EXISTENCE OF SELF-INTERSECTION LOCAL TIME OF THE MULTITYPE DAWSON-WATANABE SUPERPROCESS

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1 Introduction and preliminary definitions

Dynkin introduced in [1] a definition of self-intersection local time (SILT) for a general class of continuous measure-valued processes, and gave a criterion for existence of SILT in terms of superprocess multiple stochastic integrals. In this note we prove that if the spatial dimension is small, then the hypothesis in Dynkin's existence criterion are satisfied by the multitype Dawson-Watanabe superprocess studied in [3], and obtain in this way a sufficient condition for existence of SILT of the Dawson-Watanabe superprocess in the multitype setting.

Let us recall that the multitype Dawson-Watanabe superprocess arises as the diffusion approximation of multitype branching particle systems in $\mathbb{R}^d$ consisting of particles of $k \geq 1$ types undergoing spatial diffusion and critical multitype branching, where the diffusions, the particle lifetimes and the branching laws depend on the types. In the model we consider here the motions of particles are symmetric α_i-stable processes, $0 < \alpha_i \leq 2$, and the lifetimes of particles are exponentially distributed with parameters V_i, $i = 1, \ldots, k$. For simplicity, in this paper we restrict ourselves to the case of multitype populations which are a.s. finite, and whose branching law has a stochastic mean matrix $(m_{ij})_{1 \leq i,j \leq k}$, i.e., $m_{ij} \geq 0$ and $\sum_{j=1}^{k} m_{ij} = 1$, $i = 1, \ldots, k$.

Our multitype particle systems are properly represented by counting measures $\mu \in M_f(\mathbb{S})$, where $M_f(\mathbb{S})$ denotes the space of finite positive measures on $\mathbb{S} \equiv \mathbb{R}^d \times \{1, \ldots, k\}$. Here the first component of $(x, i) \in \mathbb{S}$ stands for the position and the second component for the type of an individual $\delta_{(x,i)}$. We denote by $\xi_t := (W_t, \eta_t)$, $t \geq 0$, the Markov process on $\mathbb{S}$ whose type component η_t follows a Markov chain with Q-matrix $(V_i(m_{ij} - \delta_{ij}))_{1 \leq i,j \leq k}$, and whose position component W_t follows an α_i-stable motion as long as $\eta_t = i$. The generator of ξ is given by

$$\mathcal{A}\phi(x,i) = \Delta_{\alpha_i}\phi(x,i) + V_i \sum_{j=1}^{k} (m_{ij} - \delta_{ij})\, \phi(x,j), \quad (x,i) \in \mathbb{S}, \quad \phi(\cdot, i) \in \mathrm{Dom}(\Delta_{\alpha_i}),$$

where Δ_{α_i} denotes the generator of the α_i-stable process, $i = 1, \ldots, k$. Writing $\mu(\phi) \equiv \int \phi(x)\, \mu(dx)$

and denoting the semigroup associated with $\mathcal{A}$ by $\mathcal{U} \equiv (\mathcal{U}_t)_{t\geq 0}$, we recall (cf. [5]) that

$$(\mathcal{U}_t \phi)(x,i) = \mathbb{E}\mathbb{Y}_t^{(x,i)}(\phi)\,,\ t \geq 0,$$

where $(\mathbb{Y}_t^{(x,i)})_{0\leq t<\infty}$ stands for the particle system started from $\delta_{(x,i)}$, that is, one type i-individual at position x. In this sense, the process ξ is the *expectation process* of the branching system; its transition probability equals $J_t^{(x,i)}(\cdot) := \mathbb{E}\mathbb{Y}_t^{(x,i)}(\cdot)$.

Roughly described, the renormalization of the branching particle system suitable for the diffusion limit consists in increasing the density of the population of all types, assigning a small mass and a short lifetime to every individual, and decreasing the mutation probabilities in such a way that the mutation rates remain constant; the spatial motion laws remain the same. If the initial populations (i.e., at time $t = 0$) of the renormalized branching particle systems converge weakly to $\mu \in M_f(\mathbb{S})$, and the branching mechanisms have finite second-order moments, this renormalization yields in the limit a continuous Markov process $X := \{X_t,\ t \geq 0\}$ with values in $M_f(\mathbb{S})$, having the Laplace functional

$$\mathbb{E}\left[e^{-X_t(\phi)}\right] = e^{-\mu(u_t)} \tag{1.1}$$

for any bounded, measurable $\phi : \mathbb{S} \to [0,\infty)$, where $u_t(x,i)$ is the unique solution of the nonlinear equation

$$\frac{\partial u_t(x,i)}{\partial t} = \mathcal{A}u_t(x,i) - C_i u_t(x,i)^2, \quad u_0(x,i) = \phi(x,i),$$

and C_i, $i = 1,\ldots,k$ are certain positive constants determined by the lifetime parameters and the variances of the branching law (see [3]); for simplicity, we shall assume $C_i = 1$, $i = 1,\ldots,k$.

Let us now recall the construction of the superprocess stochastic integrals introduced by Dynkin in [1].

Let $[0,u]$ be a fixed time interval. From (1.1) one can deduce that for every $\mu \in M_f(\mathbb{S})$, any bounded, measurable functions $\phi_i : \mathbb{S} \to \mathbb{R}$, $i = 1,\ldots,n$, and positive numbers $t_1,\ldots,t_n \in [0,u]$ the random variable

$$X_{t_1}(\phi_1)\cdots X_{t_n}(\phi_n) \tag{1.2}$$

is in $L^2(P)$, where P is the distribution of X, and that $\mathbb{E}\left[X_{t_1}(\phi_1)\cdots X_{t_n}(\phi_n)\right] = \int \left(\prod_{i=1}^n \varphi_i(y_i,s_i)\right)$ $\gamma_n(dy_1,ds_1;\ldots;dy_n,ds_n)$, where

$$\varphi(x,s) := U_{t-s}\phi(x) \tag{1.3}$$

and γ_n is the n-th moment measure of X. For our purposes we need to know the precise definition of γ_n only for the values $n = 1,2$, and in these cases,

$$\begin{aligned}
\gamma_1(A_1 \times B_1) &= 1_{B_1}(0)\mu(A_1),\\
\gamma_2(A_1 \times B_1; A_2 \times B_2) &= 1_{B_1}(0)\mu(A_1)1_{B_2}(0)\mu(A_2)
\end{aligned}$$

$$+2\int 1_{B_1}(s)1_{A_1}(y)1_{B_2}(s)1_{A_2}(y)\, J_s^x(dy)\,\mu(dx)\,dz,$$

$$A_i \in \mathcal{B}(\mathbb{S}),\ \ B_i \in \mathcal{B}([0,u]),\ \ i=1,2.$$

Let L_n^2 denote the smallest closed subspace of $L^2(P)$ containing all products of the form (1.2). We define $\mathcal{K}$ as the set of functions of the form (1.3) with ϕ bounded and measurable, and write χ_n^0 for the space of functions $\varphi : (\mathbb{S}\times[0,u])^n \to \mathbb{R}$ satisfying $(|\varphi|,|\varphi|)_n < \infty$, where

$$(\varphi,\psi)_n := \int \varphi(y_1,s_1;\ldots;y_n,s_n)\psi(y_1,s_1;\ldots;y_n,s_n)\,\gamma_{2n}(dy_1,ds_1;\ldots;dy_n,ds_n).$$

In particular, for $n=1$,

$$(\varphi,\psi)_1 = \int \varphi(z_1,t_1)\psi(z_2,t_2)\,d\gamma_2 = \int \varphi(z,0)\,\mu(dz)\int \psi(z,0)\,\mu(dz) + \int \varphi(y,s)\psi(y,s)\,\hat{\Lambda}(dy,ds),$$

where $\hat{\Lambda}$ is the measure on $\mathbb{S}\times[0,u]$ given by $\hat{\Lambda}(C) = 2\int 1_C(y,s)J_s^x(dy)\,\mu(dx)\,ds$. Therefore $\varphi\in\chi_1^0$ if and only if $\hat{\Lambda}(\varphi^2)<\infty$ and $\mu(|\varphi(0,\cdot)|)<\infty$.

The stochastic integrals are defined as follows ([1], theorems 1.2 and 1.3):

a) For $\varphi = U_{t-s}\phi \in \mathcal{K}$,

$$I_1(\varphi) \equiv \int \varphi(x,s)\,dZ_{x,s} := X_t(\phi),\quad t>0. \tag{1.4}$$

Let χ_1 denote the space of equivalence classes of χ_1^0 modulo $(\cdot,\cdot)_1$. There exist a unique isometry (which we again denote by) I_1 from χ_1 to L_1^2 obeying (1.4). Moreover, for every $\varphi\in\chi_1^0$, the process $M_t^\varphi := I_1(\varphi) = \int \varphi(x,s)1_{s\le t}\,dZ_{x,s}$ is a martingale, and every L_1^2-valued martingale can be represented in this form.

b) There exists a unique mapping $I_n : \chi_n^0 \to L_n^2$ satisfying $I_n(\varphi_1\times\cdots\times\varphi_n) = I_1(\varphi_1)\cdots I_1(\varphi_n)$ and $\mathbb{E}\left[I_n(\varphi)I_n(\psi)\right] = (\varphi,\psi)_n$ for any $\varphi_1,\ldots,\varphi_n\in\chi_1^0$ and $\varphi,\psi\in\chi_n^0$. The set $I_n(\chi_n^0)$ is dense in L_n^2.

2 Existence of SILT of X

The self-intersection local time of X can be defined heuristically by $SILT(B) = \int_B\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\delta_0(z-z')\,X_s(dz)\,X_t(dz')\,ds\,dt$, where $B\in\mathcal{B}([0,u]^2)$. This expression is formally equivalent to

$$SILT(B) = I_2(K_B^2) \equiv \int K_B^2(x_1,s_1;x_2,s_2)\,dZ_{x_1,s_1}\,dZ_{x_2,s_2}, \tag{2.1}$$

(see [1] or [2]) where $K_B^2(x_1,s_1;x_2,s_2) = \int_B dt_1\,dt_2\int_{\mathbb{S}}\prod_{i=1}^2 J_{t_i-s_i}^{x_i}(dz)$. The stochastic integral in the right-hand side of (2.1) makes sense provided $K_B^2\in\chi_2^0$, in which case we say that the self-intersection local time of X (of order two) exists. We will prove the following theorem.

Theorem 1 *Let $X = \{X_t,\ t \geq 0\}$ be the multitype Dawson-Watanabe superprocess whose Laplace functional is given by (1.1), and $\Lambda = [0, u]$. Suppose $B \subset \mathcal{B}(\Lambda \times \Lambda)$ is such that $\overline{B} \cap \{(x, y) \in \Lambda \times \Lambda : x = y\} = \emptyset$. If the measures $\mu(\{i\} \times \cdot)$, $i = 1, 2$, have bounded densities with respect to d-dimensional Lebesgue measure, and $d < 4 \min\{\alpha_1, \ldots, \alpha_k\}$, then the self-intersection local time of X exists.*

We define the measure m on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ by $m(B \times C) = \lambda(B)\nu(C)$, $B \in \mathcal{B}(\mathbb{R}^d)$, $C \subset \{1, \ldots, k\}$, where ν is the counting measure on $\{1, \ldots, k\}$ and λ is the Lebesgue measure on $\mathbb{R}^d$. The transition kernels $J_t^{(x,h)}(\cdot)$ of the expectation process are absolutely continuous with respect to m, and the corresponding densities $J_t((x, h), (z, i))$ are given by [6]

$$J_t((x, h), (z, i)) = \int_{D_{[0,\infty)}(\{1,\ldots,k\})} (q^{\alpha_1}_{L_1(t,\eta)} * \cdots * q^{\alpha_k}_{L_k(t,\eta)})(x, z) 1_{\{\eta_0 = h, \eta_t = i\}}(\eta) P^\eta_h(d\eta),$$

where P^η_i is the distribution of the Markov chain η starting in i, $L_i(t, \eta)$ is the amount of time that η spends in type i during the time interval $[0, t]$, and $\{q^{\alpha_i}_r,\ r > 0\}$ are the α_i-stable densities, $i = 1, \ldots, k$. We define $J_t \equiv 0$ if $t < 0$.

In order to prove Theorem 1 we will show that

$$\sup_{s,y} \int_{\mathbb{S}\times\mathbb{S}} G(s, y; z)G(s, y; \zeta)H(z, \zeta)\, dz\, d\zeta < \infty, \tag{2.2}$$

where, for $0 \leq s \leq t$,

$$\begin{aligned} G(s, (y, j); (z, i)) &= \int_\Lambda J_{t-s}((y, j), (z, i))\, dt, \\ H((z, i), (\zeta, l)) &= \int_{\Lambda\times\mathbb{S}} G(s, (y, j); (z, i))G(s, (y, j); (\zeta, l))\, m(d(y, j))\, ds. \end{aligned}$$

According to [1] (Theorem 1.5), under our assumptions condition (2.2) implies $K^2_B \in \chi^0_2$.

2.1 Proof of Condition (2.2)

In the remaining part of the paper we assume without loss of generality that $k = 2$ and $\alpha_1 \leq \alpha_2$. We denote by $p^{(i)}_t(x, y)$, $t > 0$, $i = 1, 2$, $x, y \in \mathbb{R}^d$, the transition densities of the position process $\{W_t,\ t \geq 0\}$, which, by the law of total probability, satisfy

$$\begin{aligned} p^{(i)}_t(x, y)\, dy &:= P[W_t \in dy | (W_0, \eta_0) = (x, i)] \\ &= \sum_{j=1}^{2} P[W_t \in dy, \eta_t = j | (W_0, \eta_0) = (x, i)] \\ &= \sum_{j=1}^{2} J_t((x, i), (y, j))\, dy. \end{aligned} \tag{2.3}$$

By conditioning on the time of first jump of η, and using commutativity of convolutions, it easily follows that

$$p_t^{(i)}(x,y) = e^{-V_i(1-m_{ii})t} q_t^{\alpha_i}(x,y) + \int_0^t \int_{\mathbb{R}^d} q_r^{\alpha_1}(x,z) q_{t-r}^{\alpha_2}(z,y)\, dz\, \theta_t^i(dr),\ i=1,2, \tag{2.4}$$

where $\theta_t^i(dr)$ is the conditional distribution of $L_i(t,\eta)$ given that a change of type occurred in the interval $(0,t]$. In order to estimate the product of stable densities inside the integral above, we need the following result from [6].

Lema 2 *Let $0 \le \alpha_1 \le \alpha_2 \le 2$. There exists a constant $K \ge 1$ such that $q_t^{\alpha_2}(x) \le K q_{t^{\alpha_1/\alpha_2}}^{\alpha_1}(x)$ for all $t>0$ and $x \in \mathbb{R}^d$. Moreover, if $t \ge 1$ then $q_t^{\alpha_2}(x) \le K t^{d(1/\alpha_1 - 1/\alpha_2)} q_t^{\alpha_1}(x) \le K t^{d/\alpha_1} q_t^{\alpha_1}(x)$, $x \in \mathbb{R}^d$.*

Using (2.4) and Lemma 2 we deduce that for $0<t<1$,

$$\begin{aligned}
p_t^{(i)}(x,y) &\le e^{-V_i(1-m_{ii})t} K q_{t^{\alpha_1/\alpha_i}}^{\alpha_1}(x,y) + \int_0^t \int_{\mathbb{R}^d} q_r^{\alpha_1}(x,z) K q_{(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1}(z,y)\, dz\, \theta_t^i(dr) \\
&\le K \left[q_{t^{\alpha_1/\alpha_i}}^{\alpha_1}(x,y) + \int_0^t \left(q_r^{\alpha_1} * q_{(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1} \right)(x,y) \theta_t^i(dr) \right] \\
&\le 2K \left[q_{t^{\alpha_1/\alpha_i}}^{\alpha_1}(x,y) + \int_0^t q_{r+(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1}(x,y) \theta_t^i(dr) \right].
\end{aligned} \tag{2.5}$$

Similarly, for $t>1$,

$$\begin{aligned}
&p_t^{(i)}(x,y) \\
&\le e^{-V_i(1-m_{ii})t} K t^{d/\alpha_1} q_t^{\alpha_1}(x,y) + \int_0^t \int_{\mathbb{R}^d} \left(1_{(0,t-1]} + 1_{(t-1,t]} \right)(r) q_r^{\alpha_1}(x,z) q_{t-r}^{\alpha_2}(z,y)\, dz\, \theta_t^i(dr) \\
&\le K t^{d/\alpha_1} q_t^{\alpha_1}(x,y) + \int_0^{t-1} \int_{\mathbb{R}^d} q_r^{\alpha_1}(x,z) K (t-r)^{d/\alpha_1} q_{t-r}^{\alpha_1}(z,y)\, dz\, \theta_t^i(dr) \\
&\quad + \int_{t-1}^t \int_{\mathbb{R}^d} q_r^{\alpha_1}(x,z) K q_{(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1}(z,y)\, dz\, \theta_t^i(dr) \\
&\le K \left[t^{d/\alpha_1} q_t^{\alpha_1}(x,y) + q_t^{\alpha_1}(x,y) \int_0^{t-1} (t-r)^{d/\alpha_1} \theta_t^i(dr) + \int_{t-1}^t q_{r+(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1}(x,y) \theta_t^i(dr) \right] \\
&\le K \left[t^{d/\alpha_1} q_t^{\alpha_1}(x,y) + t^{d/\alpha_1} q_t^{\alpha_1}(x,y) \int_0^t \theta_t^i(dr) + \int_{t-1}^t q_{r+(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1}(x,y) \theta_t^i(dr) \right] \\
&\le 2K \left[t^{d/\alpha_1} q_t^{\alpha_1}(x,y) + \int_{t-1}^t q_{r+(t-r)^{\alpha_1/\alpha_2}}^{\alpha_1}(x,y) \theta_t^i(dr) \right].
\end{aligned} \tag{2.6}$$

Let us proceed to the proof of (2.2). We start by noting that, due to (2.3), $G(s,(y,j);(z,i)) = \int_s^u J_{t-s}((y,j),(z,i))\,dt \le \int_0^{u-s} p_t^{(j)}(y,z)\,dt$ for $j=1,2$. Therefore, putting $a_+ := \max\{a,0\}$, $a \in \mathbb{R}$,

$$
\begin{aligned}
&H((z,i),(\zeta,l)) \\
&= \sum_{j=1}^{2} \int_0^u \int_{\mathbb{R}^d} G(s,(j,y);(i,z))G(s,(j,y);(l,\zeta))\,dy\,ds \\
&\le \sum_{j=1}^{2} \int_0^u \int_{\mathbb{R}^d} \left(\int_0^{u-s} p_{t_1}^{(j)}(y,z)\,dt_1\right)\left(\int_0^{u-s} p_{t_2}^{(j)}(y,\zeta)\,dt_2\right)dy\,ds \\
&= \sum_{j=1}^{2} \left(\int_0^{(u-1)_+} + \int_{(u-1)_+}^{u}\right) \int_{\mathbb{R}^d} \left(\int_0^{u-s} p_{t_1}^{(j)}(y,z)\,dt_1\right)\left(\int_0^{u-s} p_{t_2}^{(j)}(y,\zeta)\,dt_2\right)dy\,ds \\
&= \sum_{j=1}^{2} \int_0^{(u-1)_+} \int_{\mathbb{R}^d} \left(\int_0^1 p_{t_1}^{(j)}(y,z)\,dt_1 + \int_1^{u-s} p_{t_1}^{(j)}(y,z)\,dt_1\right) \\
&\quad \cdot \left(\int_0^1 p_{t_2}^{(j)}(y,\zeta)\,dt_2 + \int_1^{u-s} p_{t_2}^{(j)}(y,\zeta)\,dt_2\right)dy\,ds \\
&\quad + \sum_{j=1}^{2} \int_{(u-1)_+}^{u} \int_{\mathbb{R}^d} \left(\int_0^{u-s} p_{t_1}^{(j)}(y,z)\,dt_1\right)\left(\int_0^{u-s} p_{t_2}^{(j)}(y,\zeta)\,dt_2\right)dy\,ds.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
H((z,i),(\zeta,l)) \le \sum_{j=1}^{2} \Bigg(&\int_0^{(u-1)_+} \int_0^1 \int_0^1 \int_{\mathbb{R}^d} p_{t_1}^{(j)}(w,z)p_{t_2}^{(j)}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1 \\
&+ \int_0^{(u-1)_+} \int_1^{u-s_1} \int_0^1 \int_{\mathbb{R}^d} p_{t_1}^{(j)}(w,z)p_{t_2}^{(j)}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1 \\
&+ \int_0^{(u-1)_+} \int_0^1 \int_1^{u-s_1} \int_{\mathbb{R}^d} p_{t_1}^{(j)}(w,z)p_{t_2}^{(j)}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1 \\
&+ \int_0^{(u-1)_+} \int_1^{u-s_1} \int_1^{u-s_1} \int_{\mathbb{R}^d} p_{t_1}^{(j)}(w,z)p_{t_2}^{(j)}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1 \\
&+ \int_{(u-1)_+}^{u} \int_0^{u-s_1} \int_0^{u-s_1} \int_{\mathbb{R}^d} p_{t_1}^{(j)}(w,z)p_{t_2}^{(j)}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1 \Bigg).
\end{aligned}
$$

For an integrable function $h : [a,b] \to \mathbb{R}$ and $x \in [a,b]$, we denote $(\int_a^x + \int_x^b)h(r)\,dr := \int_a^x h(r)\,dr + \int_x^b h(r)\,dr$. From the above estimates we obtain

$$
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(s,(j,y);(i,z))G(s,(j,y);(l,\zeta))H((i,z),(l,\zeta))\,dz\,d\zeta
$$

$$
\begin{aligned}
\leq\;& \sum_{j=1}^{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_0^{(u-s)\wedge 1}+\int_{(u-s)\wedge 1}^{u-s}\right)p^{(j)}_{t_3}(y,z)\,dt_3\left(\int_0^{(u-s)\wedge 1}+\int_{(u-s)\wedge 1}^{u-s}\right)p^{(j)}_{t_4}(y,\zeta)\,dt_4\\
&\cdot\Bigg(\int_0^{(u-1)_+}\int_0^1\int_0^1\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_0^{(u-1)_+}\int_1^{u-s_1}\int_0^1\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_0^{(u-1)_+}\int_0^1\int_1^{u-s_1}\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_0^{(u-1)_+}\int_1^{u-s_1}\int_1^{u-s_1}\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_{(u-1)_+}^{u}\int_0^{u-s_1}\int_0^{u-s_1}\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\Bigg)\,dz\,d\zeta\\
=\;& \sum_{j=1}^{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Bigg(\int_0^{(u-s)\wedge 1}p^{(j)}_{t_3}(y,z)dt_3\int_0^{(u-s)\wedge 1}p^{(j)}_{t_4}(y,\zeta)\,dt_4\\
&+\int_0^{(u-s)\wedge 1}p^{(j)}_{t_3}(y,z)\,dt_3\int_{(u-s)\wedge 1}^{u-s}p^{(j)}_{t_4}(y,\zeta)\,dt_4+\int_{(u-s)\wedge 1}^{u-s}p^{(j)}_{t_3}(y,z)\,dt_3\int_0^{(u-s)\wedge 1}p^{(j)}_{t_4}(y,\zeta)\,dt_4\\
&+\int_{(u-s)\wedge 1}^{u-s}p^{(j)}_{t_3}(y,z)\,dt_3\int_{(u-s)\wedge 1}^{u-s}p^{(j)}_{t_4}(y,\zeta)\,dt_4\Bigg)\\
&\cdot\Bigg(\int_0^{(u-1)_+}\int_0^1\int_0^1\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_0^{(u-1)_+}\int_1^{u-s_1}\int_0^1\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_0^{(u-1)_+}\int_0^1\int_1^{u-s_1}\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_0^{(u-1)_+}\int_1^{u-s_1}\int_1^{u-s_1}\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\\
&+\int_{(u-1)_+}^{u}\int_0^{u-s_1}\int_0^{u-s_1}\int_{\mathbb{R}^d}p^{(j)}_{t_1}(w,z)p^{(j)}_{t_2}(w,\zeta)\,dw\,dt_1\,dt_2\,ds_1\Bigg)\,dz\,d\zeta\\
=\;& \sum_{j=1}^{2}\Bigg(\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\\
&+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_1^{u-s_1}\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}
\end{aligned}
$$

$$+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^{1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_1^{u-s_1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_{(u-1)_+}^{u}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^{u-s_1}\int_0^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_0^{(u-s)\wedge 1}\int_0^{1}\int_0^{1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_0^{(u-s)\wedge 1}\int_1^{u-s_1}\int_0^{1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_0^{(u-s)\wedge 1}\int_0^{1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_0^{(u-s)\wedge 1}\int_1^{u-s_1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_{(u-1)_+}^{u}\int_{(u-s)\wedge 1}^{u-s}\int_0^{(u-s)\wedge 1}\int_0^{u-s_1}\int_0^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_{(u-s)\wedge 1}^{u-s}\int_0^{1}\int_0^{1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_{(u-s)\wedge 1}^{u-s}\int_1^{u-s_1}\int_0^{1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_{(u-s)\wedge 1}^{u-s}\int_0^{1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_{(u-s)\wedge 1}^{u-s}\int_1^{u-s_1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_{(u-1)_+}^{u}\int_0^{(u-s)\wedge 1}\int_{(u-s)\wedge 1}^{u-s}\int_0^{u-s_1}\int_0^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_{(u-s)\wedge 1}^{u-s}\int_0^{1}\int_0^{1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_{(u-s)\wedge 1}^{u-s}\int_0^{1}\int_0^{1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_{(u-s)\wedge 1}^{u-s}\int_0^{1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_0^{(u-1)_+}\int_{(u-s)\wedge 1}^{u-s}\int_{(u-s)\wedge 1}^{u-s}\int_1^{u-s_1}\int_1^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}$$

$$+\int_{(u-1)_+}^{u}\int_{(u-s)\wedge 1}^{u-s}\int_{(u-s)\wedge 1}^{u-s}\int_0^{u-s_1}\int_0^{u-s_1}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Bigg)$$

$$\cdot p_{t_1}^{(j)}(w,z)p_{t_2}^{(j)}(w,\zeta)p_{t_3}^{(j)}(y,z)p_{t_4}^{(j)}(y,\zeta)\,dw\,dz\,d\zeta\,dt_1\,dt_2\,dt_3\,dt_4\,ds_1$$

$$=: \sum_{j=1}^{2}\sum_{i=1}^{20} I_i(j).$$

We will show how to estimate from above the integral $I_1(j)$; the remaining integrals can be estimated in a similar way. Applying repeatedly (2.5) renders

$$\begin{aligned}
I_1(j) \quad \leq \quad & 16K^4\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^1\int_0^1\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dw\,dz\,d\zeta\,dt_1\,dt_2\,dt_3\,dt_4\,ds_1\\
&\cdot\left(q^{(1)}_{t_1^{\alpha_1/\alpha_j}}(w,z)+\int_0^{t_1} q^{(1)}_{r_1+(t_1-r_1)^{\alpha_1/\alpha_2}}(w,z)\,\theta^j_{t_1}(dr_1)\right)\\
&\cdot\left(q^{(1)}_{t_2^{\alpha_1/\alpha_j}}(w,\zeta)+\int_0^{t_2} q^{(1)}_{r_2+(t_2-r_2)^{\alpha_1/\alpha_2}}(w,\zeta)\,\theta^j_{t_2}(dr_2)\right)\\
&\cdot\left(q^{(1)}_{t_3^{\alpha_1/\alpha_j}}(y,z)+\int_0^{t_3} q^{(1)}_{r_3+(t_3-r_3)^{\alpha_1/\alpha_2}}(y,z)\,\theta^j_{t_3}(dr_3)\right)\\
&\cdot\left(q^{(1)}_{t_4^{\alpha_1/\alpha_j}}(y,\zeta)+\int_0^{t_4} q^{(1)}_{r_4+(t_4-r_4)^{\alpha_1/\alpha_2}}(y,\zeta)\,\theta^j_{t_4}(dr_4)\right).
\end{aligned}$$

Using Chapman-Kolmogorov's equation (three times), the scaling property of stable densities, and that $t \leq t^\alpha$ for $0 < t \leq 1$ and $\alpha > 0$, we obtain

$$\begin{aligned}
&I_1(j)\\
&\leq \quad 16K^4 q_1^{(1)}(0)\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^1\int_0^1\Big((t_1^{\alpha_1/\alpha_j}+t_2^{\alpha_1/\alpha_j}+t_3^{\alpha_1/\alpha_j}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\\
&\quad+\int_0^{t_4}(t_1^{\alpha_1/\alpha_j}+t_2^{\alpha_1/\alpha_j}+t_3^{\alpha_1/\alpha_j}+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_4}(dr_4)\\
&\quad+\int_0^{t_3}(t_1^{\alpha_1/\alpha_j}+t_2^{\alpha_1/\alpha_j}+r_3+(t_3-r_3)^{\alpha_1/\alpha_2}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_3}(dr_3)\\
&\quad+\int_0^{t_4}\int_0^{t_3}(t_1^{\alpha_1/\alpha_j}+t_2^{\alpha_1/\alpha_j}+r_3+(t_3-r_3)^{\alpha_1/\alpha_2}+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_3}(dr_3)\theta^j_{t_4}(dr_4)\\
&\quad+\int_0^{t_2}(t_1^{\alpha_1/\alpha_j}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+t_3^{\alpha_1/\alpha_j}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\\
&\quad+\int_0^{t_4}\int_0^{t_2}(t_1^{\alpha_1/\alpha_j}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+t_3^{\alpha_1/\alpha_j}+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\theta^j_{t_4}(dr_4)\\
&\quad+\int_0^{t_3}\int_0^{t_2}(t_1^{\alpha_1/\alpha_j}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+r_3+(t_3-r_3)^{\alpha_1/\alpha_2}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^{t_4}\int_0^{t_3}\int_0^{t_2}(t_1^{\alpha_1/\alpha_j}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+r_3+(t_3-r_3)^{\alpha_1/\alpha_2}\\
&+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+t_2^{\alpha_1/\alpha_j}+t_3^{\alpha_1/\alpha_j}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\\
&+\int_0^{t_4}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+t_2^{\alpha_1/\alpha_j}+t_3^{\alpha_1/\alpha_j}+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_3}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+t_2^{\alpha_1/\alpha_j}+r_3+(t_3-r_3)^{\alpha_1/\alpha_2}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_3}(dr_3)\\
&+\int_0^{t_4}\int_0^{t_3}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+t_2^{\alpha_1/\alpha_j}+r_3+(t_3-r_3)^{\alpha_1/\alpha_2}\\
&+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_3}(dr_3)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_2}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+t_3^{\alpha_1/\alpha_j}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\\
&+\int_0^{t_4}\int_0^{t_2}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+t_3^{\alpha_1/\alpha_j}\\
&+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_3}\int_0^{t_2}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+r_3\\
&+(t_3-r_3)^{\alpha_1/\alpha_2}+t_4^{\alpha_1/\alpha_j})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\\
&+\int_0^{t_4}\int_0^{t_3}\int_0^{t_2}\int_0^{t_1}(r_1+(t_1-r_1)^{\alpha_1/\alpha_2}+r_2+(t_2-r_2)^{\alpha_1/\alpha_2}+r_3\\
&+(t_3-r_3)^{\alpha_1/\alpha_2}+r_4+(t_4-r_4)^{\alpha_1/\alpha_2})^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\cdot\theta^j_{t_4}(dr_4))dt_1dt_2dt_3dt_4ds_1\\
\leq\;& 16K^4q_1^{(1)}(0)\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^1\int_0^1\Big((t_1+t_2+t_3+t_4)^{-d/\alpha_1}\\
&+\int_0^{t_4}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_4}(dr_4)+\int_0^{t_3}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_3}(dr_3)\\
&+\int_0^{t_4}\int_0^{t_3}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_3}(dr_3)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_2}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_2}(dr_2)+\int_0^{t_4}\int_0^{t_2}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_3}\int_0^{t_2}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\\
&+\int_0^{t_4}\int_0^{t_3}\int_0^{t_2}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\theta^j_{t_4}(dr_4)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)+\int_0^{t_4}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_3}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_3}(dr_3)\\
&+\int_0^{t_4}\int_0^{t_3}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_3}(dr_3)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_2}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\\
&+\int_0^{t_4}\int_0^{t_2}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\theta^j_{t_4}(dr_4)\\
&+\int_0^{t_3}\int_0^{t_2}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\\
&+\int_0^{t_4}\int_0^{t_3}\int_0^{t_2}\int_0^{t_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\theta^j_{t_1}(dr_1)\theta^j_{t_2}(dr_2)\theta^j_{t_3}(dr_3)\cdot\theta^j_{t_4}(dr_4))\\
\le\ & 256K^4q_1^{(1)}(0)\int_0^{(u-1)_+}\int_0^{(u-s)\wedge 1}\int_0^{(u-s)\wedge 1}\int_0^1\int_0^1(t_1+t_2+t_3+t_4)^{-d/\alpha_1}dt_1dt_2dt_3dt_4ds_1.
\end{aligned}
$$

Estimating in the same way the remaining integrals $I_i(j)$ renders

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(s,(j,y);(i,z))G(s,(j,y);(l,\zeta))H((i,z),(l,\zeta))\,dz\,d\zeta\\
&\quad\le\ \text{Const.}\int_0^u\int_0^{u-s}\int_0^{u-s}\int_0^{u-s_1}\int_0^{u-s_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}dt_1\,dt_2\,dt_3\,dt_4\,ds_1.
\end{aligned}
$$

Finally, let us show that if $d<4\alpha_1$ then the right-hand side of the last inequality is finite. Indeed,

$$
\begin{aligned}
&\int_0^u\int_0^{u-s}\int_0^{u-s}\int_0^{u-s_1}\int_0^{u-s_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\,dt_1\,dt_2\,dt_3\,dt_4\,ds_1\\
=\ &\text{Const.}\Big(4(3u-s)^{5-d/\alpha_1}+2(u-s)^{5-d/\alpha_1}+u(2u-2s)^{4-d/\alpha_1}+2u^{5-d/\alpha_1}\\
&-(2u)^{5-d/\alpha_1}-2u(u-s)^{4-d/\alpha_1}-(4u-2s)^{5-d/\alpha_1}-4(2u-s)^{5-d/\alpha_1}-(2u-2s)^{5-d/\alpha_1}\Big).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_{0\le s\le u}\int_0^u\int_0^{u-s}\int_0^{u-s}\int_0^{u-s_1}\int_0^{u-s_1}(t_1+t_2+t_3+t_4)^{-d/\alpha_1}\,dt_1\,dt_2\,dt_3\,dt_4\,ds_1\\
&\le\ \text{Const.}\Big(4(3u)^{5-d/\alpha_1}+2u^{5-d/\alpha_1}+u(2u)^{4-d/\alpha_1}+2u^{5-d/\alpha_1}\Big)\\
&<\ \infty
\end{aligned}
$$

if $4-d/\alpha_1>0$. □

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