 is a Lévy measure. That is, we show that the function $f \longmapsto \exp \int_B g(f,x)\nu_1(dx)$ for all $f \in B^*$ is a characteristic functional of some Banach-valued random variable (see (3)). Then

$$\begin{aligned} &\exp \int_B g(f,x)\,\nu_1(dx) \\ =\ &\exp\left\{\int_{(0,\infty)} (Ee^{if(X(s))}-1)\nu_Z(ds) - i\int_{(0,\infty)} \int_{\|x\|\le 1} f(x)\mu_s(dx)\nu_Z(ds)\right\} \\ =\ &\exp\left\{\int_{(0,\infty)} (e^{s\log\hat{\mu}(f)}-1)\nu_Z(ds)\right\}\cdot \\ &\exp\left\{-if\left(\int_{(0,\infty)} \int_{\|x\|\le 1} x\mu_s(dx)\nu_Z(ds)\right)\right\}. \end{aligned}$$

The last two factors are characteristic functionals, since the first one corresponds to the characteristic functional of a subordinated process at time 1 obtained by subordination of X_s by $Z_t - t\gamma_Z^0$ and the second one corresponds to the characteristic functional of a degenerated distribution. In view of (25) and Remark 2(e) we get (19), (20) and (21).

c) Assume that $\gamma_Z^0 = 0$ and $\int_{(0,1]} s^{1/2}\nu_Z(ds) < \infty$. Then $A_Y = 0$ by (19) and $\nu_Y(dx) = \int_{(0,1]} \mu_s(dx)\nu_Z(ds)$ by (20). We have

$$\int_{\|x\|\le 1} \|x\|\,\nu_Y(dx) = \int_{(0,\infty)} \int_{\|x\|\le 1} \|x\|\,\mu_s(dx)\nu_Z(ds) < \infty$$

by (18) and Lemma 8 relation (12). Now, it follows from (21) and (25) that $\gamma_Y - \int_{\|x\|\le 1} x\nu_Y(dx) = 0$ and moreover

$$Ee^{if(Y_t)} = \exp\left\{t\int_B (e^{if(x)} - 1)\nu_Y(dx)\right\}.$$

By Proposition 3 we conclude that $\{Y_t\}$ has bounded variation on each interval $[0,t]$. ■

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