# *FUNCTIONS OF SINGULAR RANDOM MATRICES : A BAYESIAN APPLICATION*

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# Functions of singular random matrices: A Bayesian application

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**Abstract**

The present article describes how the Jacobian is found for certain functions of a singular random matrix, both in the general case and in that of a non-negative definite random matrix. In particular, we find the Jacobian of the $V = S^2$ transformation when $S$ is non-negative definite, and in general, the Jacobian of the $Y = X^+$ transformation, in which $X^+$ is the generalised, or Moore-Penrose, inverse of $X$. Expressions for the densities of the generalised inverse of the central Beta and F singular random matrices are proposed. Finally, two applications in the field of Bayesian inference are presented.

**Key Words:** Matrix-variate inverse Beta and F distributions, Jacobian, Hausdorff measure, inverse singular distribution, inverse Wishart and Pseudo-Wishart singular distributions, Bayesian inference.

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## 1 Introduction

Although inverse distributions appear in various areas of statistical analysis, there can be no doubt that it is in the field of Bayesian inference that such distributions play a key role. In particular, inverse Wishart, Beta and Dirichlet distributions have been used as a priori distributions; they also appear as a posteriori distributions in different Bayesian applications, see for example Press [Sections 8.6.1 and 8.6.2] [19], Box and Tiao [p.460] [2] and Xu [23].

One problem that has not been addressed previously is the one that arises when these distributions are singular, as their inverse does not initially exist. Nevertheless, this obstacle can be overcome by considering the generalised inverse. Unfortunately, little research has been carried out in this respect, and not even a general study of singular distributions has been attempted, except for some ideas proposed by Khatri, see Rao [p. 527] [20] and Cramér [p. 297] [5]. Some coordinated work has recently been done in this field, see Uhlig [22], Díaz-García et al. [8], Díaz-García and Gutiérrez-Jáimez [9], Díaz-García et al. [11] and Díaz-García and Gutiérrez-Jáimez [10].

For a given distribution of $Y$ and in order to determine the distribution of $X = Y^+$, where $Y^+$ denotes the generalised, or Moore-Penrose inverse, the first step is to determine the Jacobian of this transformation, together with the corresponding measure with respect to which this distribution exists.

This article examines various results previously obtained for the non-singular case and extends them to the singular case. In particular, Section 2 establishes some necessary notation and compiles the results essential for the development of the rest of the study. The results referring to the calculation of Jacobians are presented in Section 3, which also provides a general result and, as particular cases, the Jacobians of the transformations $X = Y^+$ and $X = Y^2$ in which $Y$ is a non-negative definite matrix. This section ends by extending the general result and providing an example for the generalised inverse case, in which $Y$ is any matrix of incomplete rank. As an application of these Jacobians, Section 4 proposes explicit expressions for the generalised inverse Wishart, Pseudo-Wishart, Beta and $F$ densities and their corresponding measures. Finally, Section 5 establishes two applications of generalised inverse distributions in the context of Bayesian inference.

## 2 Notation and Preliminary Results

Let us use $\mathcal{L}^+_{m,N}(q)$ to denote the linear space of all the real matrices $N \times m$ of rank $q \leq \min(N, m)$ with $q$ different singular values. The matrix set $H_1 \in \mathcal{L}_{m,N}$ such that $H_1'H_1 = I_m$ is a manifold denoted by $\mathcal{V}_{m,N}$, known as the Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices, denoted by $\mathcal{O}(m)$. $\mathcal{S}_m$ then denotes the homogeneous space of symmetric positive definite matrices $m \times m$; then, $\mathcal{S}_m$denotes the $(mq - q(q-1)/2)$-dimensional manifold of positive semidefinite symmetric matrices of rank $q$, with $q$ different positive eigenvalues and in which $\mathcal{D}(m)$ denotes the set of diagonal matrices with $d_i$ elements, such that $d_1 > d_2 > \cdots > d_m > 0$.

A singular random matrix $X \in \mathcal{L}^+_{m,N}(q)$ (or $X \in \mathcal{S}^+_m(q)$) has no a density with respect to the Lebesgue measure in $I\!R^{N\times m}$, but it does possess density on a subspace $\mathcal{M} \subset I\!R^{N\times m}$, see Rao [p. 527][20], Díaz-García et al. [8], Uhlig [22] and Cramér [p. 297] [5]. Formally, $X$ has density with respect to the Hausdorff measure, which coincides with the Lebesgue measure, when the latter is defined on the subspace $\mathcal{M}$, see Billingley [p. 247] [1], Uhlig [22], Díaz-García et al. [8], Díaz-García et al. [11] and Díaz-García and Gutiérrez-Jáimez [10].

In order to obtain explicit expressions for the densities with respect to the Hausdorff measure, we require a set of base coordinates to enable us to define the volume elements $(dX)$ in an explicit way. For this purpose, the spectral decomposition and the singular values decomposition of the matrix $X$ (according to $X \in \mathcal{S}^+_m(q)$ or $X \in \mathcal{L}^+_{m,N}(q)$, respectively), enables us to propose an explicit form of the Hausdorff measure $(dX)$, in both cases. As can be seen in the following two results, another instrument that has been found highly useful is the exterior product of differential forms, from which it is possible to determine the Jacobians of singular transformations, together with the explicit form of the Hausdorff measure, see James [16], Farrell [13], Muirhead [Chapter 2, 1982], Uhlig [22], Díaz-García et al. [8] and Díaz-García and Gutiérrez [9] and Díaz-García et al. [11].

**Proposition 2.1 (Expectral decomposition).** *Let $S \in \mathcal{S}^+_m(q)$, then there are $W_1 \in \mathcal{V}_{q,m}$ and $L = \operatorname{diag}(l_1, \ldots, l_q) \in \mathcal{D}(q)$, such that $S = W_1LW_1'$, is called the non-singular part of the spectral decomposition. Let*

*$W_2 \in \mathcal{V}_{m-q,m}$ (a function of $W_1$), such that $W = (W_1|W_2) \in \mathcal{O}(m)$. In columns, $W_1 = (w_1 \cdots w_q)$ and $W_2 = (w_{q+1} \cdots w_m)$, and so*

$$(dS) = 2^{-q}|L|^{m-q}\prod_{i<j}^{q}(l_i - l_j)(dL)(W_1'dW_1) \tag{2.1}$$

*where*

$$(W_1'dW_1) \equiv \bigwedge_{i=1}^{q}\bigwedge_{j=i+1}^{N} w_j'dw_i$$

*defines an invariate measure on $\mathcal{V}_{q,m}$, James [16], Farrell [13] and $(dL) \equiv \bigwedge_{i=1}^{q} dl_i$, Uhlig [22] and Díaz-García and Gutiérrez [9].*

**Proposition 2.2 (Singular values decomposition).** *Let $X \in \mathcal{L}_{m,N}^{+}(q)$, then $X = H_1 D P_1'$, is called the non-singular part of the singular values decomposition, where $H_1 \in \mathcal{V}_{q,N}$, $P_1 \in \mathcal{V}_{q,m}$ and $D = \operatorname{diag}(D_1, \dots, D_q) \in \mathcal{D}(q)$. Then*

$$(dX) = 2^{-q}|D|^{N+m-2q}\prod_{i<j}^{q}(D_i^2 - D_j^2)(dD)(H_1'dH_1)(P_1'dP_1), \tag{2.2}$$

*where $(dD) = \bigwedge_{i=1}^{q} dD_i$, Díaz-García et al. [8].*

**Proposition 2.3 (Central Wishart and Pseudo-Wishart singular distributions).** *Assume $Y \sim \mathcal{N}_{N\times m}^{k,r}(\mu, \Sigma, \Xi)$, with $r(\Sigma) = r \leq m$, $r(\Xi) = k \leq N$ and let $q$ =min$(r,k)$, then the density of $S = Y'\Xi^{-}Y \in \mathcal{S}_m^{+}(q)$ is given by*

$$dF_S(S) = \frac{\pi^{k(q-r)/2}\left(\prod_{i=1}^{q} l_i^{(k-m-1)/2}\right)}{2^{kr/2}\Gamma_q\left[\frac{1}{2}k\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^{-}S\right)(dS) \tag{2.3}$$

*where $S = W_1 L W_1'$, is the non-singular part of the spectral decomposition, with $W_1 \in \mathcal{V}_{q,m}$ and $L = \operatorname{diag}(l_1, \dots, l_q) \in \mathcal{D}(q)$, $\Sigma^{-}$ is a symmetric generalised inverse of $\Sigma$, $\Xi^{-}$ is a symmetric generalised inverse of $\Xi = Q'Q$ with $Q$ $k \times N$ matrix, $r(Q) = k$, $\lambda_i$, $i = 1, \dots r$ are the non-null eigenvalues of $\Sigma$ and $(dS)$ is given by (2.1), Díaz-García et al. [8].*

**Remark 2.1.** *If $S$ has a density (2.3), this fact is described by $S \sim \mathcal{W}_m(q, k, \Sigma)$ if $k \geq r$ ($N \geq m$) for the case of the Wishart singular distribution and by $S \sim \mathcal{P}W_m(q, k, \Sigma)$ if $k < r$ ($N < m$) for the case of the Pseudo-Wishart singular distribution.*

**Proposition 2.4 (Central matrix-variate Beta singular distribution).** *Assume that $U \in \mathcal{S}_m^+(q)$ is a random matrix. $U$ is said to have an $m$-dimensional central matrix-variate Beta singular distribution of rank $q$ and parameters $n/2$ and $p/2$, denoting $U \sim \mathcal{B}_m(q, n/2, p/2)$, if its density function is given by*

$$dF_U(U) = \pi^{(-mn+nq)/2} \frac{\Gamma_m\left[\frac{1}{2}(n+p)\right]}{\Gamma_q\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]} \prod_{i=1}^{q} l_i^{(n-m-1)/2} |I_m - U|^{(p-m-1)/2} (dU)$$

*where $U = H_1 L H_1'$ is the non-singular part of the spectral decomposition, with $H_1 \in \mathcal{V}_{q,m}$ and $L = \operatorname{diag}(l_1, \dots, l_q)$, $1 > l_1 > \cdots > l_q > 0$, and $(dU)$ is given by (2.1), see Díaz-García and Gutiérrez [9].*

**Proposition 2.5 (Central matrix-variate F singular distribution).** *The matrix $F \in \mathcal{S}_m^+(q)$, is said to have an $m$-dimensional central matrix-variate matrix $F$ singular distribution of rank $q$ and parameters $n/2$ and $p/2$, denoting $F \sim \mathcal{F}_m(q, n/2, /2)$, if its density function is given by*

$$dT_F(F) = \pi^{(-mn+nq)/2} \frac{\Gamma_m\left[\frac{1}{2}(n+p)\right]}{\Gamma_q\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]} \prod_{i=1}^{q} \gamma_i^{(n-m-1)/2} |I_m + F|^{-(p+n)/2} (dF)$$

*where $F = H_1 D_\gamma H_1'$ is the non-singular part of the spectral decomposition, with $H_1 \in \mathcal{V}_{q,m}$ and $D_\gamma = \operatorname{diag}(\gamma_1, \dots, \gamma_q) \in \mathcal{D}(q)$, and $(dF)$ is given by (2.1), see Díaz-García and Gutiérrez [9].*

Finally, if $A$ is any matrix, $A^+$ denotes the generalised, or the Moore-Penrose inverse of $A$, see Campbell and Meyer [pp. 8-9] [4].

## 3 Jacobians of Matrix Transformations

**Theorem 3.1.** *Assume $S \in \mathcal{S}_m^+(q)$. Then $S = H_1 D_\lambda H_1'$ denotes the non-singular part of the spectral decomposition of $S$, with $H_1 \in \mathcal{V}_{q,m}$ and $D_\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_q) \in \mathcal{D}(q)$. Moreover, assume that $g(x)$ is a differentiable* *function in $x$ and that $G(D_\lambda) = \operatorname{diag}(g(\lambda_1), \dots, g(\lambda_q))$. Thus we define $G(S) = H_1 G(D_\lambda) H_1'$. Then*

$$(dG(S)) = \prod_{i=1}^{q} \left( \frac{g(\lambda_i)}{\lambda_i} \right)^{m-q} \prod_{i<j}^{q} \frac{g(\lambda_i) - g(\lambda_j)}{\lambda_i - \lambda_j} \prod_{i=1}^{q} \frac{dg(\lambda_i)}{d\lambda_i} (dS) \tag{3.1}$$

*Proof.* From proposition 2.1,

$$(dS) = 2^{-m} \prod_{i=1}^{q} \lambda_i^{m-q} \prod_{i<j}^{q} (\lambda_i - \lambda_j)(H_1' dH_1) \wedge (dD_\lambda). \tag{3.2}$$

Similarly

$$(dG(S)) = 2^{-m} \prod_{i=1}^{q} g(\lambda_i)^{m-q} \prod_{i<j}^{q} (g(\lambda_i) - g(\lambda_j))(H_1' dH_1) \wedge (dG(D_\lambda)),$$

and moreover

$$\begin{aligned}
(dG(D_\lambda)) &= (\operatorname{diag}(dg(\lambda_1), \dots, dg(\lambda_q))) \\
&= \left( \operatorname{diag} \left( \frac{dg(\lambda_1)}{d\lambda_1} d\lambda_1, \dots, \frac{dg(\lambda_q)}{d\lambda_q} d\lambda_q \right) \right) \\
&= \prod_{i=1}^{q} \frac{dg(\lambda_i)}{d\lambda_i} \bigwedge_{i=1}^{q} d\lambda_i \\
&= \prod_{i=1}^{q} \frac{dg(\lambda_i)}{d\lambda_i} (dD_{\lambda_i}),
\end{aligned}$$

and so

$$(dG(S)) = 2^{-m} \prod_{i=1}^{q} g(\lambda_i)^{m-q} \prod_{i<j}^{q} (g(\lambda_i) - g(\lambda_j)) \prod_{i=1}^{q} \frac{dg(\lambda_i)}{d\lambda_i} (H_1' dH_1) \wedge (dD_\lambda),$$

from which

$$\begin{aligned}
&(H_1' dH_1) \wedge (dD_\lambda) \\
&\quad = 2^m \left( \prod_{i=1}^{q} g(\lambda_i)^{m-q} \prod_{i<j}^{q} (g(\lambda_i) - g(\lambda_j)) \prod_{i=1}^{q} \frac{dg(\lambda_i)}{d\lambda_i} \right)^{-1} (dG(S)).
\end{aligned} \tag{3.3}$$

By substituting (3.3) into (3.2), the desired result is obtained. ☐

For the case in which $S \in \mathcal{S}_m$, the result in Theorem 1 was proposed as a problem by Srivastava and Khatri [problem 1.35(i), p.39][21].

**Remark 3.1.** *Note that many results in the literature are obtained as a particular case of Theorem 3.1. Thus, for example, we have the positive semidefinite square root of a matrix, $V = S^{1/2}$, in which $g(x) = x^{1/2}$, the result of which is of great interest when the polar decomposition of a matrix is considered. For the case in which $S \in \mathcal{S}_m$, this result was examined by Hertz [15] and Cadet [3]. This result can also be expressed in its inverse form, as $S = V^2$, such as when $S \in \mathcal{S}_m$ was studied, among others, by Olkin and Rubin [18]. The singular case $S \in \mathcal{S}_m^+(q)$ has been studied by Díaz-García and González-Farías [7].*

Another particular case of Theorem 3.1 occurs when $g(x) = x^{-1}$ and $S \in \mathcal{S}_m$, in which case we obtain the Jacobian of the transformation $V = S^{-1}$, a result that has been obtained by many authors, including Deemer and Olkin [6], Press [p. 47] [19] and Muirhead [p. 59] [17]. Assuming that $S \in \mathcal{S}_m^+(q)$ and $g(x) = x^{-1}$, we obtain the transformation $S = V^+$, the Jacobian of which has been determined by Díaz-García et al. [11] and which, to serve as an example of Theorem 3.1, is established below.

**Corollary 3.1.** *Assume $V = S^+$ with $S \in \mathcal{S}_m^+(q)$. Then*

$$(dV) = \prod_{i=1}^{q} \lambda_i^{-2m+q-1}(dS) \tag{3.4}$$

*where $\lambda_i$, $i = 1, \dots, q$, are the no-null eigenvalues of $S$.*

*Proof.* Let $S = H_1 D_\lambda H_1'$. Then by application of Theorem 3.1, $V = H_1 G(D_\lambda) H_1'$, with

$$G(D_\lambda) = \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_q^{-1}).$$

Now

$$\prod_{i=1}^{q} \frac{dg(\lambda_i)}{d\lambda_i} = \prod_{i=1}^{q} \frac{d\lambda_i^{-1}}{\lambda_i} = \prod_{i=1}^{q} (-1)\frac{1}{\lambda_i^2},$$

From where, ignoring the sign,

$$
\begin{aligned}
(dV) &= \prod_{i=1}^{q}\left(\frac{\frac{1}{\lambda_i}}{\lambda_i}\right)^{m-q}\prod_{i<j}^{q}\frac{\frac{1}{\lambda_i}-\frac{1}{\lambda_j}}{\lambda_i-\lambda_j}\prod_{i=1}^{q}\frac{1}{\lambda_i^2}(dS)\\
&= \prod_{i=1}^{q}\left(\frac{1}{\lambda_i^2}\right)^{m-q+1}\prod_{i<j}^{q}\frac{1}{\lambda_i\lambda_j}(dS)\\
&= \prod_{i=1}^{q}\left(\frac{1}{\lambda_i^2}\right)^{2(m-q+1)}\prod_{i=1}^{q}\left(\frac{1}{\lambda_i^2}\right)^{q-1}(dS)
\end{aligned}
$$

□

**Corollary 3.2.** *Assume that $V = S^2$ with $S \in \mathcal{S}_m^+(q)$. Then*

$$(dV) = 2^q \prod_{i=1}^{q}\lambda_i^{m-q+1}\prod_{i<j}^{q}(\lambda_i+\lambda_j)(dS) \tag{3.5}$$

*where $\lambda_i$, $i = 1,\dots,q$, are the non-null eigenvalues of $S$.*

*Proof.* The proof follows from Theorem 3.1. □

**Remark 3.2.** *Observe that in Corollaries 3.1 and 3.2, when $S \in \mathcal{S}_m$, $q = m$, we obtain the following results, respectively,*

1. $$(dV) = \prod_{i=1}^{m}\lambda_i^{-(m+1)}(dS) = |S|^{-(m+1)}(dS)$$

   *see Deemer and Olkin [6], Press[p. 47] [19] or Muirhead [p. 59] [17].*

2. $$(dV) = 2^m\prod_{i=1}^{m}\lambda_i\prod_{i<j}^{m}(\lambda_i+\lambda_j)(dS) = \prod_{i\leq j}^{m}(\lambda_i+\lambda_j)(dS),$$

   *see Olkin and Rubin [18].*

According to the case, we can take any differentiable function and apply Theorem 3.1 to it. In particular, it would be of interest to determine the distribution of the logarithm of a matrix, $V = \log S$. In the unidimensional case, this function has been applied in order to determine, among other objects, the Log-Normal, Log-Gamma and Log-Elliptic distributions. For the Normal and Elliptic distributions, it was also extended to the vector and matrix cases, such that if $X$ is a random vector or matrix, we are interested in the distribution of $\log X$. Assuming $X$ to be an $m$-dimensional random vector, the transformation has been defined as $\log X = (\log x_1, \dots, \log x_m)'$, see for example Press [p. 149] [19]. Alternatively, we could consider defining $\log X$, in an analogous way to the definition given in Theorem 3.1. To extend this idea to any matrix, the following result is required.

**Theorem 3.2.** *Assume that $X \in \mathcal{L}^+_{m,n}(q)$. Then $X = H_1 D_\alpha P_1'$ denotes the non-singular part of the singular values decomposition of $X$, with $H_1 \in \mathcal{V}_{q,n}$, $P_1 \in \mathcal{V}_{q,m}$ and $D_\alpha = \operatorname{diag}(\alpha_1, \dots, \alpha_q) \in \mathcal{D}(q)$. Additionally, assume that $g(x)$ is a function that can be differentiated in $x$ and $G(D_\alpha) = \operatorname{diag}(g(\alpha_1), \dots, g(\alpha_q))$. We thus define $G(X) = H_1 G(D_\lambda) P_1'$. Then*

$$(dG(X)) = \prod_{i=1}^{q} \left(\frac{g(\alpha_i)}{\alpha_i}\right)^{n+m-2q} \prod_{i<j}^{q} \frac{g^2(\alpha_i) - g^2(\alpha_j)}{\alpha_i^2 - \alpha_j^2} \prod_{i=1}^{q} \frac{dg(\alpha_i)}{d\alpha_i} (dX). \quad (3.6)$$

*Proof.* The proof is analogous to that of Theorem 3.1, but now taking Proposition 2.2 into consideration. □

In Theorem 3.2, once again, we could consider any differentiable $g(\cdot)$ function. As an example of Theorem 2, we now examine the case in which $g(x) = x^{-1}$, thus obtaining the transformation $Y = X^+ \in \mathcal{L}^+_{m,n}(q)$.

**Corollary 3.3.** *Let $Y = X^+$ with $X \in \mathcal{L}^+_{m,n}(q)$. Then*

$$(dY) = \prod_{i=1}^{q} \alpha_i^{-2(n+m-q)} (dX) \quad (3.7)$$

*where $\alpha_i$, $i = 1, \dots, q$, are the non-null singular values of $X$.*

*Proof.* The proof follows from Theorem 3.2 and is analogous to that given for Corollary 3.1, see also Díaz-García et al. [11]. □

## 4 Inverse Singular Distributions

In this section, we determine the distributions of the generalised inverse of the Wishart and Pseudo-Wishart, the Beta and the $F$ matrices.

**Theorem 4.1 (Generalised Inverse Wishart and Pseudo-Wishart Distributions).** *Let $S \sim \mathcal{W}_m(q,k,\Sigma)$ (or $\sim \mathcal{P}W_m(q,k,\Sigma)$) and define $Z = S^+$. Then, the density function $Z$ is given by*

$$dF_Z(Z) = \frac{\pi^{k(q-r)/2}\left(\prod_{i=1}^{q} t_i^{-(k+3m-2q+1)/2}\right)}{2^{kr/2}\Gamma_q\left[\frac{1}{2}k\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \operatorname{etr}\left(-\tfrac{1}{2}\Sigma^- Z^+\right)(dZ), \quad (4.1)$$

*where $Z = H_1TH_1$ is the non-singular part of the spectral decomposition of $Z$, with $T = \operatorname{diag}(t_1,\dots,t_q) \in \mathcal{D}(q)$, $H_1 \in \mathcal{V}_{q,m}$, $\lambda_i$, $i = 1,\dots r$ are the non-null eigenvalues of $\Sigma$ and where the measure $(dU)$ is explicitly given by*

$$(dU) = 2^{-m}\prod_{i=1}^{q} t_i^{m-q}\prod_{i<j}(t_i - t_j)(H_1' dH_1) \wedge \bigwedge_{i=1}^{q} dt_i. \quad (4.2)$$

*Proof.* The proof is immediate from Proposition 2.3 and from Corollary 3.1. □

**Remark 4.1.** *Following the notation in Press [p. 117] [19], we have, after defining $\Sigma^- = G$ and $k = k - m - 1$, the distribution of $Z$ is called the Central Generalised Inverse Wishart (or Pseudo-Wishart) of rank $q$, with $k$ degrees of freedom and a matrix of scale $G$, this fact being denoted by $Z \sim \mathcal{W}_m^+(q,k,G)$ (or by $Z \sim \mathcal{P}W_m^+(q,k,G)$). This distribution and some of its properties have been studied by various authors considering the Wishart central non-singular case ($m \le N = k$, $q = r = m$), see example Press [Section 5.2] [19], Box and Tiao [p.460] [2] or Gupta and Nagar [Section 3.4] [14]. It has been studied in every case by Díaz-García et al. [11].*

**Theorem 4.2 (Generalised Inverse Beta Distribution).** *Assume that $U \sim \mathcal{B}_m(q,n/2,p/2)$ and define $W = U^+$. Then, $W$ is said to have an $m$-dimensional generalised inverse Beta distribution of rank $q$ and parameters $n/2$ and $p/2$, this fact being denoted by $W \sim \mathcal{B}_m^+(q,n/2,p/2)$. Moreover, its density function is given by*

$$dF_W(W) = \pi^{(-mn+nq)/2}\frac{\Gamma_m\left[\frac{1}{2}(n+p)\right]}{\Gamma_q\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]}\left(\prod_{i=1}^{q}\kappa_i^{-(n+3m-2q+1)/2}\right)$$
$$|I_m - W^+|^{(p-m-1)/2}(dW)$$

*where $W = H_1D_\kappa H_1'$ is the non-singular part of the spectral decomposition, with $H_1 \in \mathcal{V}_{q,m}$, $D_\kappa = \operatorname{diag}(\kappa_1,\dots,\kappa_q) \in \mathcal{D}(q)$, $\kappa_1 > \cdots > \kappa_q > 1$ and $(dW)$ is given in a way analogous to that of (4.2).*

*Proof.* The proof follows from Proposition 2.4 and from Corollary 3.1, noting that $\kappa_i^{-1} = l_i$, $i = 1, \dots, q$. $\square$

The distribution in Theorem 4.2 generalizes various results that have been studied in the literature, for the case in which $W \in \mathcal{S}_m$, see for example Fang and Zhang [p. 115] [12] and Xu [23], among others.

The generalised inverse $F$ distribution is obtained in an analogous way to that of the generalised inverse Beta distribution.

**Theorem 4.3 (Generalised Inverse F Distribution).** *Define $\Delta = F^+$ such that $F \sim \mathcal{F}_m(q, n/2, p/2)$. Then, $\Delta$ is said to have an $m$-dimensional generalised inverse $F$ distribution of rank $q$ and parameters $n/2$ and $p/2$, this fact being denoted by $\Delta \sim \mathcal{F}_m^+(q, n/2, p/2)$. Moreover, its density function is given by*

$$dT_{\Delta}(\Delta) = \pi^{(-mn+nq)/2} \frac{\Gamma_m\left[\frac{1}{2}(n+p)\right]}{\Gamma_q\left[\frac{1}{2}n\right]\Gamma_m\left[\frac{1}{2}p\right]} \left(\prod_{i=1}^{q} \rho_i^{-(n+3m-2q+1)/2}\right)$$

$$|I_m + \Delta^+|^{(p-n)/2}(d\Delta)$$

*where $\Delta = H_1 D_\rho H_1'$ is the non-singular part of the spectral decomposition, with $H_1 \in \mathcal{V}_{q,m}$, $D_\rho = \operatorname{diag}(\rho_1, \dots, \rho_q) \in \mathcal{D}(q)$ and $(dW)$ is given in a way analogous to that of (4.2).*

*Proof.* The proof is immediate from Proposition 2.5 and from Corollary 3.1. $\square$

Finally, note that the distributions of the eigenvalues of the matrices $Z$ and $\Delta$ in Theorems 4.2 and 4.3, respectively, are found in analogous form in the proof of Theorem 4 in Díaz-García and Gutiérrez [9].

## 5 Some Applications

In this section, we apply some of the results described in previous sections in the context of Bayesian inference. These results were presented by Xu [23] for the non-singular case.

**Theorem 5.1.** *Assume that* $A \sim \mathcal{W}_m^+(r,n,V)$*,* $V > I_m$ *and that the a priori distribution of* $V$ *is* $\mathcal{B}_m^+(r,a,b)$*, such that* $a+b-m+r = (n-m-1)/2$ *and* $n \geq 3m+1$*. Then the a posteriori density of* $V|A$ *is*

$$dP(V|A) = \frac{\prod_{i=1}^r (\varkappa_i - 1)^{b-(m+1)/2}}{2^{br}\Gamma_r[b]\prod_{i=1}^r l_i^b} \operatorname{etr}\left(-\tfrac{1}{2}(V-I_m)A^+\right)(dV)$$

*where* $V = R_1 D_\varkappa R_1'$ *is the non-singular part of the spectral decomposition with* $R_1 \in \mathcal{V}_{r,m}$ *and* $D_\varkappa = \operatorname{diag}(\varkappa_1,\dots,\varkappa_r) \in \mathcal{D}(r)$*,* $\varkappa_1 > \cdots > \varkappa_r > 1$*;* $A = H_1 L H_1'$ *is the non-singular part of the spectral decomposition with* $H_1 \in \mathcal{V}_{r,m}$*,* $L = \operatorname{diag}(l_1,\dots l_r) \in \mathcal{D}(r)$ *and* $(dV)$ *is defined in an analogous way to (4.2).*

*Proof.* Assume that $p(A|V)$ denotes the conditional density of $A|V$ and that $\pi(V)$ is the a priori density of $V$. Then the density of $A$ is given by

$$\begin{aligned} dP(A) &= \int_{V>I} p(A|V)\pi(V)(dV)(dA) \\ &= \mathbf{C}\int_{V>I}\prod_{i=1}^r \varkappa_i^{(n-m-1)/2-a-(3m-2r+1)/2}|I_m - V^+|^{b-(m+1)/2} \\ &\qquad\qquad \operatorname{etr}\left(\tfrac{1}{2}VA^+\right)(dV)(dA) \end{aligned}$$

where

$$\mathbf{C} = \frac{\pi^{(-ma+ar)/2}\Gamma_m[a+b]}{2^{(n-m-1)/2}\Gamma_m\left[\frac{1}{2}(n-m-1)\right]\Gamma_r[a]\Gamma_m[b]\prod_{i=1}^r l_i^{(n+2m-2r)/2}}.$$

Now, note that

$$|I_m - V^+| = |I_m - R_1 D_\varkappa^{-1} R_1'| = |I_r - D_\varkappa^{-1}| = |V - I_m|\prod_{i=1}^r \varkappa_i^{-1},$$

from which, given that $a+b-m+r = (n-m-1)/2$,

$$dP(A) = \mathbf{C}\int_{V>I}|V-I_m|^{b-(m+1)/2}\operatorname{etr}\left(\tfrac{1}{2}VA^+\right)(dV)(dA).$$

Let us now define $W = V - I$. Note that this transformation is one-to-one on the $\mathcal{S}_m^+(r)$ manifold, even when $V - I_m \in \mathcal{S}_m$. Then $(dW) = (dV)$, restricting W to the $\mathcal{S}_m^+(r)$ manifold. Thus

$$\begin{aligned}
dP(A) &= \mathbf{C}\int_{W\in\mathcal{S}_m^+(q)}\prod_{i=1}^r \alpha_i^{b-(m+1)/2}\operatorname{etr}\left(\tfrac{1}{2}(I_m+W)A^+\right)(dW)(dA)\\
&= \mathbf{C}\operatorname{etr}\left(\tfrac{1}{2}A^+\right)\int_{W\in\mathcal{S}_m^+(q)}\prod_{i=1}^r \alpha_i^{b-(m+1)/2}\operatorname{etr}\left(\tfrac{1}{2}WA^+\right)(dW)(dA)\\
&= \mathbf{C}\ 2^{br}\Gamma_r[b]\prod_{i=1}^r l_i^b\operatorname{etr}\left(\tfrac{1}{2}A^+\right).
\end{aligned}$$

From which, finally, the a posteriori distribution of $V|A$ is

$$\begin{aligned}
dP(V|A) &= \frac{p(A|V)\pi(V)}{p(A)}(dV)\\
&= \frac{\prod_{i=1}^r(\varkappa_i-1)^{b-(m+1)/2}}{2^{br}\Gamma_r[b]\prod_{i=1}^r l_i^b}\operatorname{etr}\left(-\tfrac{1}{2}(V-I_m)A^+\right)(dV)
\end{aligned}$$

□

**Remark 5.1.** *Note that the conclusion in Theorem 5.1 can be expressed, alternatively, as*

$$(V-I_m)|A\sim\mathcal{W}_m(r,2b,A)$$

*But note, too, that even when $W=(V-I_m)>0$, this transformation is one-to-one on the $\mathcal{S}_m^+(r)$ manifold.*

**Theorem 5.2.** *If $A\sim\mathcal{W}_m(r,n,B^+)$, $n\geq 2m$, $B>I_m$, such that the a priori distribution for $B$ is $\mathcal{B}_m^+(r,a,b)$, with $a+b-m-r=n/2$. Then the a posteriori distribution for $B$ given $A$ is*

$$dP(B|A)=\frac{\pi^{b(m-r)}|B-I_m|^{b-(m+1)/2}}{2^{br}\Gamma_m[b]\prod_{i=1}^r l_i^{-b}}\operatorname{etr}\left(-\tfrac{1}{2}(B-I_m)A\right)(dB).$$

*That is*

$$(B-I_m)|A\sim\mathcal{W}_m(r,2b,A^+)\quad \textit{for}\quad (B-I)|_{\mathcal{S}_m^+(r)}$$

*where $A=H_1LH_1'$ is the non-singular part of the spectral decomposition with $H_1\in\mathcal{V}_{r,m}$, $L=\operatorname{diag}(l_1,\dots l_r)\in\mathcal{D}(r)$ and $(dB)$ is defined in an analogous way to (4.2).*

*Proof.* The proof is analogous to that given for Theorem 5.1. □

Similar results can be expected from assuming the generalised inverse $F$ distribution represented in Theorems 5.1 and 5.2 to correspond to the a priori distributions of $V$ and $B$, respectively.

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