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WISHART AND PSEUDO-WISHART DISTRIBUTIONS AND SOME APPLICATIONS TO SHAPE THEORY, II

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Abstract

In this paper, we determine the density of a singular elliptically contoured matrix. From this, the study of Wishart and Pseudo- Wishart distributions, whether central or non-central, whether singular or non-singular, is extended to the case of elliptical models. Some applications of these results are studied in the context of shape theory. Particular attention is paid to singular size-and-shape and size-and-shape cone densities.

Keywords: Elliptic Distribution, Wishart distribution, Pseudo-Wishart distribution, Singular distribution, Shape distributions.

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1. Introduction

Consider the random matrix $Z \in \mathbb{R}^{p\times s}$ where, if Z has a density function with respect to the Lebesgue measure in $\mathbb{R}^{ps}$, the ps-dimensional Euclidean space, this function is given by $f_Z(Z) = h(\operatorname{tr} Z'Z)$. The distribution of Z is called vector-spherical distribution by Fang and Zhang (10, p. 96). Note that if Z is vectorized, that is, if all the columns of Z are located one below the other, and this vector is defined as

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$z = \operatorname{vec} Z \in I\!R^{ps}$ (see Gupta and Varga (14, p. 13)), the density function of z is given by $f_z(z) = h(z'z) = h(\operatorname{vec}' Z \operatorname{vec} Z)$, because $\operatorname{vec}' Z \operatorname{vec} Z = \operatorname{tr} Z'Z$ (see Theorem 2.1.1 in Gupta and Varga (14, p. 20)), which we call spherical distribution.

Let $A \in I\!R^{N\times p}$, $B' \in I\!R^{s\times m}$, with a rank of A, $r(A) = k \leq \min(N, p)$, $r(B) = r \leq \min(m, s)$, and $\mu \in I\!R^{N\times m}$ are matrices of constants such that $\Sigma = AA'$ and $\Theta = BB'$. Then, from Theorem 2.1.1 in Gupta and Varga (14, p. 20), $Y = AZB' + \mu$ has an elliptically-contoured matrix distribution with characteristic function $\psi_Y(T) = \operatorname{etr}(i\mu'T)\phi(\operatorname{tr} T\Sigma T'\Theta)$ which denotes

$$Y \sim \mathcal{E}_{N\times m}(\mu, \Sigma, \Theta, \phi).$$

Observe that $r(\Sigma) = r(AA') = r(A) = r$ and $r(\Theta) = r(BB') = r(B) = k$, that is, $\Sigma \geq 0$ and $\Theta \geq 0$. Therefore, the distribution rank of Y is kr, that is, Y has a singular distribution. Thus, $Y \in I\!R^{N\times m}$ ($\operatorname{vec} Y \in I\!R^{Nm}$) does not have density with respect to the Lebesgue measure in $I\!R^{Nm}$. Nevertheless, the density function does exist in the subspace $\mathcal{M}$ of $I\!R^{Nm}$ (see Cramér (2, p. 297), Gupta and Varga (14, p. 26) and Fang and Zhang (10, pp. 59 and 70)). Formally, as described in Section 2, the density of Y exists with respect to the Hausdorff measure, which coincides with that of Lebesgue when this is defined on the above-mentioned subspace $\mathcal{M}$ (see Theorem 19.1 in Billingsley (1, p. 209) and Remarks 2.2 and 2.3 in Díaz-García, Gutiérrez and Mardia (6)).

In this context, if Y has a distribution as defined above, this is denoted

$$Y \sim \mathcal{E}^{k,r}_{N\times m}(\mu, \Sigma, \Theta, h)$$

where the product of the supra-indices is the rank of the distribution and let c is a real constant, such that $f_Y(Y) = c\ h(\cdot)$, is the density function with respect to the Hausdorff measure. This measure, of course, can be replaced by the Lebesgue measure if we specify the subspace $\mathcal{M}$ or if $k = N$ and $r = m$.

Let us now consider the nonsingular part of the singular value decomposition (SVD) of a matrix $E = U_1DW_1'$, where $E \in I\!R^{N\times m}$, D is diagonal and $D_{11} > \cdots > D_{qq}$ and $r(E) = q = \min(k, r)$; $U_1 \in V_{q,N}$, the Stiefel manifold, and $W_1 \in V_{q,m}$ (see Rao (19, p. 42)). By defining $S = Y'\Theta^- Y$, it is possible, by means of SVD, to find a single expression for the density function of S by considering all the possible order relations between N, m, k and r, taking into account that $k \leq N$ and $r \leq m$. By analogy with the case in which Y has a matrix variate normal distribution, matrix S is termed the singular or nonsingular generalised Wishart or Pseudo-Wishart matrix, depending on the order relation between N and m or between k and r (Wishart or Pseudo-Wishart) and on that between N and k or between m and r (singular or nonsingular). In practice, twelve cases of the S matrix may be given. The density of Y exists with respect to the Hausdorff measure and, as a consequence of this, the density of S also exists, in general, with respect to the Hausdorff measure.

The density of S, when Y has a matrix variant normal distribution with $\Sigma > 0$ and $N > m$, has been studied by various authors, both in the central and noncentral cases; see, for example, Srivastava and Khatri (17), James (15) and Muirhead (16), among many others. In the same context, considering $N < m$ and only the central case, the density of S with $\Theta = I_N$ has been studied by Uhlig (21) and Díaz-García and Gutiérrez (5). A detailed study of the density of S (in which the normality of Y was also assumed), considering all the possible cases arising in the definition of S, was described in Díaz-García, Gutiérrez and Mardia (6). Various authors have studied the distribution of S, when Y has an elliptically-contoured distribution, $N > m$, $\Theta = I_N$ and $\mu = 0$. These studies are summarized in Fang and Zhang (10) and Gupta and Varga (14). In the noncentral, nonsingular, generalised Wishart case, with $\Theta = I_N$, the density of S has been studied by Fan (9) and Teng, Fang and Deng (20).

Unfortunately, even in classical multivariate analysis based on matrix normal distribution, the study of singular distributions has received little attention, with the exception of some studies by Khatri presented in Rao (19) and a few recent publications (see Uhlig (21), Díaz-García and Gutiérrez (5) and Díaz-García, Gutiérrez and Mardia (6). These cases of singularity were resolved by other means, such as type F matrix distributions associated with MANOVA (see theorems 10.4.1 and 10.4.4 in Muirhead (16, pp. 449-451). Some other types of singularity have been resolved by different methods applied to discriminate between variables (columns) or individuals (rows) in the observation matrix X in order to obtain a nonsingular matrix. It is important to note that such a singularity may be due to a numerical dependence between the rows and/or columns of the observation matrix, and/or there may really exist a linear dependence between the variables or individuals. The latter case could occur when we seek to represent a matrix in a lower dimension, for example in Factor Analysis, Principal Component Analysis, Analysis of Correspondences, Multidimensional Scaling and Analysis of Canonic Correlation. In such cases, in general, the confirmatory analysis is based on the distribution of the eigenvalues of a certain matrix associated with the corresponding technique which is a function of the data matrix, X. All these cases of singularity are also present within generalised multivariant analysis (in which an elliptical matrix distribution is assumed) and, indirectly, constitute one of the topics examined in the present article.

The fundamental aim of this article is to extend the results presented in Díaz-García, Gutiérrez and Mardia (6) to the case of elliptical models. We determine the density of a singular elliptically-contoured matrix, taking into account the linear dependence between the rows or columns of Y (see Theorem 2.1). From this density, we define the generalised S Wishart (Pseudo-Wishart) matrix. Making certain assumptions about the function h, described above, we find the density of S for the central/noncentral and singular/nonsingular cases (see Theorem 3.1 and Corollary 3.1). As these densities are still a function of h, various results, described in the literature, have been obtained as particular cases, see Corollaries 2.1, 3.1 and 3.4. Also, we study other particular cases of a singular elliptically-contoured distribution, that is, the matrix variate symmetric Kotz type distribution and the matrix variate symmetric Pearson Type VII distribution. Furthermore, and as particular cases of the above, we study matrix variate normal, t-, and Cauchy distributions (see Corollary 3.2).

In the context of shape theory, the generalised Wishart (Pseudo-Wishart) distribution extends the results obtained for the normal case to that of singular elliptical models (see Goodall and Mardia (12) and (13)). Specifically, this density could perform the role of a size-and-shape density (see Díaz-García, Gutiérrez and Mardia (6)). Alternatively, as a size-and-shape distribution, it is possible to use the joint density of the singular values of matrix Y (or the joint density of the eigenvalues of matrix S), called size-and-shape cone density in Goodall (11) and Goodall and Mardia (13). Unfortunately, even in the normal case, these distributions are difficult to compute, even in the case of low dimensions, a problem that has led to some approximate distributions of such cases being studied, as suggested by Goodall and Mardia (13), see also Section 9.5 in Muirhead (16, pp. 390-405). Nevertheless, it is necessary to identify the explicit form of such densities in order to propose approximate densities. The size-and-shape cone density, in the central case, has been studied by Díaz-García et al (7) for elliptical models.

Finally, Section 4 proposes the density of S as the size-and-shape density. Theorem 4.1 determines the singular size-and-shape cone density in terms of matrix Y and, making use of certain properties of symmetric polynomials (Davis (4)), obtains two particular cases, one of which was presented in Teng, Fang and Deng (20) (see Corollary 4.1 and Remark 4.1).

2. Singular Elliptically Contoured Distribution

The following result is an extension of Theorem 2.1 given in Díaz-García, Gutiérrez and Mardia (6) to the case of an elliptical model. In this theorem, an alternative proof is given, using the characteristic function (see Srivastava and Khatri (17, pp. 42-43)).

THEOREM 2.1 *Let* $Y \sim \mathcal{E}_{N\times m}(\mu, \Sigma, \Theta, h)$ *with* $\Sigma : m \times m$*,* $r(\Sigma) = r \leq m$ *and/or* $\Theta : N \times N$*,* $r(\Theta) = k \leq N$*. This is termed the Singular Elliptically-Contoured Distribution and is expressed as:*

$$Y \sim \mathcal{E}^{k,r}_{N\times m}(\mu, \Sigma, \Theta, h)$$

omitting the supra-indices when $r = m$ *and* $k = N$*. Moreover, the density function is given by:*

$$\frac{1}{\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)\left(\displaystyle\prod_{j=1}^{k}\delta_j^{r/2}\right)} h\left(\operatorname{tr}\Sigma^{-}(Y-\mu)'\Theta^{-}(Y-\mu)\right) \tag{1}$$

$$\left.\begin{array}{rcl} H_1'(Y-\mu)P_2' & = & 0 \\ H_2'(Y-\mu)P_1' & = & 0 \\ H_2'(Y-\mu)P_2' & = & 0 \end{array}\right\} \quad a.\ s. \tag{2}$$

where A^{-} *is a symmetric generalised inverse of* A*,* λ_i *and* δ_j *are the nonzero eigenvalues of* Σ *and* Θ*, respectively, and* $H_1 \in V_{k,N}$*,* $H_2 \in V_{N-k,N}$*,* $P_1' \in V_{r,m}$ *and* $P_2' \in V_{m-r,m}$*.*

Proof. Let $P = (P_1'|P_2') \in \mathcal{O}(m)$, the group of orthogonal matrices and $H = (H_1|H_2) \in \mathcal{O}(N)$, with $P_1' \in V_{r,m}$, $P_2' \in V_{m-r,m}$, $H_1 \in V_{N,k}$ y $H_2 \in V_{N-k,k}$ such that

(Srivastava and Khatri (17, p. 42))

$$\Sigma = P \begin{pmatrix} D_\Sigma & 0 \\ 0 & 0 \end{pmatrix} P' \quad \text{and} \quad \Theta = H \begin{pmatrix} D_\Theta & 0 \\ 0 & 0 \end{pmatrix} H'$$

where $D_\Sigma = \operatorname{diag}(\lambda_1, \dots, \lambda_r)$ and $D_\Theta = \operatorname{diag}(\delta_1, \dots, \delta_k)$, λ_i and δ_j are the nonzero eigenvalues of Σ and Θ, respectively. Then, from the characteristic function of Y:

$$\psi_Y(T) = \operatorname{etr}(i\mu'T)\phi(\operatorname{tr} T\Sigma T'\Theta).$$

Note that

$$\begin{aligned} i \operatorname{tr} \mu'T &= i \operatorname{tr} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \mu'(H_1 \vdots H_2) \begin{pmatrix} H_1' \\ H_2' \end{pmatrix} T(P_1' \vdots P_2') \\ &= i \operatorname{tr} \begin{pmatrix} P_1\mu'H_1 & P_1\mu'H_2 \\ P_2\mu'H_1 & P_2\mu'H_2 \end{pmatrix} \begin{pmatrix} H_1'TP_1' & H_1'TP_2' \\ H_2'TP_1' & H_2'TP_2' \end{pmatrix} \\ &= i \operatorname{tr} \begin{pmatrix} \Delta_{11}' & \Delta_{12}' \\ \Delta_{21}' & \Delta_{22}' \end{pmatrix} \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad = \quad i \operatorname{tr} \Delta' W \end{aligned}$$

and that

$$\begin{aligned} \operatorname{tr} T\Sigma T'\Theta &= \operatorname{tr} H'TP \begin{pmatrix} D_\Sigma & 0 \\ 0 & 0 \end{pmatrix} P'T'H \begin{pmatrix} D_\Theta & 0 \\ 0 & 0 \end{pmatrix} \\ &= W \begin{pmatrix} D_\Sigma & 0 \\ 0 & 0 \end{pmatrix} W' \begin{pmatrix} D_\Theta & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Thus the characteristic function of Y can be expressed as

$$\begin{aligned} \psi_Y(T) &= E(\operatorname{etr}(iY'T)) \\ &= E(\operatorname{etr}(i(H'YP)'(H'TP))) \\ &= \operatorname{etr}(i \operatorname{tr} \Delta' W)\phi \left[W \begin{pmatrix} D_\Sigma & 0 \\ 0 & 0 \end{pmatrix} W' \begin{pmatrix} D_\Theta & 0 \\ 0 & 0 \end{pmatrix} \right] \end{aligned}$$

which is valid for all values of W. Consider the following transformation:

$$H'YP = \begin{pmatrix} H_1' \\ H_2' \end{pmatrix} Y \begin{pmatrix} P_1' & P_2' \end{pmatrix} = \begin{pmatrix} H_1'YP_1' & H_1'YP_2' \\ H_2'YP_1' & H_2'YP_2' \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} = R. \tag{3}$$

Then, from Theorem 2.3.3 in Gupta and Vargas (14, p. 32)

$$R_{11} \sim \mathcal{E}_{K\times r}(\Delta_{11}, D_\Sigma, D_\Theta, h)$$

$$\left.\begin{array}{lcl} R_{12} & = & \Delta_{12} \\ R_{21} & = & \Delta_{21} \\ R_{22} & = & \Delta_{22} \end{array}\right\} \text{ a. s.}$$

In other words, the density of R or, equivalently, that of Y, is given by the following (see Theorem 2.2.1, Gupta and Vargas (14, p. 26)):

$$\frac{1}{|D_\Sigma|^{K/2}|D_\Theta|^{r/2}} h\left(\operatorname{tr}\left(D_\Sigma^{-1}(R_{11}-\Delta_{11})' D_\Theta^{-1}(R_{11}-\Delta_{11})\right)\right) \tag{4}$$

$$\left.\begin{array}{lcl} R_{12}-\Delta_{12} & = & 0 \\ R_{21}-\Delta_{21} & = & 0 \\ R_{22}-\Delta_{22} & = & 0 \end{array}\right\} \text{ a. s.} \tag{5}$$

An expression as a function of Y may be obtained, noting that $|D_\Sigma| = \prod_{i=1}^{r} \lambda_i$ and $|D_\Theta| = \prod_{j=1}^{K} \delta_j$. Furthermore, given that $\Sigma = P_1' D_\Sigma P_1$ and $\Theta = H_1 D_\Theta H_1'$, it follows that $\Sigma^- = P_1' D_\Sigma^{-1} P_1 = P_1'(P_1 \Sigma P_1')^{-1} P_1$ and $\Theta^- = H_1 D_\Theta^{-1} H_1' = H_1(H_1' \Theta H_1)^{-1} H_1'$ define symmetric generalised inverses of Σ and Θ, respectively. Note, also, that

$$\begin{aligned} &\operatorname{tr}(D_\Sigma^{-1}(R_{11}-\Delta_{11})' D_\Theta^{-1}(R_{11}-\Delta_{11})) \\ &\qquad = \operatorname{tr}(P_1'(P_1\Sigma P_1')^{-1}P_1(Y-\mu)' H_1(H_1'\Theta H_1)^{-1}H_1'(Y-\mu) \\ &\qquad = \operatorname{tr}(\Sigma^-(Y-\mu)'\Theta^-(Y-\mu)) \end{aligned}$$

and the desired result is obtained. ■

Note that the expression for densities (1)-(2), (4)-(5) and for the symmetric generalised inverses is not unique, as P and H are not unique. Nevertheless, such densities can be used to calculate probabilities and for the inference of the density parameters, without these depending on the particular values of P and H. Note, also, that taking into account the notation used by Billingsley (see Billingsley (1, pp. 208-218)), defining the set $A \in \mathbb{R}^{N\times m}$ by (5) or by (2) and considering its elements (a_{ij}) to be the coordinates of a point on a (subspace) surface of dimension kr in $\mathbb{R}^{Nm}$ (in this particular case, the surface defines a hyperplane, see Cramér (2, pp. 16-17)), the density (4) or (1) exists with respect to the Hausdorff measure, which coincides with the Lebesgue measure, when the latter is defined on the above-mentioned hyperplane (see

Remark 2.3 in Díaz-García, Gutiérrez and Mardia (6) and Theorem 19.1 in Billingsley (1, p. 209)).

A geometric interpretation of density (1) -(2) ((4)-(5)) may be made, noting that, when R is defined by (3), this is equivalent to

$$\begin{aligned}
\operatorname{vec} R &= \operatorname{vec}(H'YP) \\
&= (P' \otimes H') \operatorname{vec} Y \\
&= \begin{pmatrix} P_1 & \otimes & H_1' \\ P_1 & \otimes & H_2' \\ P_2 & \otimes & H_1' \\ P_2 & \otimes & H_2' \end{pmatrix} \operatorname{vec} Y \\
&= \begin{pmatrix} (P_1 & \otimes & H_1') & \operatorname{vec} Y \\ (P_1 & \otimes & H_2') & \operatorname{vec} Y \\ (P_2 & \otimes & H_1') & \operatorname{vec} Y \\ (P_2 & \otimes & H_2') & \operatorname{vec} Y \end{pmatrix} = \begin{pmatrix} \operatorname{vec} R_{11} \\ \operatorname{vec} R_{12} \\ \operatorname{vec} R_{21} \\ \operatorname{vec} R_{22} \end{pmatrix}.
\end{aligned}$$

Note, too, that the covariance matrix of Y (R), $\operatorname{Cov}(\operatorname{vec} Y')$, is proportional to the matrix $\Sigma \otimes \Theta$, whose rank is $r(\Sigma \otimes \Theta) = r(\Sigma)r(\Theta) = kr$. From Cramér (2, p. 297), the rank of the distribution is $kr < Nm$, from which we deduce that Y has a singular distribution and therefore that there are $Nm - kr$ linear relationships between the coordinates of $\operatorname{vec} Y$ $(\operatorname{vec} R)$ that contain the whole distribution mass. Moreover, these $Nm - kr$ linear relationships are such that

$$\begin{pmatrix} P_1 & \otimes & H_2' \\ P_2 & \otimes & H_1' \\ P_2 & \otimes & H_2' \end{pmatrix} (\operatorname{vec} Y - \operatorname{vec} \mu) = 0 \tag{6}$$

are satisfied with a probability equal to one. In summary, the density (2) is interpreted as being the density on the subspace of dimension kr defined by (6). In other words, it is the density with a total mass on the hyperplane of dimension kr, as defined by (6) (see Cramér (2, pp. 297-298)).

Corollary 2.1 *Assume that* $Y \sim \mathcal{E}_{N\times m}^{k,r}(\mu, \Sigma, \Theta, h)$. *If, in particular,* Y *has a* *matrix normal distribution, its density function is given by (see Díaz-García, et al (5))*

$$\frac{1}{(2\pi)^{kr/2}\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)\left(\prod_{i=1}^{k}\delta_j^{r/2}\right)}\operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-}(Y-\mu)'\Theta^{-}(Y-\mu)\right)$$

$$\begin{aligned} H_1'(Y-\mu)P_2' &= 0 \\ H_2'(Y-\mu)P_1' &= 0 \\ H_2'(Y-\mu)P_2' &= 0 \end{aligned}$$

Proof. The proof is immediate, noting in Theorem 2.1 that, $h(u)=\dfrac{\exp(-u/2)}{(2\pi)^{kr/2}}$. ∎

3. Generalised Wishart and Pseudo-Wishart Distributions

From Theorems 3.1 and 3.2 (in Díaz-García, Gutiérrez and Mardia (6)), with $(dD)=\bigwedge_{i=1}^{q} dD_{ii}$ and expressing the nonnormalised invariant probability measure on $V_{q,N}$ by $(U_1'dU_1)$ (see Muirhead (16, pp. 67-72)), the jacobian $J(E\to U_1,D,W_1)$ is given by

$$(dE)=2^{-q}|D|^{N+m-2q}\prod_{i<j}^{q}(D_{ii}^2-D_{jj}^2)(dD)(U_1'dU_1)(W_1'dW_1).$$

Similarly, if we define the matrix $F=E'E=W_1DU_1'U_1DW_1'=W_1LW_1'$, with $L=D^2$, we find that the jacobian $J(E\to F,U_1)$ is defined as

$$(dE)=2^{-q}|L|^{(N-m-1)/2}(dF)(U_1'dU_1)$$

The following result is an extension of Theorem 3.3 in Díaz-García, Gutiérrez and Mardia (6) to the case of an elliptical model. It also extends Theorem 1 from Teng, Fang and Deng (20) to the singular generalised Wishart and the singular and nonsingular generalised Pseudo-Wishart noncentral cases.

THEOREM 3.1. *Assume that $Y\sim\mathcal{E}_{N\times m}^{k,r}(\mu,\Sigma,\Theta,h)$, with h expanding in series of powers in $\mathbb{R}$. Let, also, $q=min(r,k)$; then the density of $S=Y'\Theta^{-}Y$ is given by*

$$\frac{\pi^{qk/2}|L|^{(k-m-1)/2}}{\Gamma_q(\frac{1}{2}k)\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{h^{(2t)}(\operatorname{tr}(\Sigma^{-}S+\Omega))}{t!}\frac{C_\kappa(\Omega\Sigma^{-}S)}{\left(\frac{1}{2}k\right)_\kappa} \tag{7}$$

$$P_2(S-\mu'\Theta^{-}\mu)P_2'=0. \tag{8}$$

where $S = W_1 L W_1'$, A^- *is a symmetric generalised inverse of* A, $\Omega = \Sigma^- \mu' \Theta^- \mu$, $C_\kappa(B)$ *are the zonal polynomials of* B *corresponding to the partition* $\kappa = (t_1, \dots, t_l)$ *of* t, *with* $\sum_1^l t_i = t$, $(\frac{1}{2}k)_\kappa$ *is the generalised hypergeometric coefficient (see James (15)) and* $h^{(j)}(\cdot)$ *is the* j*-th derivate of* h *with respect to* $v = \operatorname{tr} \Sigma^- S$.

Proof. Note that from the factorization of the complete rank of $\Theta = Q'Q$, $Q : k \times N$ and Q^-, a generalised inverse of Q

$$(Q^-)'\Theta Q^- = (Q^-)'(QQ')Q^- = (QQ^-)'(QQ^-) = I_k,$$

given that $r(\Theta) = r(QQ^-) = k$ and QQ^- is of the order $k \times k$.

Defining $X = (Q^-)'Y$ by Theorem 2.1.1 (Gupta and Varga (14)) and by Theorem 2.1, we find that

$$X \sim \mathcal{E}_{k \times m}^{k,r}(\mu_x, \Sigma, I_k, h),$$

in which $\mu_x = (Q^-)'\mu$. Thus

$$S = Y'\Theta^- Y = ((Q^-)'Y)'\Theta^-((Q^-)'Y) = X'X.$$

Let us now consider the SVD of the matrix $X = U_1 D W_1'$. Then

$$S = W_1 D U_1' U_1 D W_1' = W_1 D^2 W_1' = W_1 L W_1'$$

where $L = D^2$, whose jacobian is given by

$$(dX) = 2^{-q}|L|^{(k-m-1)/2}(dS)(U_1' dU_1)$$

in which $q = r(S) = r(Y) = r(X) = \min(r, k)$. From Theorem 2.1, the density of X is given by

$$\frac{1}{\left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)} h\left(\operatorname{tr} \Sigma^-(X - \mu_x)'(X - \mu_x)\right)$$

$$(X - \mu)P_2' = 0 \quad \text{a. s.}$$

Taking the nondegenerate part, we find that the joint density of U_1, S and W_1 is

$$\frac{2^{-q}|L|^{(k-m-1)/2}}{\left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)} h\left(\operatorname{tr}\left(\Sigma^- S + \Omega\right) + \operatorname{tr}\left(-2\mu_x \Sigma^- W_1 D U_1'\right)\right)(dS)(U_1' dU_1),$$

where $\Omega = \Sigma^{-}\mu'\Theta^{-}\mu$.

Let us now assume that h can be expanded in series of powers (Fan (8)), that is,

$$h(v) = \sum_{t=0}^{\infty} a_t \frac{v^t}{t!}.$$

The joint density of U_1, S and W_1 is

$$\frac{2^{-q}|L|^{(k-m-1)/2}}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} \frac{a_t}{t!} \left(\operatorname{tr}\left(\Sigma^{-}S + \Omega\right) + \operatorname{tr}\left(-2\mu_x\Sigma^{-}W_1DU_1'\right)\right)^t (dS)(U_1'dU_1).$$

After developing the binomial, we obtain

$$\frac{2^{-q}|L|^{(k-m-1)/2}}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} \frac{a_t}{t!} \sum_{\eta=0}^{t} \binom{t}{\eta} \left(\operatorname{tr}\left(\Sigma^{-}S + \Omega\right)\right)^{t-\eta}$$

$$(\operatorname{tr}\ (-2\mu_x\Sigma^{-}W_1DU_1'))^{\eta}(dS)(U_1'dU_1).$$

The integral with respect to $U_1 \in V_{q,k}$ is zero when t is odd (see Eqs. (34)-(46) in James (15)); thus the marginal of S can be expressed as

$$\frac{2^{-q}|L|^{(k-m-1)/2}}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} \frac{a_t}{t!} \sum_{\eta=0}^{\left[\frac{t}{2}\right]} \binom{t}{\eta} \left(\operatorname{tr}\left(\Sigma^{-}S + \Omega\right)\right)^{t-2\eta}$$

$$\int\limits_{U_1\in V_{q,k}} (\operatorname{tr}\ (-2\mu_x\Sigma^{-}W_1DU_1'))^{2\eta}(dS)(U_1'dU_1)$$

where $[\cdot]$ represents the integer part of the quotient.

By integrating (see Lemma 9.5.3 in Muirhead (16, p. 397) and Eq. (22) in James (15)), we obtain:

$$\int\limits_{U_1\in V_{q,k}} (\operatorname{tr}\ (-2\mu_x\Sigma^{-}W_1DU_1'))^{2\eta}\,(U_1'dU_1)$$

$$\begin{aligned} &= \frac{2^q\pi^{qk/2}}{\Gamma_q[\frac{1}{2}k]} \sum_{\kappa} \frac{\left(\frac{1}{2}\right)_{\eta} C_\kappa(4\mu_x\Sigma^{-}W_1DDW_1'\Sigma^{-}\mu_x')}{\left(\frac{1}{2}k\right)_{\kappa}} \\ &= \frac{2^q\pi^{qk/2}}{\Gamma_q[\frac{1}{2}k]} \sum_{\kappa} \frac{4^{\eta}\left(\frac{1}{2}\right)_{\eta}}{\left(\frac{1}{2}k\right)_{\kappa}} C_\kappa(\Omega\Sigma^{-}S). \end{aligned}$$

Therefore the marginal of S is

$$\frac{\pi^{qk/2}|L|^{(k-m-1)/2}}{\Gamma_q[\frac{1}{2}k]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty} \frac{a_t}{t!} \sum_{\eta=0}^{\left[\frac{t}{2}\right]} \binom{t}{\eta} 4^{\eta}\left(\tfrac{1}{2}\right)_{\eta} \left(\operatorname{tr}\left(\Sigma^{-}S + \Omega\right)\right)^{t-2\eta} \sum_{\kappa} \frac{C_\kappa(\Omega\Sigma^{-}S)}{\left(\frac{1}{2}k\right)_{\kappa}}.$$

Bearing in mind (see Muirhead (16, p. 21)) that

$$4^{\eta}\left(\tfrac{1}{2}\right)_{\eta} = \frac{(2\eta)!}{\eta!} = 2^{\eta}(2\eta-1)!!$$

the nondegenerate marginal of S is (see Fan (8) and Teng, Fang and Deng (20))

$$\frac{\pi^{qk/2}|L|^{(k-m-1)/2}}{\Gamma_q(\frac{1}{2}k)\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{h^{(2t)}(\operatorname{tr}(\Sigma^{-}S+\Omega))}{t\,!}\frac{C_\kappa(\Omega\Sigma^{-}S)}{(\frac{1}{2}k)_\kappa}.$$

To obtain the degenerate part, note that

$$P_2X'XP_2' = P_2SP_2' = P_2\mu_x'\mu_xP_2'$$

Given that $\mu_x = (Q^{-})'\mu$ and $\mu_x'\mu_x = \mu'\Theta^{-}\mu$, then $P_2SP_2' = P_2\mu'\Theta^{-}\mu P_2'$. In other words,

$$P_2(S-\mu'\Theta^{-}\mu)P_2' = 0 \quad \text{a.s.}$$

and thus the desired result is obtained. ∎

Note that the singularity of the matrix S may be the consequence of three circumstances: the linear dependence between rows, the linear dependence between columns of matrix Y or the singularity in the definition of S (generalised Pseudo-Wishart matrix), that is, when $N < m$ or $k < r$. Thus, when $q = k < m$, S does not have a density with respect to the Lebesgue measure in $\mathbb{R}^{m(m+1)/2}$ (note, the measure (dS) defined on $\mathcal{L}_m^+$, the manifold of positive definite symmetric matrices with distinct eigenvalues, is equivalent to the ordinary Lebesgue measure, taking the elements of S to be the coordinates of points on an $m(m+1)/2$-dimension surface in $\mathbb{R}^{m^2}$). In this case, the density of S exists with respect to the measure (dS) defined on $\mathcal{L}_{m,q}^+$, the manifold of positive semidefined symmetric matrices with q nonzero distinct eigenvalues. The explicit definition of (dS) is obtained by taking the decomposition $S = W_1LW_1'$ to be a system of coordinates for this manifold such that

$$(dS) = 2^{-q}\prod_{i=1}^{q} l_i^{m-q}\prod_{i<j}^{q}(l_i-l_j)(W_1'dW_1)\bigwedge_{i=1}^{q} dl_i \tag{9}$$

where, as above, L is diagonal with $l_1 > \cdots > l_q$ (see Díaz-García, Gutiérrez and Mardia (6)). By analogy to the case of the Y matrix (see Section 2), this measure is a particular case of the Hausdorff measure, which is eqivalent to the Lebesgue measure when the latter is defined on $\mathcal{L}^{+}_{m,q}$, taking the elements of S to be coordinates of points on an $mq-q(q-1)/2$-dimension surface in $I\!R^{m^2}$. When $r<m$, the degenerate part (7) appears explicitly; in this case, an explicit form of the measure with respect to which the density given in (8) exists is given by the product of a count measure for P_2SP_2' with a single point on the support given by $P_2\mu\Theta^-\mu P_2'$ and the Hausdorff measure (or the Lebesgue measure, as appropriate) defined above. Therefore density (7) is interpreted as being the density on subspace (8) with respect to the volume given in (9).

Corollary 3.1 *In Theorem 3.1, note that:*

1. *If $k=N$, $r=m$ with $N\geq m$ then $q=m$. Then*
$$f_S(S)=\frac{\pi^{Nm/2}|S|^{(N-m-1)/2}}{\Gamma_m\left(\frac{1}{2}N\right)|\Sigma|^{N/2}}\sum_{t=0}^{\infty}\frac{h^{2t}(\operatorname{tr}(\Sigma^{-1}S+\Omega))}{t!}\sum_{\kappa}\frac{C_\kappa(\Omega\Sigma^{-1}S)}{\left(\frac{1}{2}k\right)_\kappa}.$$
(See Teng, Fang and Deng (20)).

2. *If $h(u)=(2\pi)^{-kr/2}\exp(-u/2)$. Then*
$$f_S(S)=\begin{cases}\dfrac{\pi^{k(q-r)/2}|L|^{(k-m-1)/2}}{2^{qk/2}\Gamma_q\left(\frac{1}{2}k\right)\left(\prod_{i=1}^r\lambda_i^{k/2}\right)}\operatorname{etr}\left(\frac{1}{2}\Sigma^-S+\frac{1}{2}\Omega\right){}_0F_1\left(\frac{1}{2}k;\frac{1}{4}\Omega\Sigma^-S\right)\\ \qquad\qquad\qquad\qquad\qquad P_2SP_2'=0\end{cases}$$
(See Díaz-García, et al (6)).

Proof.

1. The proof is immediate from Theorem 3.1, noting that $P=P_1\in\mathcal{O}(m)$, $\prod_{i=1}^r\lambda_i^{k/2}=|\Sigma|^{N/2}$, $|L|^{(k-m-1)/2}=|W_1'LW_1|^{(N-m-1)/2}=|S|^{(N-m-1)/2}$, and that $W_1=W\in\mathcal{O}(m)$.

2. Observe that $\dfrac{dh^{2t}(u)}{du^{2t}}=\dfrac{1}{(2\pi)^{kr/2}}\exp(-u/2)\left(\dfrac{1}{2}\right)^{2t}$. Then, from Theorem 3.1, the nonsingular part of density S can be written as
$$\frac{\pi^{k(q-r)/2}|L|^{(k-m-1)/2}}{2^{kr/2}\Gamma_q\left(\frac{1}{2}k\right)\left(\prod_{i=1}^r\lambda_i^{k/2}\right)}\operatorname{etr}\left(\tfrac{1}{2}\Sigma^-S+\Omega\right)\sum_{t=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(\Omega\Sigma^-S)}{2^{2t}t!\left(\frac{1}{2}k\right)_\kappa}.$$
The result follows, noting that $C_\kappa(aX)=C_\kappa(X)/a^k$ and that ${}_0F_1(b;X)=\displaystyle\sum_{l=0}^{\infty}\sum_{\kappa}\frac{C_\kappa(X)}{l!(b)_\kappa}$ ■

We now present another two particular cases of elliptically-contoured distribution: the matrix variate symmetric Kotz type distribution and the matrix variate symmetric Pearson Type VII distribution (see Gupta and Varga (14, pp. 75-76)).

COROLLARY 3.2. *Let* $Y \sim \mathcal{E}^{k,r}_{N\times m}(\mu,\Sigma,\Theta,h)$, *with* h *expanding in series of powers in* $I\!R$. *Then,*

1. *If* Y *has a matrix variate symmetric Pearson Type VII distribution, the density of* $S = Y'\Theta^{-}Y$ *is given by*

$$\frac{\pi^{k(q-r)/2}\Gamma[b]|L|^{(k-m-1)/2}}{a^{rk/2}\Gamma[b-rk/2]\Gamma_q(\frac{1}{2}k)\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{(b)_{2t}\left(1+\dfrac{\operatorname{tr}(\Sigma^{-}S+\Omega)}{a}\right)^{-(b+2t)}}{t!}$$

$$\frac{C_\kappa\left(\frac{1}{a^2}\Omega\Sigma^{-}S\right)}{\left(\frac{1}{2}k\right)_\kappa}$$

$$P_2(S-\mu'\Theta^{-}\mu)P_2' = 0.$$

in which $a, b \in I\!R$ *and* $(b)_{2t} = b(b+1)\cdots(b+2t-1)$.

2. *If* Y *has a matrix variate symmetric Kotz type distribution, the density of* $S = Y'\Theta^{-}Y$ *is given by*

$$\frac{\pi^{k(q-r)/2}a\ b^{(2c+kr-2)/2a}\Gamma[\frac{1}{2}kr]|L|^{(k-m-1)/2}}{\Gamma[(2c+kr-2)/2a]\Gamma_q(\frac{1}{2}k)\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}$$

$$\sum_{t=0}^{\infty}\sum_{\kappa}\sum_{z=0}^{\infty}\frac{(-b)_z(\omega)_{2t}(\operatorname{tr}(\Sigma^{-}S+\Omega))^{az+c-1}}{(z+2t)!}\ \frac{C_\kappa(b^2(\operatorname{tr}(\Sigma^{-}S+\Omega))^{2(a-1)}\Omega\Sigma^{-}S)}{\left(\frac{1}{2}k\right)_\kappa}$$

$$P_2(S-\mu'\Theta^{-}\mu)P_2' = 0.$$

where $a, b, c \in I\!R$, $a > 0$, $b > 0$, $2c + kr > 2$ *and* $\omega = a(z+2t) + c - 1$.

Proof. The proof is immediate from Theorem 3.1, noting that:

1. For the Pearson Type VII case,

$$h(v) = \frac{\Gamma[b]}{(\pi\ a)^{kr/2}\Gamma[b-kr/2]}(1+v/a)^{-b}$$

from which

$$h^{(2t)}(v) = \frac{\Gamma[b]}{(\pi\ a)^{kr/2}\Gamma[b-kr/2]}\frac{(b)_{2t}}{a^{2t}}(1+v/a)^{-(b+2t)}.$$

2. For the Kotz case

$$\begin{aligned} h(v) &= \frac{a\ b^{(2c+kr-2)/2a}\Gamma[\frac{1}{2}kr]}{\pi^{kr/2}\Gamma[(2c+kr-2)/2a]}v^{c-1}\exp(-bv^a) \\ &= \frac{a\ b^{(2c+kr-2)/2a}\Gamma[\frac{1}{2}kr]}{\pi^{kr/2}\Gamma[(2c+kr-2)/2a]}\sum_{l=0}^{\infty}\frac{(-b)^l v^{al+c-1}}{l\ !} \end{aligned}$$

from which, effecting a change to the sum parameter, we obtain

$$h^{(2t)}(v) = \frac{a\ b^{(2c+kr-2)/2a}\Gamma[\frac{1}{2}kr]}{\pi^{kr/2}\Gamma[(2c+kr-2)/2a]}\sum_{z=0}^{\infty}\frac{(-b)^{z+2t}(a(z+2t)+c-1)_{2t}}{v^{-(a(z+2t)+c-1-2t)}(z+2t)!}$$

■

Remark 3.1. Note that, when $b=(rk+a)/2$ in the singular symmetric Pearson Type VII distribution we obtain the singular distribution t of a random matrix with a degrees of freedom. Note, too, that if we take $a=1$ in the definition of the distribution t of a random matrix, we obtain the Cauchy singular distribution of a random matrix. And if we take $a=c=1$ and $b=1/2$ in the Kotz type singular symmetric distribution, we obtain the matrix variate normal singular distribution. Making the same substitutions in the corresponding distributions of S, we obtain the generalised Wishart or Pseudo-Wishart for each of the particular cases, see Corollary 3.1.

Corollary 3.3. *Assume that in Theorem 3.1, $\mu=0$. Then, the density of S is given by*

$$\frac{\pi^{qk/2}|L|^{(k-m-1)/2}}{\Gamma_q(\frac{1}{2}k)\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}S)$$

$$P_2SP_2' = 0.$$

Corollary 3.4 *For Corollary 3.3:*

1. *If Y has a matrix normal distribution with $r=m$, $k=N$ such that $N<m$ then $q=N$, and then*

$$f_S(S) = \frac{\pi^{N(N-m)/2}|L|^{(N-m-1)/2}}{2^{Nm/2}\Gamma_N\left(\frac{1}{2}N\right)|\Sigma|^{N/2}}\operatorname{etr}\left(\tfrac{1}{2}\Sigma^{-}S\right).$$

(See Uhlig (21)).

2. *If Y has a matrix t distribution, with $r = m = q$, $k = N$, $m \leq N$. Then*

$$f_S(S) = \frac{a^{a/2}\Gamma\left(\frac{1}{2}(a+Nm)\right)|S|^{(N-m-1)/2}}{\Gamma\left(\frac{1}{2}a\right)\Gamma_m\left(\frac{1}{2}N\right)|\Sigma|^{N/2}}(a + \operatorname{tr}\Sigma^{-1}S)^{-(a+Nm)/2}.$$

(See Sutradhar and Ali (18)).

Proof. The proof follows from Corollary 3.3, noting again that $P = P_1 \in \mathcal{O}(m)$, $\prod_{i=1}^{r}\lambda_i^{k/2} = |\Sigma|^{N/2}$, $|L|^{(k-m-1)/2} = |S|^{(N-m-1)/2}$, and

1. taking $h(u) = (2\pi)^{-Nm/2}\exp(u/2)$
2. taking $h(u) = \dfrac{\Gamma\left(\frac{1}{2}(a+Nm)\right)}{(a\pi)^{Nm/2}\Gamma\left(\frac{1}{2}a\right)}(1+u/a)^{-(a+Nm)/2}$. ∎

4. Shape Theory

By analogy with the Gaussian case, in the context of shape theory the densities given in Theorem 3.1 and in Corollary 3.1 are termed size-and-shape densities (see Díaz-García, Gutiérrez and Mardia (6)). In this section, we determine the joint density of the singular values $D_{11}, \dots, D_{qq}$ of X, termed the size-and-shape cone density as an extension of the singular noncentral case of Theorem 2.1 in Díaz-García, Gutiérrez and Ramos (7). This can be obtained from the density of the eigenvalues of the matrix $S = W_1 L W_1'$, noting that $L^{1/2} = D$, or directly from the SVD of X.

THEOREM 4.1. *The size-and-shape cone density is given by*

$$\frac{2^q\pi^{q(k+m)/2}\prod_{i=1}^{q}D_{ii}^{k+m-2q}\prod_{i<j}^{q}(D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}m]\Gamma_q[\frac{1}{2}k]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{\theta,\kappa}^{\infty}\sum_{\phi\in\theta,\kappa}\frac{h^{(2t+l)}(\operatorname{tr}\Omega)}{t!\;l!}$$
$$\frac{\Delta_\phi^{\theta,\kappa}C_\phi(D^2)C_\phi^{\theta,\kappa}(\Sigma^-,\Omega\Sigma^-)}{\left(\frac{1}{2}k\right)_\kappa C_\phi(I_m)}$$
$$(D - D_{\mu_x})P_2' = 0 \qquad a.s.$$

where D_{μ_x} is the diagonal matrix in the SVD of the matrix μ_x, $X = U_1 D W_1'$ and $q = \min(k, r)$. The notation of the sum operators, $\Delta_\phi^{\theta,\kappa}$ and $C_\phi^{\theta,\kappa}$ is given in Davis (4) (see also Chikuse (3)).

Proof. Consider the SVD of the matrix $X = U_1 D W_1'$. The joint density of U_1, D and W_1 (the nondegenerate part) is given by:

$$\frac{\displaystyle\prod_{i=1}^{q} D_{ii}^{k+m-2q} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)}{2^q \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)} h(\mathrm{tr}(\Sigma^- W_1 D^2 W_1' + \Omega - 2\Sigma^- W_1 D U_1' \mu_x))$$

$$(dD)(U_1' dU_1)(W_1' dW_1).$$

Assume that h can be expanded in series of powers and integrated with respect to $U_1 \in V_{q,m}$, proceed in an analogous form to the proof of Theorem 3.1. We then find that the joint density of W_1 and D is given by

$$\frac{\pi^{qk/2} |D|^{(k+m-2q)} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}k] \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)}$$

$$\sum_{t=0}^{\infty} \sum_{\kappa} \frac{h^{(2t)}(\mathrm{tr}(\Sigma^- W_1 D^2 W_1' + \Omega))}{t!} \frac{C_\kappa(\Omega \Sigma^- W_1 D^2 W_1')}{\left(\frac{1}{2}k\right)_\kappa} (dD)(W_1' dW_1).$$

By expanding $h^{(2t)}$ into series of powers

$$h^{(2t)}(v) = \sum_{l=0}^{\infty} \frac{b_l}{l!} v^l$$

the joint density of W_1 and D can be expressed as

$$\frac{\pi^{qk/2} |D|^{(k+m-2q)} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}k] \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)}$$

$$\sum_{t,l=0}^{\infty} \sum_{\kappa} \frac{b_l}{l!} \sum_{f=0}^{l} \frac{\binom{l}{f} (\mathrm{tr}\,\Omega)^{l-f}}{t! \left(\frac{1}{2}k\right)_\kappa} (\mathrm{tr}\,\Sigma^- W_1 D W_1')^f C_\kappa(\Omega \Sigma^- W_1 D^2 W_1')(dD)(W_1' dW_1).$$

After expanding $(\mathrm{tr}\,\Sigma^- W_1 D W_1')^f$ into zonal polynomials and integrating with respect to $W_1 \in V_{q,m}$, with the help of Eq. (4.13) from Davis (4), we obtain

$$\int_{W_1 \in V_{q,m}} (\operatorname{tr} \Sigma^- W_1 D W_1')^f C_\kappa(\Omega \Sigma^- W_1 D^2 W_1')(W_1' dW_1)$$

$$= \sum_\theta \int_{W_1 \in V_{q,m}} C_\theta(\Sigma^- W_1 D W_1') C_\kappa(\Omega \Sigma^- W_1 D^2 W_1')(W_1' dW_1)$$

$$= \frac{2^q \pi^{qm/2}}{\Gamma_q[\frac{1}{2}m]} \sum_\theta \int_{\mathcal{O}(m)} C_\theta(\Sigma^- W_1 D W_1') C_\kappa(\Omega \Sigma^- W_1 D^2 W_1')(dW)$$

$$= \frac{2^q \pi^{qm/2}}{\Gamma_q[\frac{1}{2}m]} \sum_\theta \sum_{\phi \in \theta,\kappa} \frac{\Delta_\phi^{\theta,\kappa} C_\phi(D^2) C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-)}{C_\phi(I_m)}.$$

The final equality is a consequence of (5.1) in Davis (4). Therefore, applying the notation used by Davis (4), the joint density (of the nondegenerate part) of $D_{11}, \dots, D_{qq}$ is given by

$$\frac{2^q \pi^{q(k+m)/2} |D|^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}m] \Gamma_q[\frac{1}{2}k] \left(\prod_{i=1}^r \lambda_i^{k/2} \right)} \sum_{\theta,\kappa}^\infty \sum_{\phi \in \theta,\kappa} \frac{b_l}{l!} \sum_{f=0}^l \frac{\binom{l}{f} (\operatorname{tr} \Omega)^{l-f} \Delta_\phi^{\theta,\kappa}}{t! \, \left(\frac{1}{2}k\right)_\kappa}$$
$$\frac{C_\phi(D^2) C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-)}{C_\phi(I_m)}.$$

Finally,

$$\frac{2^q \pi^{q(k+m)/2} \prod_{i=1}^q D_{ii}^{k+m-2q} \prod_{i<j}^q (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}m] \Gamma_q[\frac{1}{2}k] \left(\prod_{i=1}^r \lambda_i^{k/2} \right)} \sum_{\theta,\kappa}^\infty \sum_{\phi \in \theta,\kappa} \frac{h^{(2t+l)}(\operatorname{tr} \Omega)}{t! \, l!}$$
$$\frac{\Delta_\phi^{\theta,\kappa} C_\phi(D^2) C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-)}{\left(\frac{1}{2}k\right)_\kappa C_\phi(I_m)}.$$

For the degenerate part, consider the SVD of $\mu_x = U_{1_{\mu_x}} D_{\mu_x} W_{1_{\mu_x}}'$. Then,

$$(D - D_{\mu_x}) P_2' = 0 \qquad a.s.$$

and thus the desired result is obtained. ■

An important consequence of Theorem 4.1 in shape theory is that when $\Sigma = \sigma^2 I_m$, this result is obtained by making use of certain properties of symmetric polynomials, Davis (4).

Corollary 4.1. *In the hypotheses of Theorem 4.1, assume that $\Sigma = \sigma^2 I_m$. Then,* *the size-and-shape cone density is given by*

$$\frac{2^q \pi^{q(k+m)/2} \prod_{i=1}^{q} D_{ii}^{k+m-2q} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}m]\Gamma_q[\frac{1}{2}k]\sigma^{mk}} \sum_{\kappa}^{\infty} \frac{h^{(2t)}(\operatorname{tr}\Omega + \frac{1}{\sigma^2}\sum_{i=1}^{q} D_{ii}^2)}{t!} \frac{C_\kappa(\frac{1}{\sigma^2}D^2)C_\kappa(\Omega)}{\left(\frac{1}{2}k\right)_\kappa C_\kappa(I_m)}.$$

Proof. Note that the degenerate part disappears,

$$\prod_{i=1}^{r} \lambda_i^{k/2} = |\Sigma|^{k/2} \quad \Omega = \sigma^{-2}\mu_x'\mu_x = \sigma^{-2}\mu'\Theta^-\mu$$

and from (5.7) and (5.2), Davis (4),

$$\begin{aligned} C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-) &= C_\phi^{\theta,\kappa}\left(\frac{1}{\sigma^2}I_m, \frac{1}{\sigma^2}\Omega\right) \\ &= \left(\frac{1}{\sigma}\right)^{2t+l} \frac{\Delta_\phi^{\theta,\kappa} C_\phi(I_m)}{C_\kappa(I_m)} C_\kappa(\Omega). \end{aligned}$$

Therefore, from Theorem 4.1, we have

$$\begin{aligned} I &= \sum_{\theta,\kappa}^{\infty} \sum_{\phi\in\theta,\kappa} \frac{h^{(2t+l)}(\operatorname{tr}\Omega)}{t!\ l!} \frac{\Delta_\phi^{\theta,\kappa} C_\phi(D^2) C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-)}{\left(\frac{1}{2}k\right)_\kappa C_\phi(I_m)} \\ &= \sum_{\theta,\kappa}^{\infty} \sum_{\phi\in\theta,\kappa} \frac{h^{(2t+l)}(\operatorname{tr}\Omega)}{t!\ l!\ (\sigma^2)^{2t+l}} \frac{\left(\Delta_\phi^{\theta,\kappa}\right)^2 C_\phi(D^2) C_\kappa(\Omega)}{\left(\frac{1}{2}k\right)_\kappa C_\kappa(I_m)}. \end{aligned}$$

Note that $\sum_{\phi\in\theta,\kappa} \left(\Delta_\phi^{\theta,\kappa}\right)^2 C_\phi(D^2) = C_\kappa(D^2)C_\theta(D^2)$ (see Eq. (5.10), Davis (4)), from which we obtain

$$I = \sum_{\theta,\kappa}^{\infty} \frac{h^{(2t+l)}(\operatorname{tr}\Omega)}{t!\ l!\ (\sigma^2)^{2t+l}} \frac{C_\kappa(D^2)C_\theta(D^2)C_\kappa(\Omega)}{\left(\frac{1}{2}k\right)_\kappa C_\kappa(I_m)}.$$

After summing on θ and on l, noting that

1. $\sum_{\theta} C_\theta(D^2) = (\operatorname{tr} D^2)^l$ and that
2. $h(v) = \sum_{l=0}^{\infty} \frac{h^{(l)}(a)}{l!}(v-a)^l$, with $a = \operatorname{tr}\Omega$, $v = \operatorname{tr}\Omega + \frac{1}{\sigma^2}\sum_{i=1}^{q} D_{ii}^2$ and $h(v) = h^{(2t)}(v)$

the desired result is obtained. ∎

Remark 4.1. As a particular case of Corollary 4.1, we obtained Theorem 2 in Teng, Fang and Deng (20). By taking $D^2 = \Lambda$ with $(dD) = 2^{-q}|\Lambda|^{-1/2}(d\Lambda)$, $\sigma = 1$, $k = n$ and $r(X) = q = m \leq n$.

Remark 4.2. Proceeding as in Corollary 3.2, it is simple to obtain particular examples of the size-and-shape cone density. For this purpose we only need to evaluate the $(2t + l)$ derivative of the function h.

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