COMPUTATION OF EIGENFRECUENCIES FOR ELASTIC BEAMS, A COMPARATIVE APPROACH

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Abstract

In this manuscript we present an extensive study on mathematical and numerical modeling of flexural vibrations of elastic beams. We consider classical one dimensional models. Three fundamental effects are considered; Bending, Rotary Inertia and Shear. Based on the Wave Propagation Method (WPM), we propose an asymptotic method corrected with a root finding technique to compute eigenfrequencies to any desired accuracy. This method is applied successfully to equations involving bending and either rotary inertia and shear.

Chapter 1

Introduction

Flexural motion of elastic beams is a problem of interest in structural engineering. In particular, engineers need to calculate the natural frequencies, or eigenfrequencies, of beam elements, that is, frequencies at which the elastic beam freely vibrate. The reason is that another part of the system may force it to vibrate at a frequency near one of its natural frequencies. If so, resonance brings about a large amplification of the forcing amplitude with potentially disastrous consequences.

The most realistic and accurate approach for computing eigenfrequencies is to model the elastic beam based on the fundamentals of elasticity theory, see Malvern [7],Fung [8]. Then compute eigenfrequencies by means of the finite element method (FEM) Zienkiewicz & Taylor[10], Bathe [11]. The model is three dimensional and consequently, the computational cost is high.

In applications one dimensional models are preferred. Three fundamental effects are considered; Bending, Rotary Inertia and Shear. Following Russell [4] or Achenbach [1] we may consider the energy of the system to obtain the *Timoshenko system (TS)*

$$\rho\frac{\partial^2 Y}{\partial t^2} - K\frac{\partial^2 Y}{\partial x^2} + K\frac{\partial \varphi}{\partial x} = 0$$

$$I_\rho\frac{\partial^2 \varphi}{\partial t^2} - EI\frac{\partial^2 \varphi}{\partial x^2} + K\left(\varphi - \frac{\partial Y}{\partial x}\right) = 0$$

Here $Y(x,t)$ represents the vertical displacement of the *elastic axis* of the beam, and $\varphi(x,t)$ is the rotation angle due to bending and shear.

The physical constants in the model are: $\rho \equiv$ linear density, $EI \equiv$ flexural rigidity, $I_\rho \equiv$ rotary inertia and $K \equiv$ shear modulus.

By formal differentiation the system can be uncoupled to obtain the *Timoshenko equation (TE)*

$$\rho\frac{\partial^2 Y}{\partial t^2} - I_\rho\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\frac{\partial^4 Y}{\partial x^4} + \frac{\rho}{K}\left(I_\rho\frac{\partial^4 Y}{\partial t^4} - EI\frac{\partial^4 Y}{\partial t^2 \partial x^2}\right) = 0. \tag{1.1}$$

In this equation $-I_\rho\dfrac{\partial^4 Y}{\partial t^2 \partial x^2}$ is the contribution of rotary inertia and the term due to shear is $\dfrac{\rho}{K}\left(I_\rho\dfrac{\partial^4 Y}{\partial t^4} - EI\dfrac{\partial^4 Y}{\partial t^2 \partial x^2}\right)$. If both effects are neglected we obtain the well known Euler-Bernoulli equation

$$\rho\frac{\partial^2 Y}{\partial t^2} + EI\frac{\partial^4 Y}{\partial x^4} = 0 \ .$$

Equation (1.1) is also obtained from equilibrium considerations if shear and moment are assumed to arise from a distributed applied normal load and moment, each per unit length, see Traill-Nash & Collar [6]. Therein a frequency study is presented for the Timoshenko equation, the work is limited by the computational resources of the time. A more qualitatively study of the fundamental effects in beam theory is presented in Stephen [5]

The works mentioned above,are only a few of the vast literature on the subject. Nevertheless, the fundamental problem from the modeling point of view, still of current interest, is to asses the accuracy of the different models obtained from the Timoshenko system and Timoshenko equation when some, or all, of the effects are considered. Our purpose is to discuss this problem in the context of natural frequencies and modes of vibration of the Timoshenko equation. Also, we present a method to compute eigenfrecuencies to any desired accuracy.

For the Euler-Bernoulli equation, Chen & Coleman [2] apply the Wave Propagation Method (WPM) to estimate high order eigenfrecuencies, by means of a formal perturbation approach the estimates are improved to include all low order eigenfrecuencies. Instead of this perturbation approach, we propose a bracketing technique. We show that the WPM can be used to enclose the eigenfrecuencies, which are then found by a simple iterative method to any desired accuracy. An advantage of this approach is that generalizes to more general beam equations. In particular, we consider what we call the quasi Timoshenko equations, that is, equations which involve bending and either rotary inertia or shear.

Computing eigenfrecuencies involves the solution of an eigenvalue problem for a differential operator, to make the problem well posed boundary conditions need to be prescribed. Following Chen & Coleman[2] we consider the following configurations: Clamped-Clamped, Clamped-Simply supported, Clamped-Roller Supported, Clamped-Free. It will become apparent that the method applies to any other configuration.

Of interest in its own right, is to provide asymptotic estimates for eigenfrecuencies, see Geist & McLaughlin [3] for such a study in the case of a free-free Timoshenko system. In the proposed method, when enclosing eigenfrecuencies we provide these estimates.

To test and validate our results, we carry out an exhaustive computational study. We consider beams or four geometries: circular, elliptic, rectangular and square. For each geometry we consider three different materials: aluminium, concrete and steel. To test the modeling virtues of 1-D models, we compute the eigenfrecuencies using 3-D elasticity theory and FEM. For cross validation, eigenfrecuencies for 1-D models are computed with FEM and with the asymptotic method to be introduced.

To conclude this introduction, let us discuss in some detail the content of this work.

In Chapter 1 we present the physical foundations of one dimensional elastic beam models. The purpose is to illustrate the main properties of the Timoshenko system and Timoshenko equation, and the role played for the different physical effects under consideration. The Timoshenko system turns out to be a coupled hyperbolic system for displacement an rotation angle. If the solution is smooth the system can be uncoupled leading to the Timoshenko equation. Remarkably, we shall see that the rotation angle also satisfies the Timoshenko equation.

The eigenvalue problem for the Timoshenko equation is the content of Chapter 2. There, we present the mathematical formulation of the problem, and introduce the quasi Timoshenko Equations with corresponding eigenvalue problems. Hereafter we work with the equation in adimensional form and compute normalized eigenfrecuencies. Also in the chapter, there are two sections describing the basics of FEM computation, 1-D and 3-D. We shall note the numerical and computational requirements when using FEM.

In the context of the Euler-Bernoulli equation in the clamped-clamped configuration, we present a simplified version of WPM in Chapter 3. Eigenfrecuencies are the roots of trascendental equations. Rouhly speaking, WPM approximates these trascendental equations, by equations that are solved in explicit form. For the same model we introduce an asymptotic method complemented with a root finding technique to obtain high accuracy for eigenfrecuencies. The WPM is used, not to approximate the roots, but to enclose them. The method allows us to compute eigenfrecuencies of any order at virtually no computational cost. Moreover, asymptotic estimates are derived leading to a qualitatively description of the relationship between the physical parameters and eigenfrecuencies of beams.

In Chapter 4 we complete the study of the Euler-Bernoulli equation for the remaining configurations, and show the same analysis for the incomplete Timoshenko equations.

Extension of our work, as well as some problems of future research are part of the content of Chapter 5 entitled Concluding Comments.

The numerical results for all beams are presented in the last chapter. Frequencies are normalized, thus frequencies for an actual beam can be easily derived.

Chapter 2

One dimensional models

In practice, elastic beams are modeled as 1-dimensional structures assuming that cross sections move as a whole. The effects to be considered are: bending, rotary inertia and shear. For modelling we select an element of the beam and obtain the equations of motion following two approaches. In the first approach we state equilibrium of forces in the element leading us to the *Timoshenko equation*. For the second approach we propose an energy functional whose minimizer satisfies the so called *Timoshenko System*. The former is a fourth order hyperbolic equation on displacement, the latter is a second order hyperbolic systems on displacement and angle of rotation. As a consequence, there is a great difference on the formulation of the initial and boundary value problems of the equations themselves, as well as the formulation of the corresponding eigenvalue problems. We shall explore these differences in what follows.

It is our purpose to provide a tool to compute eigenfrecuencies of beams in a straightforward fashion, hence units and dimensions are necessary. We list the physical parameters under consideration, as well as their dimensions. The dimensions of the quantities are given in term of mass, length, and time, denoted as usual by M, L, T respectively; dimensionless quantities are denoted by a unit.

Symbol	Dimensions	Definition
EI	ML^3T^{-2}	Bending stiffness
K	MLT^{-2}	Shear modulus
$F(x,t)$	MLT^{-2}	Distributed moment, per unit length
$f(x,t)$	MT^{-2}	Distributed force, per unit length
κ	L	Radius of gytarion of beam section
L	L	Length of beam
M	ML^2T^{-2}	Applied moment
Q	MLT^{-2}	Applied shear force
t	T	Time
x	L	Length coordinate
$Y(x,t)$, $y(x,t)$	L	Lateral displacement
$Z(x,t)$, $z(x,t)$	L	Component of lateral displacement due to bending
ω	T^{-1}	Circular frequency of vibration
ρ	ML^{-1}	Linear density of the beam
I_ρ	ML	Rotatory inertia
θ	1	Amplitude of shearing angle

2.1 The Timoshenko equation

Consider an element of a uniform beam. The beam is originally straight and lies along the x-axis. Let us assume that the shear force Q and the moment M are positive in the $y-$ direction, and in the sense from x to y, respectively. If the shear stiffness of the section is K, the angle of shear is Q/C; if this angle is added to that arising from bending, namely the slope $\partial Z/\partial x$, we have as the total angle

$$\frac{\partial Y}{\partial x} = \frac{\partial Z}{\partial x} + \frac{Q}{K}$$

further, since EI is the bending stiffness, the Euler-Bernoulli curvature formula is

$$\frac{\partial^2 Z}{\partial x^2} = \frac{M}{EI}$$

The main assumption here is that shear and moment arise from a distributed applied normal load $f(x,t)$ and moment $F(x,t)$, each per unit length.

Equilibrium considerations for the element give

$$\frac{\partial Q}{\partial x} + f = 0$$

$$Q + \frac{\partial M}{\partial x} + F = 0$$

By differentiation and substitution

$$\frac{\partial^2 Y}{\partial x^2} = \frac{M}{EI} - \frac{1}{K} f$$

$$-f + \frac{\partial^2 M}{\partial x^2} + \frac{\partial F}{\partial x} = 0$$

and eliminating M

$$\frac{\partial^4 Y}{\partial x^4} = \frac{1}{EI} f - \frac{1}{EI}\frac{\partial F}{\partial x} - \frac{1}{K}\frac{\partial^2 f}{\partial x^2}$$

This equation defines the displacement Y in terms of the applied force and moment distributions. When this arise from inertia loads, that is, are reversed mass acceleration, we have

$$f(x,t) = -\rho \frac{\partial^2 Y}{\partial t^2}$$

$$F(x,t) = -I_\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial Z}{\partial x}\right)$$

then we have

$$\frac{\partial F}{\partial x} = -I_\rho \frac{\partial^2}{\partial t^2}\left(\frac{M}{EI}\right) = -I_\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 Y}{\partial x^2} - \frac{\rho}{K}\frac{\partial^2 Y}{\partial t^2}\right)$$

leading to the Timoshenko equation

$$\rho \frac{\partial^2 Y}{\partial t^2} - I_\rho \frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI \frac{\partial^4 Y}{\partial x^4} + \frac{\rho}{K}\left(I_\rho \frac{\partial^4 Y}{\partial t^4} - EI \frac{\partial^4 Y}{\partial t^2 \partial x^2}\right) = 0.$$

This equation incorporates bending, rotary inertia and shear.

When vibration is only due to bending we obtain the Euler-Bernoulli equation

$$\rho\frac{\partial^2 Y}{\partial t^2} + EI\frac{\partial^4 Y}{\partial x^4} = 0. \tag{2.1}$$

If shear is neglected we obtain the Rayleigh equation

$$\rho\frac{\partial^2 Y}{\partial t^2} - I_\rho\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\frac{\partial^4 Y}{\partial x^4} = 0$$

For convenience we shall refer to this equation sometimes as B+R

In some problems shear is more important than rotary inertia. Neglecting the latter we obtain the equation B+S

$$\rho\frac{\partial^2 Y}{\partial t^2} - \frac{\rho EI}{K}\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\frac{\partial^4 Y}{\partial x^4} = 0.$$

2.2 The Timoshenko system

The E-B equation is also obtained as the simplest equations in the theory of flexural motions of beams of arbitrary but uniform cross section with a plane of simmetry. More precisely, it is assumed that the dominant displacement component is parallel to the plane of simmetry. It is also assumed that the deflections are small and that cross-sectional areas remain plane and normal to the neutral axis. Equation (2.1), is the equation for a beam which is free of lateral loading.

Substituting a harmonic wave, we find for the phase velocity

$$c = \left(\frac{EI}{\rho}\right)^{1/2}\omega \tag{2.2}$$

Thus the phase velocity is proportional to the wave number, which suggests that (2.2) cannot be correct for large wavenumbers (short waves).

By taking in to account shear deformation in the description of the flexural motion of a rod we obtain a model which yields more satisfactory results for shorter wavelengths. In this model it is still assumed that plane sections remain plane; it is, however, not assumed that plane sections remain normal to the neutral plane. After deformation the neutral axis has been rotated through the small angle $\partial Y/\partial x$, while the cross section has been rotated through the angle φ. The shearing angle θ is the net decrease angle

$$\theta = \frac{\partial Y}{\partial x} - \varphi$$

The bending moment acting over the cross section is related to φ by the relation

$$M = -EI\frac{\partial \varphi}{\partial x} \tag{2.3}$$

The relation between the shear force Q and the angle θ is

$$Q = K\theta \tag{2.4}$$

where K is a numerical factor which reflects the fact that the beam is not in a state of uniform shear, but that (2.3) represents a relation between the resultant shear force and some kind of average shear angle. The factor K depends on the cross-sectional shape and on the rationale adopted in the averaging process. Fortunately there is not very much spread in the values of K obtained by different averaging processes. The factor does, however, depend noticeably on the shape of the cross section.

By employing (2.3) and (2.4) the strain energy of a finite segment can be computed as

$$U = \int_{x_1}^{x_2} \left[\frac{1}{2} EI \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} K \left(\frac{\partial Y}{\partial x} - \varphi \right)^2 \right] dx$$

The corresponding kinetic energy is

$$\mathcal{K} = \int_{x_1}^{x_2} \left[\frac{1}{2} \rho \left(\frac{\partial Y}{\partial t} \right)^2 + \frac{1}{2} I_\rho \left(\frac{\partial \varphi}{\partial t} \right)^2 \right] dx$$

The total energy is

$$E = \int_{x_1}^{x_2} \left[\frac{1}{2} \rho \left(\frac{\partial Y}{\partial t} \right)^2 + \frac{1}{2} I_\rho \left(\frac{\partial \varphi}{\partial t} \right)^2 + \frac{1}{2} EI \left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{2} K \left(\frac{\partial Y}{\partial x} - \varphi \right)^2 \right] dx$$

Subsequently Hamilton's principle and the Euler equations can be employed to obtain the following set of governing equations for a homogeneous beam

$$\rho\frac{\partial^2 Y}{\partial t^2} - K\frac{\partial^2 Y}{\partial x^2} + K\frac{\partial\varphi}{\partial x} = 0$$

$$I_\rho\frac{\partial^2\varphi}{\partial t^2} - EI\frac{\partial^2\varphi}{\partial x^2} + K\left(\varphi - \frac{\partial Y}{\partial x}\right) = 0$$

As inspection quickly shows, these equations have the form of two coupled wave equations. In general the wave speeds $\sqrt{K/\rho}$ and $\sqrt{EI/I_\rho}$ are different, resulting in a hyperbolic system with four families of characteristic curves, consisting of two pairs corresponding to wave velocities $\pm\sqrt{K/\rho}$ and $\pm\sqrt{EI/I_\rho}$.

If shear is not considered, then the angle $\theta = 0$ and $\varphi = \partial Y/\partial x$. The energy of the system is

$$E = \int_{x_1}^{x_2}\left[\frac{1}{2}\rho\left(\frac{\partial Y}{\partial t}\right)^2 + \frac{1}{2}I_\rho\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2}EI\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dx$$

which leads to the Rayleigh equation

$$\rho\frac{\partial^2 Y}{\partial t^2} - I_\rho\frac{\partial^4 Y}{\partial t^2\partial x^2} + EI\frac{\partial^4 Y}{\partial x^4} = 0$$

This model involves the rotary inertia effect. When this is neglected, the energy is

$$E = \int_{x_1}^{x_2}\left[\frac{1}{2}\rho\left(\frac{\partial Y}{\partial t}\right)^2 + \frac{1}{2}EI\left(\frac{\partial\varphi}{\partial x}\right)^2\right]dx$$

and we recover the E-B equation

$$\rho\frac{\partial^2 Y}{\partial t^2} + EI\frac{\partial^4 Y}{\partial x^4} = 0$$

2.3 Fourth order equations

If the functions are smooth, it is possible to derive the Timoshenko equation from the Timoshenko system. Indeed, let us recall the Timoshenko system

$$
\begin{aligned}
&\rho\frac{\partial^2 Y}{\partial t^2} - K\frac{\partial^2 Y}{\partial x^2} + K\frac{\partial \varphi}{\partial x} = 0 \\
&I_\rho\frac{\partial^2 \varphi}{\partial t^2} - EI\frac{\partial^2 \varphi}{\partial x^2} + K\left(\varphi - \frac{\partial Y}{\partial x}\right) = 0
\end{aligned}
\tag{2.5}
$$

Is the first equation we solve for $\dfrac{\partial \varphi}{\partial x}$ obtaining

$$\frac{\partial \varphi}{\partial x} = -\frac{\rho}{K}\frac{\partial^2 Y}{\partial t^2} + \frac{\partial^2 Y}{\partial x^2}$$

differentiating the second equation with respect to x we have

$$I_\rho\frac{\partial^3 \varphi}{\partial t^2 \partial x} - EI\frac{\partial^3 \varphi}{\partial x^3} + K\left(\frac{\partial \varphi}{\partial x} - \frac{\partial^2 Y}{\partial x^2}\right) = 0$$

after substitution

$$
\begin{aligned}
&I_\rho\left(-\tfrac{\rho}{K}\frac{\partial^4 Y}{\partial t^4} + \frac{\partial^4 Y}{\partial t^2 \partial x^2}\right) - EI\left(-\tfrac{\rho}{K}\frac{\partial^4 Y}{\partial x^2 \partial t^2} + \frac{\partial^4 Y}{\partial x^4}\right) + \\
&+ K\left(-\tfrac{\rho}{K}\frac{\partial^2 Y}{\partial t^2} + \frac{\partial^2 Y}{\partial x^2}\right) - K\frac{\partial^2 Y}{\partial x^2} = 0
\end{aligned}
$$

simplifying

$$\rho\frac{\partial^2 Y}{\partial t^2} - I_\rho\frac{\partial^4 Y}{\partial t^2 \partial x^2} + EI\left(-\frac{\rho}{K}\frac{\partial^4 Y}{\partial x^2 \partial t^2} + \frac{\partial^4 Y}{\partial x^4}\right) + \frac{\rho}{K}\left(I_\rho\frac{\partial^4 Y}{\partial t^4} - EI\frac{\partial^4 Y}{\partial x^2 \partial t^2}\right) = 0$$

as asserted.

It is remarkable that the same equation is obtained for φ. In fact, solving for $\dfrac{\partial Y}{\partial x}$ in the equation for φ in (2.5)

$$\frac{\partial Y}{\partial x} = \frac{I_\rho}{K}\frac{\partial^2 \varphi}{\partial t^2} - \frac{EI}{K}\frac{\partial^2 \varphi}{\partial x^2} + \varphi$$

differentiate with respect to x in the equation for Y in (2.5)

$$\rho\frac{\partial^3 Y}{\partial x \partial t^2} - K\frac{\partial^3 Y}{\partial x^3} + K\frac{\partial^2 \varphi}{\partial x^2} = 0$$

after substitution

$$\rho\left(\frac{I_\rho}{K}\frac{\partial^4\varphi}{\partial t^4}-\frac{EI}{K}\frac{\partial^4\varphi}{\partial t^2\partial x^2}+\frac{\partial^2\varphi}{\partial t^2}\right)-K\left(\frac{I_\rho}{K}\frac{\partial^4\varphi}{\partial x^2\partial t^2}-\frac{EI}{K}\frac{\partial^4\varphi}{\partial x^4}+\frac{\partial^2\varphi}{\partial x^2}\right)+K\frac{\partial^2\varphi}{\partial x^2}=0$$

which leads to

$$\rho\frac{\partial^2\varphi}{\partial t^2}-I_\rho\frac{\partial^4\varphi}{\partial t^2\partial x^2}+EI\left(-\frac{\rho}{K}\frac{\partial^4\varphi}{\partial x^2\partial t^2}+\frac{\partial^4\varphi}{\partial x^4}\right)+\frac{\rho}{K}\left(I_\rho\frac{\partial^4\varphi}{\partial t^4}-EI\frac{\partial^4\varphi}{\partial x^2\partial t^2}\right)=0$$

Remark. It is of interest to study these fourth order equations and compare with Timoshenko System.

Chapter 3

The Eigenvalue Problem

In this chapter we pose the eigenvalue problems of interest. Next, as a basis for comparison with the asymptotic method to be introduced, we describe the basics of 1-D FEM computations.

The benchmark model for an elastic beam is obtained from 3-D elasticity theory. Eigenfrecuencies are computed with FEM. An example of such a computation is presented in the last section.

3.1 Problem Formulation

Let us consider the Timoshenko equation.

$$\rho\frac{\partial^2 Y}{\partial t^2} - I_\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 Y}{\partial x^2}\right) + EI\frac{\partial^4 Y}{\partial x^4} + \frac{\rho}{K}\left(I_\rho\frac{\partial^4 Y}{\partial t^4} - EI\frac{\partial^2}{\partial t^2}\left(\frac{\partial^2 Y}{\partial x^2}\right)\right) = 0$$

under harmonic motion

$$Y(x,t) = y(x)e^{-i\omega t}$$

we have

$$-\rho\omega^2 y + I_\rho\omega^2\frac{d^2y}{dx^2} + EI\frac{d^4y}{dx^4} + \frac{\rho}{K}\omega^2\left(I_\rho\omega^2 y + EI\frac{d^2y}{dx^2}\right) = 0 \qquad (3.1)$$

In dimensionless form

$$\xi = x/L$$
$$\eta = y/L$$
$$\phi^2 = (\rho\omega^2 L^4)/EI$$
$$\alpha = EI/(KL^2)$$
$$\beta = I_\rho/(\rho L^2)$$

where L is some specified length, e.g. the length of the beam.

Equation (3.1) then becomes

$$\frac{d^4\eta}{d\xi^4} + \phi^2(\alpha+\beta)\frac{d^2\eta}{d\xi^2} - \phi^2\left(1 - \phi^2\alpha\beta\right)\eta = 0$$

We are interested in the following boundary conditions

A. Displacement zero: $\eta = 0$

B. Total slope zero: $\dfrac{d\eta}{d\xi} = 0$

C. Slope due to bending only zero: $\alpha\dfrac{d^3\eta}{d\xi^3} + \left(1 + \phi^2\alpha^2\right)\dfrac{d\eta}{d\xi} = 0$

D. Moment zero: $\dfrac{d^2\eta}{d\xi^2} + \phi^2\alpha\eta = 0$

E. Shear zero: $\dfrac{d^3\eta}{d\xi^3} + \phi^2(\alpha+\beta)\dfrac{d\eta}{d\xi} = 0$

To make the eigenvalue problem well posed, two boundary conditions need to be prescribed at both ends. In reference to Table , we consider the following conditions for any end of the beam;

- Clamped: A, B
- Simply sypported: A, D
- Roller supported: B, E
- Free: D, E

3.2 Quasi-Timoshenko equations

We call a quasi-Timoshenko equation, an equation of the form

$$\frac{d^4\eta}{d\xi^4} + \phi^2\gamma\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0 \tag{3.2}$$

where γ is a small nonnegative parameter representing either shear or rotary inertia. Next we present the corresponding equations and boundary conditions.

3.2.1 B + R equation

In this case there is no shear, that is, $\alpha = 0$. Equation (3.1) then becomes

$$\frac{d^4\eta}{d\xi^4} + \phi^2\beta\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0$$

Boundary conditions correspond to

A. Displacement zero: $\eta = 0$

B. Total slope zero: $\frac{d\eta}{d\xi} = 0$

C. Slope due to bending only zero: $\frac{d\eta}{d\xi} = 0$

D. Moment zero: $\frac{d^2\eta}{d\xi^2} = 0$

E. Shear zero: $\frac{d^3\eta}{d\xi^3} + \phi^2\beta\frac{d\eta}{d\xi} = 0$

3.2.2 B + S equation

When $\beta = 0$ there is no rotary inertia effect and the resulting equation is

$$\frac{d^4\eta}{d\xi^4} + \phi^2\alpha\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0$$

The corresponding boundary conditions are given by

A. Displacement zero: $\eta = 0$

B. Total slope zero: $\dfrac{d\eta}{d\xi} = 0$

C. Slope due to bending only zero: $\alpha\dfrac{d^3\eta}{d\xi^3} + \left(1 + \phi^2\alpha^2\right)\dfrac{d\eta}{d\xi} = 0$

D. Moment zero: $\dfrac{d^2\eta}{d\xi^2} + \phi^2\alpha\eta = 0$

E. Shear zero: $\dfrac{d^3\eta}{d\xi^3} + \phi^2\alpha\dfrac{d\eta}{d\xi} = 0$

3.3 1-D FEM computations

To deal with the numerical approximaton of the eigenvalue problem corresponding to equation (3.2), one of the most versatile and accurate method is the Finite Element Method (FEM). Which we use to validate our results.

For the reader's convenience, let us present a brief description of FEM.

Consider the eigenvalue problem for the incomplete Timoshenko equation

$$\frac{d^4\eta}{d\xi^4} = \phi^2\left(-\gamma\frac{d^2\eta}{d\xi^2} + \eta\right), \quad \xi \in (0,1) \tag{3.3}$$

For simplicity of exposition, we consider the equation subject to clamped-clamped boundary conditions, $\eta(0) = \eta'(0) = \eta(1) = \eta'(1)$.

We define the test space V, as the space of functions v satisfying the clamped-clamped boundary conditions $v(0) = v'(0) = v(1) = v'(1) = 0$.

Multiplying equation (3.3) by a test function v we obtain after integration by parts

$$\int\frac{d^2\eta}{d\xi^2}\frac{d^2v}{d\xi^2} = \phi^2\int\left(\gamma\frac{d\eta}{d\xi}\frac{dv}{d\xi} + \eta v\right) \tag{3.4}$$

This leads to the weak formulation of the eigenvalue problem:

Find $\eta \in V$ such that η solves (3.4) for all $v \in V$.

Next we apply the Galerkin method, that is, we construct V_h, a finite dimensional subspace of V and consider the weak formulation in a finite dimensional setting, namely, the Galerkin formulation:

Find $\eta \in V_h$ such is η solves (3.4) for all $v \in V_h$.

If V_h of dimension n, we may consider a basis, say $\{v_1, \dots v_n\}$. The problem is equivalent to finding $\eta \in V_h$ such that

$$\int \frac{d^2\eta}{d\xi^2}\frac{d^2 v_i}{d\xi^2} = \phi^2 \int \left(\gamma \frac{d\eta}{d\xi}\frac{dv_i}{d\xi} + \eta v_i\right), \quad i = 1, 2, \dots n \tag{3.5}$$

Since $\eta \in V_h$ we have

$$\eta(\xi) = \sum_{j=1}^{n} \eta_j v_j(\xi)$$

and (3.5) becomes a generalized eigenvalue problem

$$A\boldsymbol{\eta} = \phi^2 \left(\gamma B + \mathrm{C}\right) \boldsymbol{\eta}$$

where $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)^t$, and

$A = [A_{ij}]$, $A_{ij} = \int v_j'' v_i''$

$B = [B_{ij}]$, $B_{ij} = \int v_j' v_i'$

$C = [C_{ij}]$, $C_{ij} = \int v_j v_i$

It is apparent that to have a good approximation n is necessarily large. Hence the basis of V_h has to be chosen carefully.

With FEM a basis is formed by continuous functions which are compactly supported and locally polynomials. For beam equations a convenient basis is formed by using Hermite polynomials. We proceed as follows. The interval $[0, 1]$, is partitioned into nonoverlapping subintervals (elements), $I_1 = [\xi_0, \xi_1], \dots I_n = [\xi_{n-1}, \xi_n]$. Here $\xi_0 = 0$ and $\xi_n = 1$. For each node ξ_i, a pair of basis functions, φ_i, ψ_i are selected with the following properties:

1. $\varphi_i(\xi_j) = \delta_{ij}$
2. $\varphi_i'(\xi_j) = 0$
3. $supp(\varphi_i) = I_i \cup I_{i+1}$
4. $\varphi_i|_{I_j}$ a cubic polynomial for $j = i, i+1$
5. $\psi_i(\xi_j) = 0$

6. $\psi_i'(\xi_j) = \delta_{ij}$

7. $supp(\psi_i) = I_i \cup I_{i+1}$

8. $\psi_i|_{I_j}$ a cubic polynomial for $j = i, i+1$

The resulting eigenvalue problem involves banded matrices which are symmetric and positive definite. These properties make tractable the numerical approximation.

3.4 3-D Theory

Let us consider a rectangular beam in the clamped-clamped configuration with dimensions 30 cm width, 50 cm height, 300 cm length.. The material properties are: Young Modulus $E = 200,000 kg/cm^3$, Poisson ratio $\nu = 0.2$, density $d = 2.449x10^{-6} \frac{kg-s^2}{cm}$. Using the 3-D FEM model, Botello& Oñate [12], the estimated values for the first frequencies are shown in Table 2.1.

Frec.	3-D FEM (rad/seg)
1	902.690
2	2153.09
3	3680.71
4	5354.43
5	7119.67
6	9121.56
7	10783.9
8	12703.8
9	13337.9
10	14619.1

TABLE 2.1

The computation was carried out with a grid of 182406 tetrahedral elements and linear shape functions. To complete the 3-D model we need modes of vibration, a few are shown in Figure 2.1.

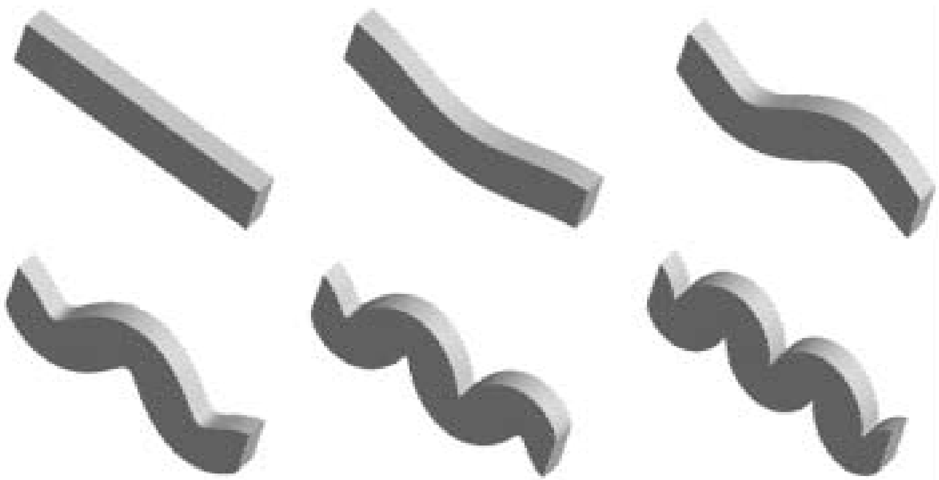

Figure 2.1. Fundamental modes of vibration.

Is worth to mention that to estimate the eigenfrecuencies and modes of vibration of a particular elastic beam, a lab experiment can be developed. Unfortunately, the cost to generate a table and figure like the ones shown above is prohibited.

To obtain a similar degree of numerical accuracy with one dimensional models we need no more than 150 elements. The number of elements reflects, not only the size of the linear system to solve, but also the numerical integrations to be computed.

One may argue that the most difficult part for applying FEM, is the partition of the domain in finite elements. For 1-D models this is trivial.

We consider the eigenfrecuencies of the 3-D model as the actual eigenfrecuencies of vibrations of elastic beams, and discuss the modelling virtues of the various one dimensional models presented above.

Chapter 4

Asymptotic Methods

With the appropriate boundary conditions, the eigenvalue problem for the quasi-Timoshenko equation has eigenvalues $0 < \phi_1 < \phi_2 < \ldots < \phi_n$, with $\phi_n \nearrow \infty$. The corresponding mode of vibration $\eta_n(x)$ is the sum of two functions, one of which decays rapidly with ϕ_n. The Wave Propagation Method (WPM) disregards the latter to estimate the eigenvalue ϕ_n. A formal perturbation is proposed in Chen & Coleman [2] to improve the estimate by WPM for eigenfrecuencies of the Euler-Bernoulli beam. Here we present an alternative approach to improve thr results of WPM also for the quasi-Timoshenko equation.

4.1 The wave propagation method (WPM)

To illustrate the WPM we consider the E-B equation

$$\frac{d^4\eta}{d\xi^4} - \phi^2\eta = 0$$

subject to clamped-clamped conditions

$$\eta(0) = \eta'(0) = \eta(1) = \eta'(1) = 0 \tag{4.1}$$

For simplicity we write

$$\eta^{(4)}(\xi) - k^4\eta(\xi) = 0, \qquad 0 < x < 1, \tag{4.2}$$

where $k^2 = \phi$, $k > 0$.

The eigenvalue problem, therefore, is to determine all values of k which satisfy equation (4.2), subject to the boundary conditions given in equation (4.1).

Is is well known that the eigenvalue problem does not have any closed form solutions. We first recall the standard, straightforward approach to determine k that is familiar. For $k > 0$ the general solution of equation (4.2) is

$$\eta(\xi) = A\cos k\xi + B\sin k\xi + Ce^{-k\xi} + De^{k\xi}$$

substitute this equation into the (C-C) boundary conditions in (4.1), yielding

$$\begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ \cos k & \sin k & e^{-k} & 1 \\ -k\sin k & k\cos k & -ke^{-k} & k \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In order to have a non-trivial solution, k satisfies the transcendental equation determined by the zero determinant condition

$$\begin{vmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ \cos k & \sin k & e^{-k} & e^{k} \\ -k\sin k & k\cos k & -ke^{-k} & ke^{k} \end{vmatrix} = 0 \qquad (4.3)$$

or

$$-2k^2\cos k + 4k^2e^{-k} - 2k^2\cos k\, e^{-2k} = 0$$

Hence, we need to find roots of the equation

$$-\cos k + 2e^{-k} - \cos k\, e^{-2k} = 0 \qquad (4.4)$$

An expression of k from equation (4.4) is not possible, an asymptotic approach to estimate the solution by means of the WPM is shown below..

Let us write the solution $\eta(\xi)$ in the form

$$\eta(\xi) = A\cos k\xi + B\sin k\xi + Ce^{-k\xi} + De^{k(\xi-1)}$$

Observe that for k large, the third term $e^{-k\xi}$ is negligible for $\xi = 1$, whereas the same is true for the fourth term $e^{-k(\xi-1)}$ if $\xi = 0$. Hence the function $\eta(\xi)$ behaves like $A\cos k\xi + B\sin k\xi + Ce^{-k\xi}$ forξ near zero, and like $A\cos k\xi + B\sin k\xi + De^{-k(\xi-1)}$ for ξ near one. This suggests to disregard the terms involving e^{-k} in the determinant equation (4.3). Thus we have

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & k & -k & 0 \\ \cos k & \sin k & 0 & e^k \\ -k\sin k & k\cos k & 0 & ke^k \end{vmatrix} = 0$$

After some simplification, we are led to solve for k the equation

$$\cos k = 0$$

Consecuently, the eigenvalue problem

$$\eta^{(4)}(\xi) - k^4\eta(\xi) = 0, \qquad 0 < \xi < 1,$$

$$\eta(0) = \eta'(0) = \eta(1) = \eta'(1) = 0$$

has non-trivial solution η when

$$\phi^2 \approx k^4 = \left[(2n+1)\frac{\pi}{2}\right]^4, \qquad n = 1, 2, \ldots$$

or

$$\phi \approx k^2 = \left[(2n+1)\frac{\pi}{2}\right]^2, \qquad n = 1, 2, \ldots$$

As we can see from Table 3.1, the adimensional frequencies in this expression gives good estimates except for a few of the smallest eigenvalues.

Frec.	1-D FEM	WPM
1	22.37329	22.20660
2	61.67282	61.68503
3	120.90339	120.90265
4	199.85946	199.85949
5	298.55557	298.55553
6	416.99089	416.99079
7	555.16548	555.16525
8	713.07941	713.07892
9	890.73277	890.73180
10	1088.12565	1088.12389

Table 3.1

Remark. The same conclusion holds for other boundary conditions, that is, the WPM gives good estimates for all but a few low order eigenfrecuencies.

4.2 High accuracy of eigenfrecuencies by the WPM and bracketing

Instead of the perturbation approach in Chen & Coleman [2], we improve the aproximation of the eigenvalues by applying a simple iterative method. We illustrate the technique with the Euler-Bernoulli equation in the C-C case.

Substituting the C-C boundary conditions for

$$\eta(\xi) = A\cos k\xi + B\sin k\xi + Ce^{-k\xi} + De^{k(\xi-1)}$$

we obtain

$$\begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ \cos k & \sin k & e^{-k} & 1 \\ -k\sin k & k\cos k & -ke^{-k} & k \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Let

$$f_d(k) = \det \begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ \cos k & \sin k & e^{-k} & 1 \\ -k\sin k & k\cos k & -ke^{-k} & k \end{bmatrix}$$

thus

$$f_d(k) = k^2 \det \begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & 1 & -1 & e^{-k} \\ \cos k & \sin k & e^{-k} & 1 \\ -\sin k & \cos k & -e^{-k} & 1 \end{bmatrix}$$

or

$$f_d(k) = -2k^2 \left[\left(1 + e^{-2k}\right)\cos k - 2e^{-k}\right] \tag{4.5}$$

>From (4.5) it suffices to find zeros of the function

$$f(k) = \left(1 + e^{-2k}\right) \cos k - 2e^{-k} \tag{4.6}$$

It is readily seen that in the intervals

$$n\pi \leq k \leq (n+1)\pi \qquad n = 1, 2, \ldots \tag{4.7}$$

$f(k)$ is strictly monotone and $f(n\pi) f((n+1)\pi) < 0$, hence, there is only one root of $f(k)$ in such intervals.

Recall that $\phi = k^2$ hence ϕ is monotone on k. Thus, the interval (4.7) provides asymptotic estimates for the eigenfrecuency

$$n^2\pi^2 \leq \phi \leq (n+1)^2\pi^2 \qquad n = 1, 2, \ldots$$

In this case improving these estimates is a trivial matter.

The roots of function (4.6) can be found by bisection to any desired accuracy. See the results in Table 3.2 for the adimensional natural frecuencies.

Frec.	1-D FEM	WPMB
1	22.37329	22.37329
2	61.67282	61.67282
3	120.90339	120.90339
4	199.85946	199.85945
5	298.55557	298.55554
6	416.99089	416.99079
7	555.16548	555.16525
8	713.07941	713.07892
9	890.73277	890.73180
10	1088.12565	1088.12389

Table 3.2

Remark. By considering the full determinat function, the WPMB is more accurate than 1-D FEM. Starting on the fourth eigenfrecuency, there is a slight difference on the estimates for the eigenfrecuencies. This is due to the accumulation of error when solving the generalized eigenvalue problem arising from FEM.

Chapter 5

Eigenfrecuencies of Elastic Beams by WPM and Bracketing

In this chapter we use WPM to estimate the intervals to initiate the iterative search of eigenfrecuencies for all beams and configurations of interest.

In what follows we denote by f_d the full determinant function of the beam in consideration, and by f the function for root finding.

For each beam and configuration we list the matrix of the corresponding homogeneous system, $f_{d,}$ f, and the intervals enclosing the eigenfrecuencies.

Computations are straightforward, when necessary we provide additional details.

5.1 The Euler-Bernoulli Equation

5.1.1 The (C-C) case

$>$From the previous chapter we have

$$f_d(k) = f(k) = \left(1 + e^{-2k}\right)\cos k - 2e^{-k}$$

and

$$n\pi \leq k \leq (n+1)\pi \qquad n = 1, 2, \ldots$$

5.1.2 The (C-S) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ \cos k & \sin k & e^{-k} & 1 \\ -k^2 \cos k & -k^2 \sin kx & k^2 e^{-k} & k^2 \end{bmatrix}$$

$$f_d(k) = 2k^3 \left[-\left(1 - e^{-2k}\right) \cos k + \left(1 + e^{-2k}\right) \sin k \right]$$

$$f(k) = -\left(1 - e^{-2k}\right) \cos k + \left(1 + e^{-2k}\right) \sin k$$

$$n\pi < k < \left(\frac{1}{2} + n\right) \pi, \qquad n = 1, 2, \ldots$$

5.1.3 The (C-R) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ -k \sin k & k \cos k & -ke^{-k} & k \\ k^3 \sin k & -k^3 \cos k & -k^3 e^{-k} & k^3 \end{bmatrix}$$

$$f_d(k) = 2k^5 \left[\left(1 - e^{-2k}\right) \cos k + \left(1 + e^{-2k}\right) \sin k \right]$$

$$f(k) = \left(1 - e^{-2k}\right) \cos k + \left(1 + e^{-2k}\right) \sin k$$

$$\left(\frac{1}{2} + n\right) \pi < k < (1 + n)\, \pi, \qquad n = 1, 2, \ldots$$

5.1.4 The (C-F) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-k} \\ 0 & k & -k & ke^{-k} \\ -k^2 \cos k & -k^2 \sin k & k^2 e^{-k} & k^2 \\ k^3 \sin k & -k^3 \cos k & -k^3 e^{-k} & k^3 \end{bmatrix}$$

$$f_d(k) = 2k^6 \left[\left(1 + e^{-2k}\right) \cos k + 2e^{-k} \right]$$

$$f(k) = \left(1 + e^{-2k}\right) \cos k + 2e^{-k}$$

$$n\pi \leq k \leq (n+1)\pi \qquad n = 0, 1, 2, \ldots$$

5.2 Quasi-Timoshenko Equations

Let recall the Timosehnko equation

$$\frac{d^4\eta}{d\xi^4} + \phi^2 (\alpha + \beta) \frac{d^2\eta}{d\xi^2} - \phi^2 \left(1 - \phi^2 \alpha\beta\right) \eta = 0$$

The quasi-Timoshenko equations are

$$\frac{d^4\eta}{d\xi^4} + \phi^2 \beta \frac{d^2\eta}{d\xi^2} - \phi^2 \eta = 0$$

$$\frac{d^4\eta}{d\xi^4} + \phi^2 \alpha \frac{d^2\eta}{d\xi^2} - \phi^2 \eta = 0$$

Both models have the form

$$\frac{d^4\eta}{d\xi^4} + \gamma\phi^2 \frac{d^2\eta}{d\xi^2} - \phi^2 \eta = 0$$

The characteristic polynomial is

$$P(r) = r^4 + \gamma\phi^2 r^2 - \phi^2$$

It has the four roots

$$r_1 = -r_2 = -i\lambda$$

$$r_3 = -r_4 = -\mu$$

Where

$$\lambda = \sqrt{\frac{1}{2}\gamma\phi^2 + \frac{1}{2}\sqrt{\left(\gamma^2\phi^4 + 4\phi^2\right)}} \tag{5.1}$$

$$\mu = \sqrt{-\frac{1}{2}\gamma\phi^2 + \frac{1}{2}\sqrt{\left(\gamma^2\phi^4 + 4\phi^2\right)}} \tag{5.2}$$

For later reference we see that

$$\lambda^2 - \mu^2 = \gamma\phi^2$$

$$\mu\lambda = \phi$$

Intervals for the eigenfrecuencies will be given in terms of λ. Let us deduce some properties of λ and μ as functions of ϕ.

>From (5.1) it is readily seen that $\lambda'(\phi) > 0$, hence λ is strictly increasing and unbounded. It can be inverted to obtain

$$\phi^2 = \frac{\lambda^4}{1 + \gamma\lambda^2} \tag{5.3}$$

Consequently, when finding an interval for λ, a corresponding interval for ϕ is derived from (5.3).

For μ we ca write

$$\mu^2 = \frac{2}{\gamma + \sqrt{\gamma^2 + \frac{4}{\phi}}}$$

We see that μ is also strictly increasing with respect to ϕ. Moreover,

$$\mu \to \frac{1}{\sqrt{\gamma}}, \quad \text{when } \phi \nearrow \infty \tag{5.4}$$

We have the mode of vibration $\eta(\xi)$

$$\eta(\xi) = A\cos\lambda\xi + B\sin\lambda\xi + Ce^{-\mu\xi} + De^{\mu(\xi-1)}$$

Because of (5.4), the term $e^{-\mu}$ does not tend to zero with ϕ, unlike the corresponding term for the Euler-Bernoulli beam. Nevertheless, for actual beams, γ is small, thus $e^{-\mu}$ is small and decreases to $e^{-1/\sqrt{\gamma}}$. Thanks to these properties, we will be able to consider $e^{-\mu}$ negligible.

5.3 The B + R Equation

Here we study the equation

$$\frac{d^4\eta}{d\xi^4} + \phi^2\beta\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0$$

The corresponding boundary conditions are

A. Displacement zero: $\eta = 0$

B. Total slope zero: $\dfrac{d\eta}{d\xi} = 0$

C. Slope due to bending only zero: $\dfrac{d\eta}{d\xi} = 0$

D. Moment zero: $\dfrac{d^2\eta}{d\xi^2} = 0$

E. Shear zero: $\dfrac{d^3\eta}{d\xi^3} + \phi^2\beta\dfrac{d\eta}{d\xi} = 0$

5.3.1 The C-C case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ \cos\lambda & \sin\lambda & e^{-\mu} & 1 \\ -\lambda\sin\lambda & \lambda\cos\lambda & -\mu e^{-\mu} & \mu \end{bmatrix}$$

$$f_d = \phi\left[-\gamma\phi\left(1 - e^{-2\mu}\right)\sin\lambda - 2\left(1 + e^{-2\mu}\right)\cos\lambda + 4e^{-\mu}\right]$$

$$f(\lambda) = \phi\gamma\left(1 - e^{-2\mu}\right)\sin\lambda + 2\left(1 + e^{-2\mu}\right)\cos\lambda - 4e^{-\mu}$$

$$\left(\frac{1}{2} + n\right)\pi < \lambda < (1 + n)\pi, \qquad n = 1, 2, 3, \ldots$$

5.3.2 The (C-S) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ \cos\lambda & \sin\lambda & e^{-\mu} & 1 \\ -\lambda^2\cos\lambda & -\lambda^2\sin\lambda & \mu^2 e^{-\mu} & \mu^2 \end{bmatrix}$$

$$f_d(\lambda) = \left(\mu^2 + \lambda^2\right)\left[\left(1 + e^{-2\mu}\right)\mu\sin\lambda - \left(1 - e^{-2\mu}\right)\lambda\cos\lambda\right]$$

$$f(\lambda) = \left(1 + e^{-2\mu}\right)\mu\sin\lambda - \left(1 - e^{-2\mu}\right)\lambda\cos\lambda$$

$$n\pi < \lambda < \left(\frac{1}{2} + n\right)\pi, \qquad n = 1, 2, 3, \ldots$$

5.3.3 The (C-R) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ -\lambda\sin\lambda & \lambda\cos\lambda & -\mu e^{-\mu} & \mu \\ \lambda^3\sin\lambda - \phi^2\beta\lambda\sin\lambda & -\lambda^3\cos\lambda + \phi^2\beta\lambda\cos\lambda & -\mu^3 e^{-\mu} - \phi^2\beta\mu e^{-\mu} & \mu^3 + \phi^2\beta\mu \end{bmatrix}$$

$$f_d(\lambda) = \lambda\mu\left(\mu^2 + \lambda^2\right)\left(\mu\cos\lambda\left(1 - e^{-2\mu}\right) + \lambda\sin\lambda\left(1 + e^{-2\mu}\right)\right)$$

$$f(\lambda) = \left(\mu\cos\lambda\left(1 - e^{-2\mu}\right) + \lambda\sin\lambda\left(1 + e^{-2\mu}\right)\right)$$

$$\left(\frac{1}{2} + n\right)\pi < \lambda < (1 + n)\pi, \qquad n = 0, 1, 2, \ldots$$

5.3.4 The C-F case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ -\lambda^2\cos\lambda & -\lambda^2\sin\lambda & \mu^2 e^{-\mu} & \mu^2 \\ \lambda^3\sin\lambda-\phi^2\beta\lambda\sin\lambda & -\lambda^3\cos\lambda+\phi^2\beta\lambda\cos\lambda & -\mu^3e^{-\mu}-\phi^2\beta\mu e^{-\mu} & \mu^3+\phi^2\beta\mu \end{bmatrix}$$

$$f_d(\lambda) = \mu\lambda\left[\left(\mu^4+\lambda^4\right)\left(1+e^{2(-\mu)}\right)\cos\lambda - \lambda\mu\left(\lambda^2-\mu^2\right)\left(1-e^{2(-\mu)}\right)\sin\lambda + 4\lambda^2\mu^2e^{-\mu}\right]$$

but from (5.1) and (5.2)

$$f_d(\lambda) = \mu\lambda\phi^2\left[\left(2+\beta\phi^2\right)\left(1+e^{2(-\mu)}\right)\cos\lambda - \phi\beta\left(1-e^{2(-\mu)}\right)\sin\lambda + 4e^{-\mu}\right]$$

$$f(\lambda) = \left(2+\beta\phi^2\right)\left(1+e^{2(-\mu)}\right)\cos\lambda - \phi\beta\left(1-e^{2(-\mu)}\right)\sin\lambda + 4e^{-\mu}$$

$$\frac{\pi}{2} < \lambda < \pi$$

$$n\pi < \lambda < \left(\frac{1}{2}+n\right)\pi \qquad n = 1, 2, 3, \ldots$$

5.4 The B+S equation

$$\beta = 0$$

$$\frac{d^4\eta}{d\xi^4} + \phi^2\alpha\frac{d^2\eta}{d\xi^2} - \phi^2\eta = 0$$

Boundary conditions

A. Displacement zero: $\eta = 0$

B. Total slope zero: $\dfrac{d\eta}{d\xi} = 0$

C. Slope due to bending only zero: $\alpha \dfrac{d^3\eta}{d\xi^3} + \left(1 + \phi^2\alpha^2\right) \dfrac{d\eta}{d\xi} = 0$

D. Moment zero: $\dfrac{d^2\eta}{d\xi^2} + \phi^2\alpha\eta = 0$

E. Shear zero: $\dfrac{d^3\eta}{d\xi^3} + \phi^2\alpha \dfrac{d\eta}{d\xi} = 0$

5.4.1 The (C-C) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ \cos\lambda & \sin\lambda & e^{-\mu} & 1 \\ -\lambda\sin\lambda & \lambda\cos\lambda & -\mu e^{-\mu} & \mu \end{bmatrix}$$

$$f_d = \phi\left[-\gamma\phi\left(1 - e^{-2\mu}\right)\sin\lambda - 2\left(1 + e^{-2\mu}\right)\cos\lambda + 4e^{-\mu}\right]$$

$$f(\lambda) = \phi\gamma\left(1 - e^{-2\mu}\right)\sin\lambda + 2\left(1 + e^{-2\mu}\right)\cos\lambda - 4e^{-\mu}$$

$$\left(\frac{1}{2} + n\right)\pi < \lambda < (1 + n)\pi, \qquad n = 1, 2, 3, \ldots$$

5.4.2 The (C-S) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ \cos\lambda & \sin\lambda & e^{-\mu} & 1 \\ \left(-\lambda^2 + \phi^2\alpha\right)\cos\lambda & \left(-\lambda^2 + \phi^2\alpha\right)\sin\lambda & \left(\mu^2 + \phi^2\alpha\right)e^{-\mu} & \mu^2 + \phi^2\alpha \end{bmatrix}$$

$$f_d(\lambda) = \left(\lambda^2 + \mu^2\right)\left[\left(1 + e^{-2\mu}\right)\mu\sin\lambda - \left(1 - e^{-2\mu}\right)\lambda\cos\lambda\right]$$

$$f(\lambda) = \left(1 + e^{-2\mu}\right)\mu\sin\lambda - \left(1 - e^{-2\mu}\right)\lambda\cos\lambda$$

$$n\pi < \lambda < \left(\frac{1}{2} + n\right)\pi, \quad n = 1, 2, 3, ...$$

5.4.3 The (C-R) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ -\lambda\sin\lambda & \lambda\cos\lambda & -\mu e^{-\mu} & \mu \\ \lambda^3\sin\lambda - \phi^2\alpha\lambda\sin\lambda & -\lambda^3\cos\lambda + \phi^2\alpha\lambda\cos\lambda & -\mu^3 e^{-\mu} - \phi^2\alpha\mu e^{-\mu} & \mu^3 + \phi^2\alpha\mu \end{bmatrix}$$

$$f_d(\lambda) = \lambda\mu\left(\mu^2 + \lambda^2\right)\left(\mu\cos\lambda\left(1 - e^{-2\mu}\right) + \lambda\sin\lambda\left(1 + e^{-2\mu}\right)\right)$$

$$f(\lambda) = \mu\cos\lambda\left(1 - e^{-2\mu}\right) + \lambda\sin\lambda\left(1 + e^{-2\mu}\right)$$

$$\left(\frac{1}{2} + n\right)\pi < \lambda < (1 + n)\pi, \quad n = 0, 1, 2, 3, ...$$

5.4.4 The (C-F) case

$$\begin{bmatrix} 1 & 0 & 1 & e^{-\mu} \\ 0 & \lambda & -\mu & \mu e^{-\mu} \\ \left(-\lambda^2 + \phi^2\alpha\right)\cos\lambda & \left(-\lambda^2 + \phi^2\alpha\right)\sin\lambda & \left(\mu^2 + \phi^2\alpha\right)e^{-\mu} & \mu^2 + \phi^2\alpha \\ \left(\lambda^2 - \phi^2\alpha\right)\lambda\sin\lambda & -\left(\lambda^2 - \phi^2\alpha\right)\lambda\cos\lambda & -\left(\mu^2 + \phi^2\alpha\right)\mu e^{-\mu} & \left(\mu^2 + \phi^2\alpha\right)\mu \end{bmatrix}$$

$$f_d(\lambda) = \mu\lambda\left[\left(\lambda^2 - \mu^2\right)\left(1 - e^{2(-\mu)}\right)\mu(\sin\lambda)\lambda + \left(1 + e^{-2\mu}\right)2\mu^2(\cos\lambda)\lambda^2 + 2\left(\lambda^4 + \mu^4\right)e^{-\mu}\right]$$

again from (5.1) and (5.2)

$$f_d(\lambda) = \phi\left[\alpha\phi^3\left(1-e^{2(-\mu)}\right)(\sin\lambda) + \left(1+e^{-2\mu}\right)2\phi^2(\cos\lambda) + 2\phi^2\left(\phi^2\alpha^2+2\right)e^{-\mu}\right]$$

thus

$$f_d(\lambda) = \phi^3\left[\alpha\phi\left(1-e^{2(-\mu)}\right)(\sin\lambda) + \left(1+e^{-2\mu}\right)2(\cos\lambda) + 2\left(\phi^2\alpha^2+2\right)e^{-\mu}\right]$$

$$f(\lambda) = \alpha\phi\left(1-e^{2(-\mu)}\right)(\sin\lambda) + \left(1+e^{-2\mu}\right)2(\cos\lambda) + 2\left(\phi^2\alpha^2+2\right)e^{-\mu}$$

In this case we write

$$f(\lambda) = \alpha\frac{\lambda^2}{\sqrt{1+\alpha\lambda^2}}\left(1-e^{2(-\mu)}\right)(\sin\lambda)+\left(1+e^{-2\mu}\right)2(\cos\lambda)+2\frac{\left(1+\alpha\lambda^2\right)^2+1}{1+\alpha\lambda^2}e^{-\mu}$$

$$\left(\frac{1}{2}+n\right)\pi < \lambda < (1+n)\pi, \quad n = 0, 1, 2, 3, ...$$

Chapter 6

Concluding Comments

In Chapter 2 we described 1-D models for the elastic beam equations. It is readily seen that when shear is neglected, the models arising from Timoshenko equation and Timoshenko system are identical. Thus, we have a 1-D model for the elastic beam incorporating bending and/or rotary inertia. When shear is the main effect to be consider we obtain different models. A comparative study is wanted. Of particular interest is to compare the Timoshenko system against the two Timoshenko equations for displacement and rotation angle.

One of the problems left to study in Chapter 3, is the eigenvalue problem for the Timoshenko equation

$$\frac{d^4\eta}{d\xi^4} + \phi^2 (\alpha + \beta) \frac{d^2\eta}{d\xi^2} - \phi^2 \left(1 - \phi^2 \alpha\beta\right) \eta = 0$$

By denoting $\lambda = \phi^2$. We obtain a quadratic eigenvalue problem.

$$\frac{d^4\eta}{d\xi^4} + \lambda (\alpha + \beta) \frac{d^2\eta}{d\xi^2} - \lambda (1 - \lambda\alpha\beta) \eta = 0$$

By using FEM it can be reduced to a linear generalized eigenvalue problem. The matrices in this problem are unstructured, and due to the change of variable, spurious eigenvalues are found. An algorithm to deal with this problem is part of our current work.

We have introduced an asymptotic method in Chapter 4 and applied it succesfully to the quasi-Timoshenko equations in Chapter 5. The method is simple, higly accurate and allows to compute frequencies of any order at virtually no cost. An extension to the Timoshenko equation and Timoshenko system is part of our ongoing research. Also, by refining the asymptotic estimates for the eigenfrecuencies, we may have a tool to deal with inverse eigenvalue problem, for instance, in applications is common to have measurments of eigenfrecuencies and modes of vibration. The problem of interest is to characterize the material, that is a problem of parameter identification.

Finally in the Appendix we have considered some common beam specimens. Our purpose is to provide to the reader a tool for making a decision on model choice and beam design, as well as computation of eigenfrecuencies.

To decide over the modelling virtues of the different 1-D models, the numerical data collected in the Appendix, can be used as a basis for a statistical study.

Chapter 7

Appendix

Here we list computations of eigenfrecuencies by 3-D FEM, 1-D FEM and the method developed refered to as the aymptotic method.

In each section a beam is presented with realistic parameters. All tables present normalized frecuencies. To find the actual freecuencies recall that

$$\phi^2 = \left(\rho\omega^2 L^4\right)/EI$$

Thus the true eigenfrecuency is

$$\omega = \left(\frac{1}{L^2}\sqrt{\frac{EI}{\rho}}\right)\phi$$

For the parameters α and β in the quasi-Timoshenko equations, recall that

$$\alpha = EI/\left(KL^2\right)$$
$$\beta = I_\rho/\left(\rho L^2\right)$$

In the description of the beam especimens ρ is denoted by D, $I_\rho = \frac{I}{A}$, and $K = GA$. All the frecuencies are adimensional in this appendix.

7.1 Circular Beam

7.1.1 Aluminium

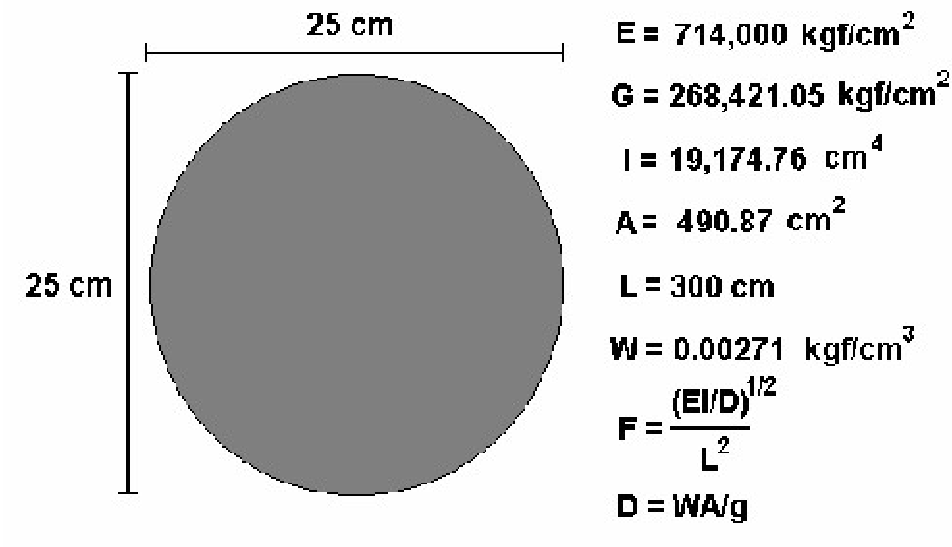

Figure 6.1. Physical parameters for a circular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	23.70021	16.53463	6.06864	3.83491
2	62.48760	51.71088	31.86995	23.42552
3	116.12042	102.90060	75.48156	63.14110
4	180.70652	166.29476	133.20936	117.69465
5	253.49447	240.08138	201.72040	183.69004
6	332.53738	318.45294	277.82712	258.12165
7	415.75745	402.86608	360.31236	339.06384
8	502.23367	490.24914	445.94407	423.96752
9	590.32520	579.69527	535.42138	512.36483
10	681.74655	672.89914	634.85981	602.21338

Table 6.1. Eigenfrecuencies by 3-D FEM for a circular aluminium beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.2. Euler-Bernoulli by Asymptotic Method for a circular aluminium beam..

Frec.	C-C	C-S	C-R	C-F
1	22.31378	15.37983	5.58959	3.51247
2	61.06537	49.50610	30.06494	21.88103
3	118.38843	102.18314	73.59222	60.68754
4	192.80737	172.21256	135.11685	117.31834
5	282.79060	258.09414	213.27423	190.64546
6	386.61971	358.14051	306.45644	279.07134
7	502.51553	470.58588	412.92404	380.89126
8	628.72830	593.67982	530.90845	494.37512
9	763.60697	725.75952	658.69546	617.84930
10	905.64495	865.29884	794.68588	749.75362

Table 6.3. B+R by Asymptotic Method for a circular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.21602	15.31673	5.58342	3.51776
2	60.09463	48.77160	29.80339	21.86698
3	114.53741	99.01098	71.94778	60.12380
4	182.58435	163.39166	129.61276	114.53490
5	261.36271	239.04633	199.98635	182.58461
6	348.18576	323.26106	280.24018	261.36267
7	440.77501	413.69751	367.83005	348.18576
8	537.32000	508.47393	460.64297	440.77501
9	636.45133	606.15066	557.02402	537.32000
10	737.17129	705.66757	655.73264	636.45133

Table 6.4. B+S by Asymptotic Method for a circular aluminium beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.5. Euler-Bernoulli by 1-D FEM for a circular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.31378	15.37983	5.58959	3.51247
2	61.06537	49.50610	30.06494	21.88103
3	118.38843	102.18313	73.59221	60.68753
4	192.80736	172.21256	135.11685	117.31833
5	282.79059	258.09413	213.27422	190.64545
6	386.61972	358.14051	306.45643	279.07133
7	502.51561	470.58594	412.92407	380.89127
8	628.72854	593.68001	530.90856	494.37518
9	763.60752	725.75996	658.69575	617.84951
10	905.64604	865.29976	794.68653	749.75411

Table 6.6. B+R by 1-D FEM for a circular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.21601	15.31673	5.58342	3.51775
2	60.09459	48.77157	29.80338	21.86698
3	114.53727	99.01087	71.94772	60.12376
4	182.58402	163.39139	129.61258	114.53476
5	261.36207	239.04579	199.98595	182.58427
6	348.18470	323.26013	280.23946	261.36203
7	440.77347	413.69612	367.82890	348.18471
8	537.31795	508.47203	460.64132	440.77347
9	636.44880	606.14827	557.02186	537.31795
10	737.16840	705.66478	655.73003	636.44879

Table 6.7. B+S by 1-D FEM for a circular aluminium beam.

7.1.2 Concrete

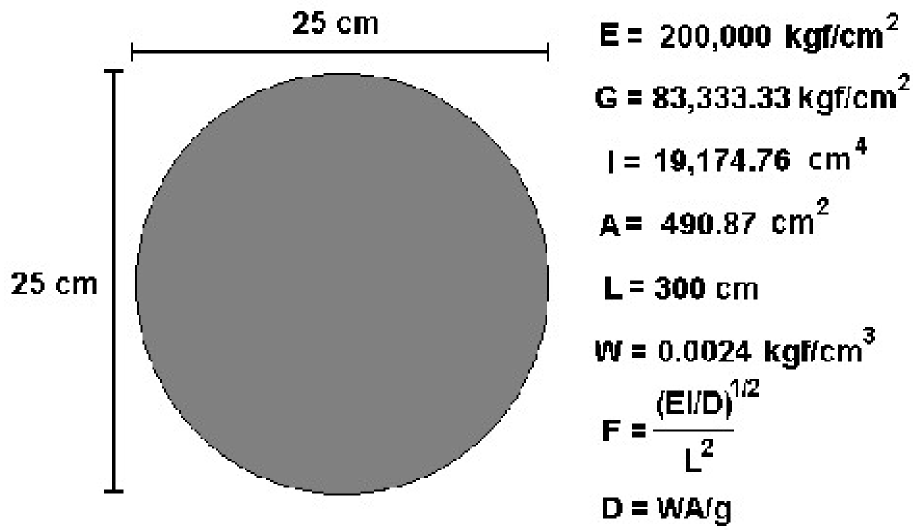

Figure 6.2. Physical parameters for a circular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.85432	15.95270	5.84632	3.69113
2	60.58961	50.09921	30.81372	22.61412
3	113.28675	100.22634	73.32198	61.20083
4	177.37539	162.88141	130.06387	114.67296
5	250.23783	235.36139	198.50061	179.93517
6	329.80304	314.98859	274.22414	254.18381
7	414.23738	400.32741	357.21785	335.34284
8	502.20602	488.96673	444.02350	421.22184
9	592.16000	580.16735	535.05380	510.84678
10	685.83876	675.52407	628.71744	602.27817

Table 6.8 Eigenfrecuencies by 3-D FEM for a circular concrete beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.9. Euler-Bernoulli by Asymptotic Method for a circular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.31378	15.37983	5.58959	3.51247
2	61.06537	49.50610	30.06494	21.88103
3	118.38843	102.18314	73.59222	60.68754
4	192.80737	172.21256	135.11685	117.31834
5	282.79060	258.09414	213.27423	190.64546
6	386.61971	358.14051	306.45644	279.07134
7	502.51553	470.58588	412.92404	380.89126
8	628.72830	593.67982	530.90845	494.37512
9	763.60697	725.75952	658.69546	617.84930
10	905.64495	865.29884	794.68588	749.75362

Tabe 6.10 B+R by Asymptotic Method for a circular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.23124	15.32656	5.58438	3.51759
2	60.24361	48.88444	29.84388	21.88321
3	115.11567	99.48816	72.19797	60.27230
4	184.07804	164.68349	130.43020	115.11328
5	264.39753	241.75140	201.90340	184.07828
6	353.45104	328.05357	283.90299	264.39750
7	448.95086	421.25408	373.92228	353.45105
8	549.02552	519.41902	469.82047	448.95086
9	652.21701	621.02564	569.87218	549.02552
10	757.43107	724.91982	672.74621	652.21701

Table 6.11. B+S by Asymptotic Method for a circular concrete beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51601
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.12. Euler-Bernoulli by 1-D FEM for a circular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.31378	15.37983	5.58959	3.51247
2	61.06537	49.50610	30.06494	21.88103
3	118.38843	102.18313	73.59221	60.68753
4	192.80736	172.21256	135.11685	117.31833
5	282.79059	258.09413	213.27422	190.64545
6	386.61972	358.14051	306.45643	279.07133
7	502.51561	470.58594	412.92407	380.89127
8	628.72854	593.68001	530.90856	494.37518
9	763.60752	725.75996	658.69575	617.84951
10	905.64604	865.29976	794.68653	749.75411

Table 6.13. B+R by 1-D FEM for a circular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.23124	15.32656	5.58438	3.51759
2	60.24365	48.88447	29.84390	21.88321
3	115.11583	99.48828	72.19803	60.27234
4	184.07844	164.68381	130.43041	115.11344
5	264.39833	241.75207	201.90388	184.07867
6	353.45243	328.05478	283.90391	264.39830
7	448.95306	421.25603	373.92383	353.45244
8	549.02876	519.42193	469.82288	448.95306
9	652.22157	621.02977	569.87568	549.02876
10	757.43724	724.92547	672.75107	652.22157

Table 6.14. B+S by 1-D FEM for a circular concrete beam.

7.1.3 Steel

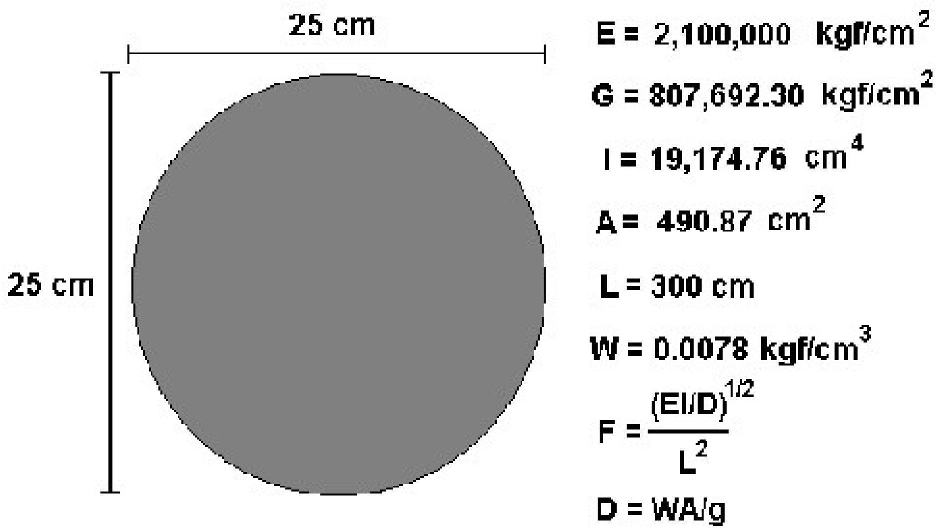

Figure 6.3. Physical parameters for a circular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	23.44496	16.36130	6.00264	3.79230
2	61.90809	51.22489	31.55397	23.18329
3	115.23385	102.08028	74.82533	62.55596
4	179.61958	165.22075	132.24545	116.76871
5	252.35974	237.83531	200.54081	182.50709
6	331.46025	317.21056	276.59151	256.82163
7	414.91473	401.78640	359.12703	337.73987
8	501.68283	489.40393	444.97842	422.81752
9	590.16101	579.21372	534.76858	511.44428
10	682.07615	672.86698	627.01453	601.61010

Table 6.15. Eigenfrecuencies by 3d FEM for a circular steel beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table. 6.16. Euler-Bernoulli by Asymptotic Method for circular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.31378	15.37983	5.58959	3.51247
2	61.06537	49.50610	30.06494	21.88103
3	118.38843	102.18314	73.59222	60.68754
4	192.80737	172.21256	135.11685	117.31834
5	282.79060	258.09414	213.27423	190.64546
6	386.61971	358.14051	306.45644	279.07134
7	502.51553	470.58588	412.92404	380.89126
8	628.72830	593.67982	530.90845	494.37512
9	763.60697	725.75952	658.69546	617.84930
10	905.64495	865.29884	794.68588	749.75362

Table 6.17. B+R by Asymptotic Method for a circular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.21952	15.31899	5.58364	3.51772
2	60.12886	48.79754	29.81270	21.87072
3	114.66989	99.12034	72.00521	60.15792
4	182.92536	163.68670	129.79979	114.66741
5	262.05284	239.66173	200.42334	182.92561
6	349.37832	324.34698	281.07179	262.05281
7	442.61957	415.40303	369.20774	349.37832
8	539.95116	510.93508	462.71042	442.61957
9	639.98311	609.48407	559.90805	539.95116
10	741.69588	709.96851	659.53922	639.98311

Table 6.18. B+S by Asymptotic Method for a circular steel beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51601
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.19. Euler-Bernoulli by 1-D FEM for a circular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.31378	15.37983	5.58959	3.51247
2	61.06537	49.50610	30.06494	21.88103
3	118.38843	102.18313	73.59221	60.68753
4	192.80736	172.21256	135.11685	117.31833
5	282.79059	258.09413	213.27422	190.64545
6	386.61972	358.14051	306.45643	279.07133
7	502.51561	470.58594	412.92407	380.89127
8	628.72854	593.68001	530.90856	494.37518
9	763.60752	725.75996	658.69575	617.84951
10	905.64604	865.29976	794.68653	749.75411

Table 6.20. B+R by 1-D FEM for a circular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.21952	15.31899	5.58364	3.51772
2	60.12889	48.79756	29.81271	21.87072
3	114.67001	99.12043	72.00525	60.15796
4	182.92565	163.68694	129.79995	114.66753
5	262.05344	239.66223	200.42369	182.92591
6	349.37935	324.34788	281.07247	262.05340
7	442.62121	415.40448	369.20889	349.37936
8	539.95359	510.93726	462.71221	442.62120
9	639.98655	609.48718	559.91066	539.95359
10	741.70059	709.97280	659.54288	639.98655

Table 6.21. B+S by 1-D FEM for a circular steel beam.

7.2 Elliptic Beam

7.2.1 Aluminium

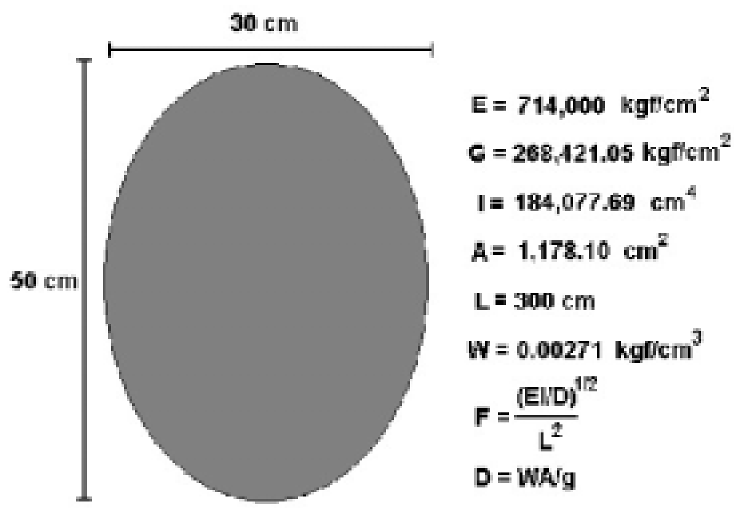

Figure 6.4. Physical parameters for an elliptic aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	2.11808	15.26753	5.78698	3.70644
2	5.12851	44.35290	28.41388	21.20357
3	88.44664	81.82225	62.33461	53.13915
4	129.55926	124.02450	102.51982	91.93490
5	173.21970	168.92491	146.22815	134.75636
6	218.61038	215.46616	191.88239	179.64530
7	264.46298	262.24403	238.42647	225.59964
8	310.27875	272.56087	285.62943	271.19464
9	344.96293	341.03230	332.05307	314.52054
10	401.64389	407.88557	370.15352	358.20506

Table 6.22. Eigenfrecuencies by 3d FEM for an elliptic aluminium beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.23. Euler-Bernoulli by Asymptotic Method for an elliptic aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.13799	15.26634	5.57845	3.50191
2	59.34318	48.20186	29.59712	21.43851
3	111.68748	96.65501	70.69769	57.93405
4	175.42061	157.18228	125.62902	108.23810
5	247.22255	226.41164	190.90204	169.24305
6	324.34553	301.50713	263.37673	237.92977
7	404.74469	380.31554	340.55576	311.89981
8	486.99844	461.31475	420.61521	389.35472
9	570.16776	543.48272	502.30087	469.00234
10	653.65735	626.16007	584.78809	549.94062

Table 6.24. B+R by Asymptotic Method for an elliptic aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	21.76280	15.02366	5.55400	3.52302
2	55.98862	45.64608	28.63435	21.38266
3	100.15368	87.04643	65.34067	56.03492
4	149.27949	134.32998	110.19374	100.14545
5	200.48204	184.29280	159.11281	149.28174
6	252.23771	235.19357	209.69357	200.48116
7	303.85629	286.19261	260.71105	252.23816
8	355.06154	336.92688	311.60457	303.85601
9	405.77508	387.26560	362.14876	355.06174
10	456.00315	437.18470	412.27688	405.77492

Table 6.25. B+S by Asymptotic Method for an elliptic aluminium beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.26. Euler-Bernoulli by 1-D FEM for an elliptic aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.13799	15.26634	5.57845	3.50191
2	59.34316	48.20185	29.59711	21.43851
3	111.68744	96.65498	70.69768	57.93404
4	175.42053	157.18221	125.62897	108.23806
5	247.22241	226.41152	190.90194	169.24298
6	324.34533	301.50694	263.37657	237.92964
7	404.74447	380.31532	340.55556	311.89962
8	486.99824	461.31454	420.61498	389.35449
9	570.16771	543.48261	502.30069	469.00213
10	653.65760	626.16020	584.78807	549.94050

Table 6.27. B+R by 1-D FEM for an elliptic aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	21.76279	15.02365	5.55400	3.52304
2	55.98856	45.64604	28.63433	21.38266
3	100.15350	87.04629	65.34059	56.03486
4	149.27914	134.32969	110.19353	100.14527
5	200.48149	184.29232	159.11243	149.28139
6	252.23696	235.19288	209.69299	200.48061
7	303.85535	286.19173	260.71027	252.23741
8	355.06048	336.92586	311.60361	303.85508
9	405.77397	387.26449	362.14769	355.06068
10	456.00210	437.18362	412.27577	405.77381

Table 6.28. B+S by 1-D FEM for an elliptic aluminium beam.

7.2.2 Concrete

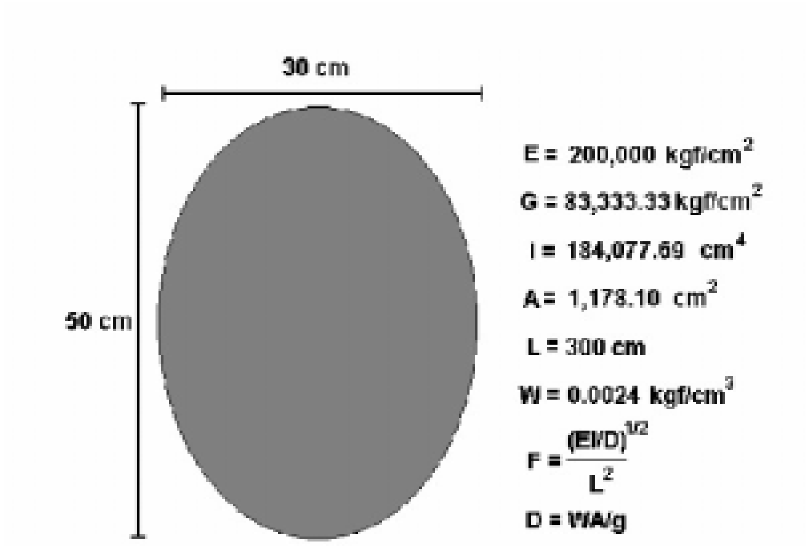

Figure 6.5. Physical parameters for an elliptic concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	20.61210	14.81345	5.58500	3.56557
2	50.58885	43.51769	27.72862	20.60535
3	88.12303	81.08336	61.48283	52.15597
4	130.05345	123.87417	101.99755	91.06099
5	174.70187	169.66846	146.49581	134.47437
6	221.29584	217.41559	193.21777	1807.83239
7	268.47307	265.58323	241.07324	227.43304
8	315.59235	313.82872	289.78634	274.59540
9	362.60833	362.03137	337.95776	320.17175
10	410.45574	414.70358	375.74439	368.92441

Table 6.29. Eigenfrecuencies by 3d FEM for an elliptic concrete beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.30. Euler-Bernoulli by Asymptotic Method for an elliptic concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.13799	15.26634	5.57845	3.50191
2	59.34318	48.20186	29.59712	21.43851
3	111.68748	96.65501	70.69769	57.93405
4	175.42061	157.18228	125.62902	108.23810
5	247.22255	226.41164	190.90204	169.24305
6	324.34553	301.50713	263.37673	237.92977
7	404.74469	380.31554	340.55576	311.89981
8	486.99844	461.31475	420.61521	389.35472
9	570.16776	543.48272	502.30087	469.00234
10	653.65735	626.16007	584.78809	549.94062

Table 6.31. B + R by Asymptotic Method for an elliptic concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.82033	15.06091	5.55781	3.52233
2	56.47721	46.01960	28.77895	21.44427
3	101.72572	88.36307	66.09956	56.52104
4	152.60885	137.25707	112.23820	101.71849
5	206.07348	189.35897	163.05519	152.61066
6	260.40775	242.74355	215.96902	206.07283
7	314.77704	296.42302	269.58390	260.40806
8	368.81361	349.93613	323.21653	314.77686
9	422.38524	403.09308	376.56474	368.81373
10	475.46903	455.83660	429.51717	422.38515

Table 6.32. B + S by Asymptotic Method for an elliptic concrete beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.33. Euler-Bernoulli by 1-D FEM for an elliptic concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.13799	15.26634	5.57845	3.50191
2	59.34316	48.20185	29.59711	21.43851
3	111.68744	96.65498	70.69768	57.93404
4	175.42053	157.18221	125.62897	108.23806
5	247.22241	226.41152	190.90194	169.24298
6	324.34533	301.50694	263.37657	237.92964
7	404.74447	380.31532	340.55556	311.89962
8	486.99824	461.31454	420.61498	389.35449
9	570.16771	543.48261	502.30069	469.00213
10	653.65760	626.16020	584.78807	549.94050

Table 6.34. B + R by 1-D FEM for an elliptic concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.82034	15.06092	5.55781	3.52234
2	56.47727	46.01964	28.77897	21.44429
3	101.72590	88.36322	66.09965	56.52111
4	152.60922	137.25738	112.23842	101.71867
5	206.07411	189.35951	163.05561	152.61104
6	260.40868	242.74438	215.96970	206.07345
7	314.77833	296.42419	269.58489	260.40899
8	368.81533	349.93769	323.21789	314.77815
9	422.38748	403.09512	376.56653	368.81545
10	475.47190	455.83922	429.51949	422.38739

Table 6.35. B + S by 1-D FEM for an elliptic concrete beam.

7.2.3 Steel

Figure 6.6. Physical parameters for an elliptic steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.00586	15.13326	5.72783	3.66534
2	51.04769	44.09382	28.20770	21.02604
3	88.27398	81.55966	62.05769	52.83389
4	129.56452	123.88241	102.29611	91.62897
5	173.43616	168.97172	146.17294	134.57226
6	219.09120	215.78037	192.05085	179.65092
7	265.23404	262.86515	234.34179	225.87940
8	311.18764	271.29362	286.43487	271.28928
9	349.77386	341.17018	333.27015	315.70100
10	403.43546	407.42472	373.19220	360.53744

Table 6.36. Eigenfrecuencies by 3d FEM for an elliptic steel beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.37. Euler Bernoulli by Asymptotic Method for an elliptic steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.13799	15.26634	5.57845	3.50191
2	59.34318	48.20186	29.59712	21.43851
3	111.68748	96.65501	70.69769	57.93405
4	175.42061	157.18228	125.62902	108.23810
5	247.22255	226.41164	190.90204	169.24305
6	324.34553	301.50713	263.37673	237.92977
7	404.74469	380.31554	340.55576	311.89981
8	486.99844	461.31475	420.61521	389.35472
9	570.16776	543.48272	502.30087	469.00234
10	653.65735	626.16007	584.78809	549.94062

Table 6.38. B + R by Asymptotic Method for an elliptic steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.77603	15.03222	5.55488	3.52286
2	56.10021	45.73143	28.66751	21.39683
3	100.50978	87.34488	65.51339	56.14593
4	150.02796	134.98845	110.65536	100.50179
5	201.73102	185.42509	159.99666	150.03010
6	254.05338	236.87221	211.09227	201.73020
7	306.27356	288.45790	262.67953	254.05380
8	358.09603	339.79824	314.17139	306.27331
9	409.43124	390.75020	365.32636	358.09620
10	460.27958	441.28294	416.06852	409.43110

Table 6.39.B + S by Asymptotic Method for an elliptic steel beam.

Eigenfrecuecnies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.40. Euler-Bernoulli by 1-D FEM for an elliptic steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.13799	15.26634	5.57845	3.50191
2	59.34316	48.20185	29.59711	21.43851
3	111.68744	96.65498	70.69768	57.93404
4	175.42053	157.18221	125.62897	108.23806
5	247.22241	226.41152	190.90194	169.24298
6	324.34533	301.50694	263.37657	237.92964
7	404.74447	380.31532	340.55556	311.89962
8	486.99824	461.31454	420.61498	389.35449
9	570.16771	543.48261	502.30069	469.00213
10	653.65760	626.16020	584.78807	549.94050

Table 6.41. B + R by 1-D FEM for an elliptic steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.77604	15.03223	5.55488	3.52283
2	56.10024	45.73145	28.66752	21.39683
3	100.50987	87.34495	65.51343	56.14595
4	150.02814	134.98860	110.65546	100.50188
5	201.73133	185.42536	159.99686	150.03028
6	254.05386	236.87263	211.09261	201.73051
7	306.27424	288.45850	262.68003	254.05427
8	358.09697	339.79908	314.17210	306.27399
9	409.43252	390.75134	365.32735	358.09714
10	460.28133	441.28450	416.06986	409.43239

Table 6.42.B + S by 1-D FEM for an elliptic steel beam.

7.3 Rectangular Beam

7.3.1 Aluminium

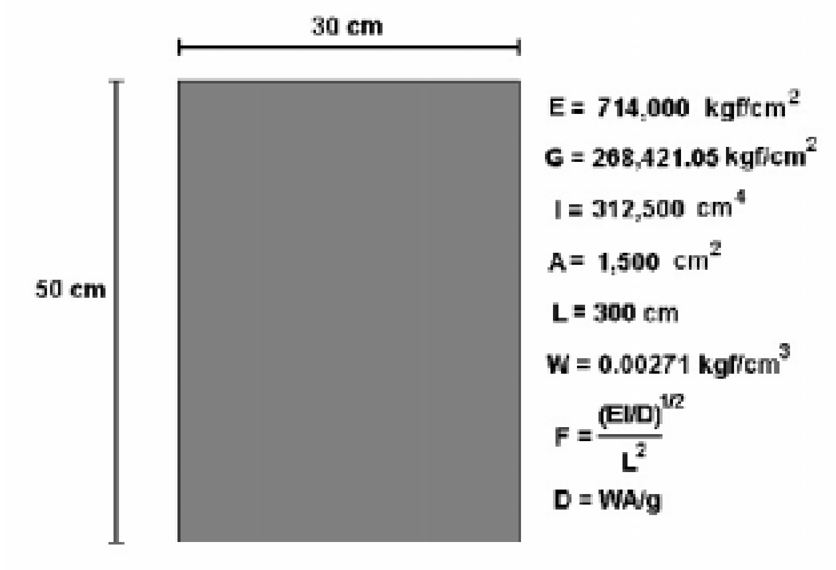

Figure 6.7. Physical parameters for a rectangular aluminium beam.

Frec	C-C	C-S	C-R	C-F
1	20.5387313	15.0374476	5.78388855	3.73412626
2	48.1941219	42.381028	27.5670382	20.731634
3	81.562351	76.43705564	58.8312543	50.5203657
4	117.708771	113.762243	94.8381509	85.5301079
5	155.81919	152.9416027	133.190066	123.195084
6	194.778896	192.8363709	172.759035	161.997719
7	234.185272	233.003559	212.988683	200.724273
8	275.085048	264.3674168	254.626938	235.821754
9	281.220628	274.4027715	280.802795	272.112973
10	311.486217	285.8272214	295.083726	285.815441

Table 6.43. Eigenfrecuencies by 3d FEM for a rectangular aluminium beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.44. Euler-Bernoulli by Asymptotic Method for a rectangular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.06115	15.21669	5.57351	3.49724
2	58.62221	47.65428	29.39604	21.25000
3	109.05242	94.47024	69.51631	56.82699
4	169.06981	151.65822	122.00913	104.84232
5	235.21602	215.64344	182.99224	161.85457
6	304.91980	283.71649	249.30278	224.82907
7	376.47229	354.03155	318.68012	291.56409
8	448.81430	425.41940	389.64397	360.53536
9	521.32563	497.17845	461.28304	430.72849
10	593.66268	568.90774	533.06361	501.48942

Table 6.45. B + R by Asymptotic Method for a rectangular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	21.56982	14.89857	5.54107	3.52537
2	54.41112	44.43719	28.15693	21.17568
3	95.29467	82.96294	62.93633	54.46632
4	139.38064	125.59972	103.98049	95.28242
5	184.37489	169.65960	147.55213	139.38491
6	229.27247	213.92657	191.80795	184.37280
7	273.73150	257.92666	235.97376	229.27375
8	317.67099	301.51369	279.77441	273.73058
9	361.11720	344.67462	323.14577	317.67173
10	404.12562	387.44505	366.10567	361.11656

Table 6.46. B + R by Asymptotic Method for a rectangular aluminium beam.

Eigenfrecuecnies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51601
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.47. Euler-Bernoulli by 1-D FEM for a rectangular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	22.06114	15.21669	5.57351	3.49724
2	58.62219	47.65427	29.39604	21.25000
3	109.05234	94.47018	69.51628	56.82697
4	169.06964	151.65809	122.00903	104.84225
5	235.21574	215.64319	182.99205	161.85442
6	304.91939	283.71612	249.30247	224.82880
7	376.47177	354.03106	318.67969	291.56370
8	448.81373	425.41884	389.64344	360.53486
9	521.32510	497.17789	461.28247	430.72791
10	593.66234	568.90731	533.06310	501.48885

Table 6.48. B + R by 1-D FEM for a rectangular aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	21.56981	14.89857	5.54107	3.52541
2	54.41111	44.43718	28.15693	21.17569
3	95.29465	82.96292	62.93631	54.46631
4	139.38060	125.59969	103.98047	95.28240
5	184.37484	169.65956	147.55209	139.38487
6	229.27243	213.92652	191.80790	184.37275
7	273.73150	257.92664	235.97373	229.27371
8	317.67108	301.51374	279.77442	273.73058
9	361.11745	344.67480	323.14587	317.67182
10	404.12613	387.44545	366.10595	361.11681

Table 6.49. B + S by 1-D FEM for a rectangular aluminium beam.

7.3.2 Concrete

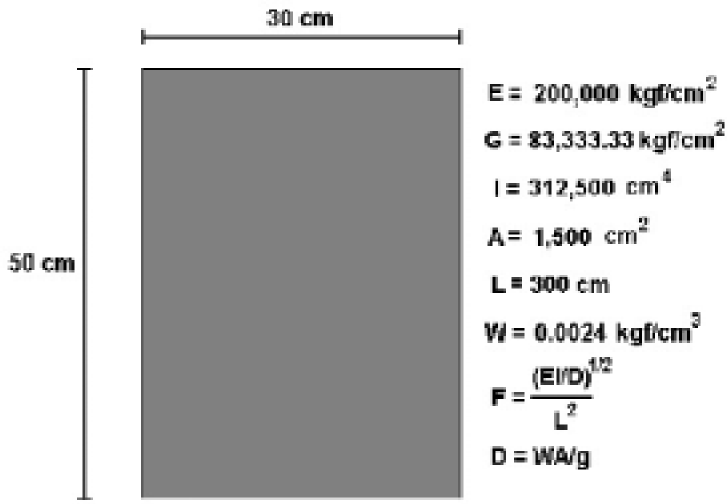

Figure 6.8. Physical parameters for a rectangular concrete beam.

Frec	C-C	C-S	C-R	C-F
1	19.69999139	15.90386958	5.58203152	3.5877475
2	46.9924446	38.52808355	26.9848011	20.18918
3	80.32273451	72.43549395	58.2970239	49.780297
4	116.8595887	107.9025479	94.8262414	85.103561
5	155.1291136	133.7726009	134.027232	123.44051
6	195.0830884	209.8516998	174.684879	163.24153
7	235.3495156	248.8229333	216.172751	203.25049
8	277.2316619	281.0478575	263.90007	241.17963
9	319.0527143	291.9073231	290.491251	290.24469
10	362.0323714	299.4087985	301.383445	297.96

Table 6.50. Eigenfrecuencies by 3d FEM for a rectangular concrete beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.51. Euler-Bernoulli by Asymptotic Method for a rectangular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.64466	14.94710	5.54612	3.52445
2	55.01206	44.89824	28.34072	21.25601
3	97.10832	84.48961	63.84420	55.06368
4	143.00945	128.80489	106.28153	97.09776
5	190.19400	174.95284	151.76324	143.01282
6	237.47630	221.53111	198.23800	190.19247
7	284.40057	267.94474	244.77751	237.47718
8	330.82626	313.97994	291.01498	284.39996
9	376.74931	359.58903	336.83743	330.82672
10	422.21245	404.79128	382.23861	376.74892

Table 6.52. B + R by Asymptotic Method for a rectangular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.64466	14.94710	5.54612	3.52445
2	55.01206	44.89824	28.34072	21.25601
3	97.10832	84.48961	63.84420	55.06368
4	143.00945	128.80489	106.28153	97.09776
5	190.19400	174.95284	151.76324	143.01282
6	237.47630	221.53111	198.23800	190.19247
7	284.40057	267.94474	244.77751	237.47718
8	330.82626	313.97994	291.01498	284.39996
9	376.74931	359.58903	336.83743	330.82672
10	422.21245	404.79128	382.23861	376.74892

Table 6.53. B + S by Asymptotic Method for a rectangular concrete beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.54. Euler-Bernoulli by 1-D FEM for a rectangular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	22.06114	15.21669	5.57351	3.49724
2	58.62219	47.65427	29.39604	21.25000
3	109.05234	94.47018	69.51628	56.82697
4	169.06964	151.65809	122.00903	104.84225
5	235.21574	215.64319	182.99205	161.85442
6	304.91939	283.71612	249.30247	224.82880
7	376.47177	354.03106	318.67969	291.56370
8	448.81373	425.41884	389.64344	360.53486
9	521.32510	497.17789	461.28247	430.72791
10	593.66234	568.90731	533.06310	501.48885

Table 6.55. B + R by 1-D FEM for a rectangular concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.64467	14.94711	5.54612	3.52449
2	55.01211	44.89828	28.34073	21.25603
3	97.10848	84.48973	63.84428	55.06374
4	143.00976	128.80515	106.28172	97.09792
5	190.19451	174.95328	151.76359	143.01314
6	237.47703	221.53176	198.23854	190.19298
7	284.40157	267.94565	244.77829	237.47792
8	330.82757	313.98114	291.01602	284.40096
9	376.75102	359.59058	336.83879	330.82804
10	422.21466	404.79329	382.24037	376.75063

Table 6.56. B + S by 1-D FEM for a rectangular concrete beam.

7.3.3 Steel

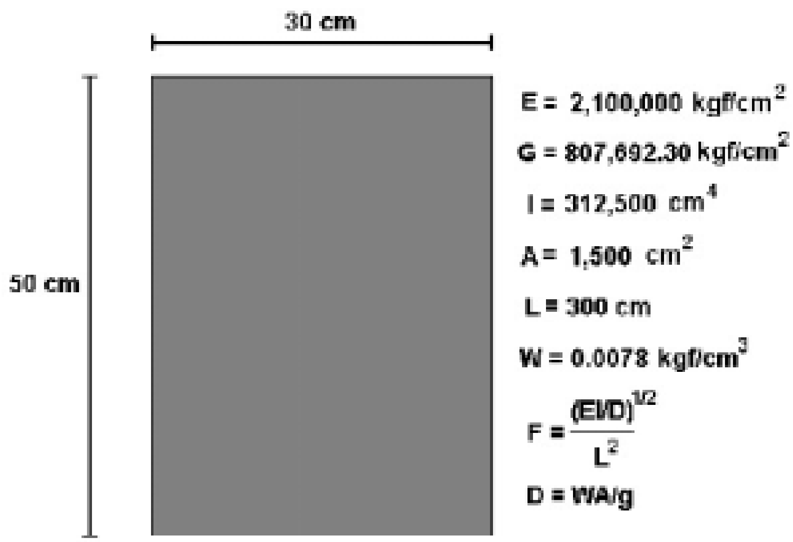

Figure 6.9. Physical parameters for a rectangular steel beam.

Frec	C-C	C-S	C-R	C-F
1	20.388347	14.91324852	5.72516373	3.69174309
2	48.044858	42.18446502	27.3912722	20.5677627
3	81.524982	76.29533602	58.6467841	50.2874357
4	117.8567	113.7737148	94.76352	85.3553533
5	156.18287	153.1663082	133.302249	123.165736
6	195.3861	193.3175953	173.114127	162.1927
7	235.04454	233.7796434	213.623882	201.218207
8	276.258	266.5977003	256.183324	237.045426
9	283.20035	275.570923	282.868829	268.644595
10	313.27375	287.1152071	296.3686	287.107559

Table 6.57. Eigenfrecuencies by 3d FEM for a rectangular steel beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.58. Euler-Bernoulli by Asymptotic Method for a rectangular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.06115	15.21669	5.57351	3.49724
2	58.62221	47.65428	29.39604	21.25000
3	109.05242	94.47024	69.51631	56.82699
4	169.06981	151.65822	122.00913	104.84232
5	235.21602	215.64344	182.99224	161.85457
6	304.91980	283.71649	249.30278	224.82907
7	376.47229	354.03155	318.68012	291.56409
8	448.81430	425.41940	389.64397	360.53536
9	521.32563	497.17845	461.28304	430.72849
10	593.66268	568.90774	533.06361	501.48942

Table 6.59. B + R by Asymptotic Method for a rectangular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.58703	14.90973	5.54224	3.52516
2	54.54811	44.54235	28.19903	21.19416
3	95.70421	83.30793	63.14243	54.60247
4	140.19349	126.31814	104.49825	95.69237
5	185.67012	170.83841	148.49279	140.19754
6	231.08980	215.61182	193.23618	185.66818
7	276.08648	260.13860	237.92092	231.09099
8	320.56688	304.25848	282.25256	276.08563
9	364.55118	347.95143	326.15694	320.56755
10	408.09244	391.24983	369.64707	364.55061

Table 6.60. B + S by Asymptotic Method for a rectangular steel beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.61. Euler-Bernoulli by 1-D FEM for a rectangular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	22.06114	15.21669	5.57351	3.49724
2	58.62219	47.65427	29.39604	21.25000
3	109.05234	94.47018	69.51628	56.82697
4	169.06964	151.65809	122.00903	104.84225
5	235.21574	215.64319	182.99205	161.85442
6	304.91939	283.71612	249.30247	224.82880
7	376.47177	354.03106	318.67969	291.56370
8	448.81373	425.41884	389.64344	360.53486
9	521.32510	497.17789	461.28247	430.72791
10	593.66234	568.90731	533.06310	501.48885

Table 6.62. Euler-Bernoulli by 1-D FEM for a rectangular steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.58702	14.90973	5.54224	3.52517
2	54.54809	44.54233	28.19903	21.19416
3	95.70415	83.30788	63.14240	54.60245
4	140.19338	126.31804	104.49818	95.69231
5	185.66996	170.83827	148.49266	140.19743
6	231.08961	215.61163	193.23601	185.66801
7	276.08628	260.13839	237.92072	231.09079
8	320.56673	304.25830	282.25236	276.08543
9	364.55114	347.95133	326.15679	320.56739
10	408.09262	391.24991	369.64706	364.55057

Table 6.63. B + S by 1-D FEM for a rectangular steel beam.

7.4 Square Beam

7.4.1 Aluminium

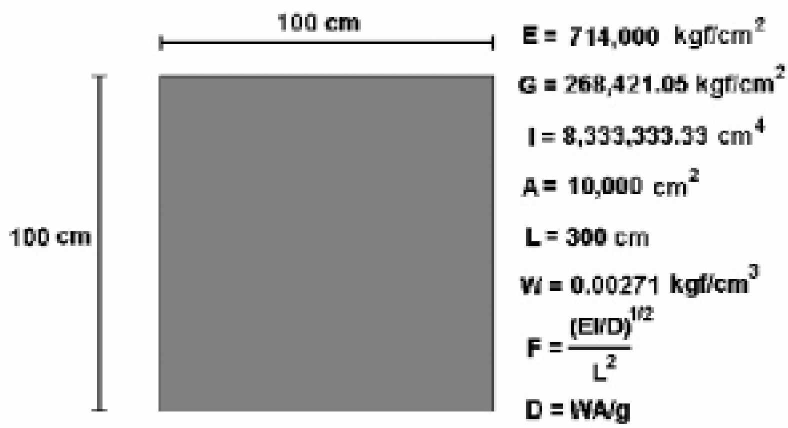

Figure 6.10. Physical parameters for a square aluminium beam.

Frec	C-C	C-S	C-R	C-F
1	15.1805293	12.09525292	5.15468483	3.53820492
2	31.1998538	29.54913735	20.4682335	15.7687474
3	50.0427722	48.92462407	39.1311953	34.2972786
4	67.8374267	69.07748879	58.9623102	52.0775146
5	78.2657039	73.45565461	79.0786069	69.2197729
6	89.4597015	86.71955145	82.0381039	77.985676
7	100.276484	91.34553335	98.2054799	89.7346982
8	111.21291	107.5873953	103.896844	97.005974
9	126.047515	112.119012	122.36316	111.765112
10	133.653793	132.9920097	127.899846	118.567016

Table 6.64. Eigenfrecuencies by 3d FEM for a square aluminium beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.65. Euler-Bernoulli by Asymptotic Method for a square aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	21.19562	14.65554	5.51529	3.44251
2	51.59430	42.26738	27.26307	19.31314
3	87.33686	76.22974	58.80387	47.23900
4	124.27723	112.20319	94.11778	79.90055
5	161.09717	148.41509	130.33093	114.67699
6	197.38756	184.30066	166.41156	150.04497
7	233.13394	219.74126	202.07662	185.39942
8	268.39189	254.76139	237.30033	220.50194
9	303.25179	289.42193	272.13257	255.30602
10	337.78184	323.78523	306.63713	289.82212

Table 6.66. B + R by Asymptotic Method for a square aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	19.58609	13.60275	5.39245	3.55430
2	42.09529	34.84500	23.82358	19.01901
3	65.32579	57.31778	46.09347	42.30733
4	87.67960	79.41722	68.43244	65.18720
5	109.55146	101.03069	90.27867	87.79740
6	130.90999	122.26758	111.68898	109.43874
7	152.03009	143.23321	132.77893	131.02414
8	172.87953	164.00357	153.63910	151.91150
9	193.61536	184.63087	174.33257	173.00451
10	214.19172	205.15114	194.90262	193.48308

Table 6.67. B + S by Asymptotic Method for a square aluminium beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.68. Euler-Bernoulli by 1-D FEM for a square aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	21.19563	14.65555	5.51529	3.44251
2	51.59434	42.26741	27.26308	19.31314
3	87.33697	76.22983	58.80393	47.23904
4	124.27742	112.20335	94.11790	79.90064
5	161.09746	148.41534	130.33113	114.67715
6	197.38796	184.30102	166.41186	150.04523
7	233.13449	219.74175	202.07704	185.39979
8	268.39262	254.76205	237.30090	220.50243
9	303.25276	289.42280	272.13333	255.30668
10	337.78313	323.78638	306.63813	289.82299

Table 6.69. B + R by 1-D FEM for a square aluminium beam.

Frec.	C-C	C-S	C-R	C-F
1	19.58614	13.60278	5.39245	3.55450
2	42.09549	34.84515	23.82366	19.01909
3	65.32617	57.31809	46.09370	42.30754
4	87.68016	79.41771	68.43285	65.18758
5	109.55221	101.03136	90.27926	87.79796
6	130.91093	122.26844	111.68975	109.43948
7	152.03124	143.23427	132.77989	131.02508
8	172.88090	164.00484	153.64027	151.91265
9	193.61700	184.63239	174.33396	173.00589
10	214.19368	205.15295	194.90428	193.48472

Table 6.65. B + S by 1-D FEM for a square aluminium beam.

7.4.2 Concrete

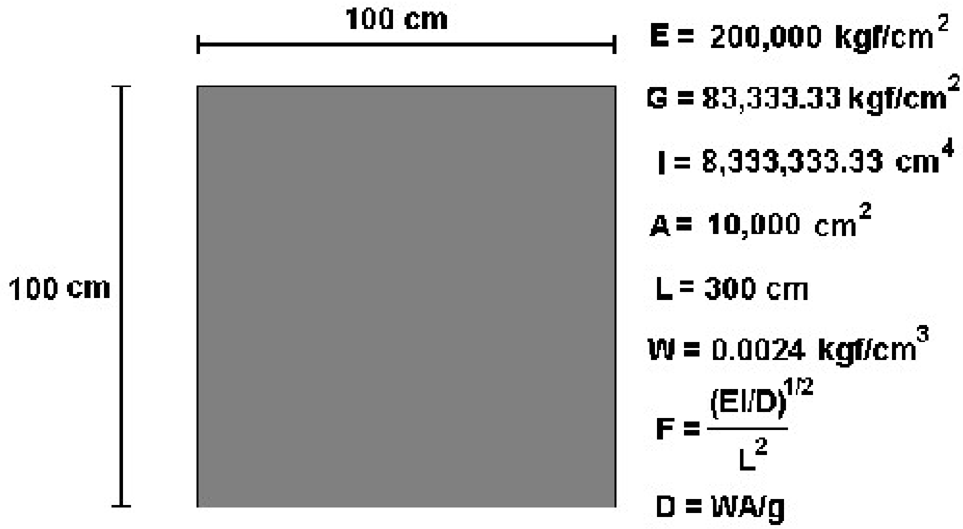

Figure 6.11. Physical parameters for a square concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	15.10463	11.74620	4.99880	3.38777
2	31.55736	38.21584	20.36515	15.59032
3	50.65098	50.59207	39.39757	34.30451
4	70.63593	71.74981	59.63792	52.66545
5	80.73805	73.21104	80.61564	71.46114
6	91.45787	89.35809	84.08807	79.77516
7	104.44166	94.56919	100.26697	92.70080
8	114.10618	110.28235	106.42972	100.95253
9	128.95531	117.63327	125.34858	116.11356
10	137.93286	136.96735	129.56734	123.98945

Table 6.71. Eigenfrecuencies by 3d FEM for a square concrete beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.72. Euler-Bernoulli by Asymptotic Method for a square concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.19562	14.65554	5.51529	3.44251
2	51.59430	42.26738	27.26307	19.31314
3	87.33686	76.22974	58.80387	47.23900
4	124.27723	112.20319	94.11778	79.90055
5	161.09717	148.41509	130.33093	114.67699
6	197.38756	184.30066	166.41156	150.04497
7	233.13394	219.74126	202.07662	185.39942
8	268.39189	254.76139	237.30033	220.50194
9	303.25179	289.42193	272.13257	255.30602
10	337.78184	323.78523	306.63713	289.82212

Table 6.73. B + R by Asymptotic Method for a square concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	19.81475	13.75305	5.41117	3.55046
2	43.24577	35.75254	24.27717	19.27055
3	67.69086	59.36945	47.54877	43.43311
4	91.33334	82.70716	71.10063	67.57625
5	114.45382	105.55393	94.18804	91.42664
6	137.04111	127.99808	116.81930	114.36695
7	159.34852	150.14484	139.10301	137.12747
8	181.37110	172.07448	161.13280	159.25990
9	203.25256	193.84369	182.97635	181.46365
10	224.96745	215.49213	204.68104	203.15521

Table 6.74. B + S by Asymptotic Method for a square concrete beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51601
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.75. Euler-Bernoulli by 1-D FEM for a square concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	21.19563	14.65555	5.51529	3.44251
2	51.59434	42.26741	27.26308	19.31314
3	87.33697	76.22983	58.80393	47.23904
4	124.27742	112.20335	94.11790	79.90064
5	161.09746	148.41534	130.33113	114.67715
6	197.38796	184.30102	166.41186	150.04523
7	233.13449	219.74175	202.07704	185.39979
8	268.39262	254.76205	237.30090	220.50243
9	303.25276	289.42280	272.13333	255.30668
10	337.78313	323.78638	306.63813	289.82299

Table 6.76. B + R by 1-D FEM for a square concrete beam.

Frec.	C-C	C-S	C-R	C-F
1	19.81472	13.75303	5.41117	3.55047
2	43.24564	35.75244	24.27712	19.27051
3	67.69060	59.36924	47.54862	43.43297
4	91.33296	82.70683	71.10036	67.57600
5	114.45333	105.55349	94.18765	91.42626
6	137.04053	127.99754	116.81880	114.36646
7	159.34786	150.14421	139.10242	137.12688
8	181.37039	172.07379	161.13213	159.25923
9	203.25184	193.84297	182.97563	181.46294
10	224.96679	215.49144	204.68032	203.15450

Table 6.77. B + S by 1-D FEM for a square concrete beam.

7.4.3 Steel

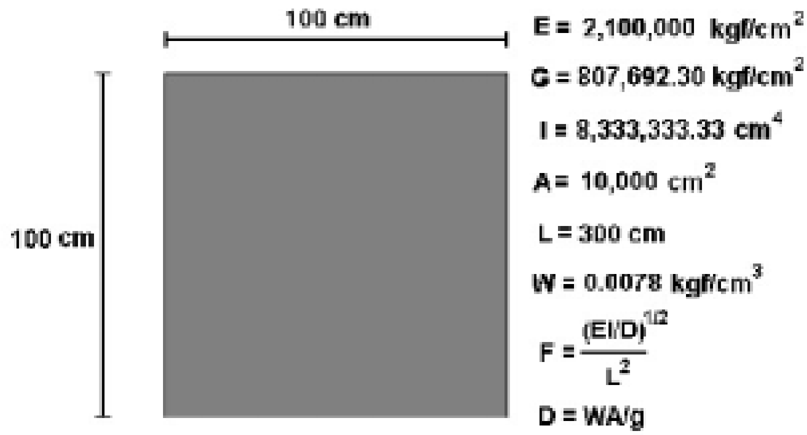

Figure 6.12. Physical parameters for a square steel beam.

Frec	C-C	C-S	C-R	C-F
1	19.69999139	15.90386958	5.58203152	3.5877475
2	46.9924446	38.52808355	26.9848011	20.18918
3	80.32273451	72.43549395	58.2970239	49.780297
4	116.8595887	107.9025479	94.8262414	85.103561
5	155.1291136	133.7726009	134.027232	123.44051
6	195.0830884	209.8516998	174.684879	163.24153
7	235.3495156	248.8229333	216.172751	203.25049
8	277.2316619	281.0478575	263.90007	241.17963
9	319.0527143	291.9073231	290.491251	290.24469
10	362.0323714	299.4087985	301.383445	297.96

Table 6.78. Eigenfrecuencies by 3d FEM for a square steel beam.

Eigenfrecuencies by Asymptotic Method

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85945	178.26973	138.79131	120.90192
5	298.55554	272.03097	222.68295	199.85953
6	416.99079	385.53142	326.31380	298.55553
7	555.16525	518.77108	449.68385	416.99079
8	713.07892	671.74995	592.79311	555.16525
9	890.73180	844.46803	755.64159	713.07892
10	1088.12389	1036.92531	938.22927	890.73180

Table 6.79. Euler-Bernoulli by Asymptotic Method for a square steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.19562	14.65554	5.51529	3.44251
2	51.59430	42.26738	27.26307	19.31314
3	87.33686	76.22974	58.80387	47.23900
4	124.27723	112.20319	94.11778	79.90055
5	161.09717	148.41509	130.33093	114.67699
6	197.38756	184.30066	166.41156	150.04497
7	233.13394	219.74126	202.07662	185.39942
8	268.39189	254.76139	237.30033	220.50194
9	303.25179	289.42193	272.13257	255.30602
10	337.78184	323.78523	306.63713	289.82212

Table 6.80. B + R by Asymptotic Method for a square steel beam.

Frec.	C-C	C-S	C-R	C-F
1	19.63821	13.63703	5.39676	3.55341
2	42.35286	35.04837	23.92605	19.07642
3	65.84948	57.77245	46.41753	42.55911
4	88.48395	80.14172	69.02162	65.71659
5	110.62680	102.02311	91.13780	88.59586
6	132.25203	123.52200	112.81316	110.52034
7	153.62962	144.74390	134.16214	132.35942
8	174.73368	165.76582	155.27609	153.51834
9	195.71807	186.64096	176.21914	174.85073
10	216.54165	207.40614	197.03549	195.59434

Table 6.81. B + S by Asymptotic Method for a square steel beam.

Eigenfrecuencies by 1-D FEM

Frec.	C-C	C-S	C-R	C-F
1	22.37329	15.41821	5.59332	3.51602
2	61.67282	49.96486	30.22585	22.03449
3	120.90339	104.24770	74.63888	61.69721
4	199.85946	178.26974	138.79132	120.90192
5	298.55557	272.03100	222.68296	199.85954
6	416.99089	385.53150	326.31384	298.55557
7	555.16548	518.77127	449.68398	416.99089
8	713.07941	671.75037	592.79340	555.16548
9	890.73277	844.46885	755.64218	713.07941
10	1088.12565	1036.92684	938.23040	890.73277

Table 6.82. Euler-Bernoulli by 1-D FEM for a square steel beam.

Frec.	C-C	C-S	C-R	C-F
1	21.19563	14.65555	5.51529	3.44251
2	51.59434	42.26741	27.26308	19.31314
3	87.33697	76.22983	58.80393	47.23904
4	124.27742	112.20335	94.11790	79.90064
5	161.09746	148.41534	130.33113	114.67715
6	197.38796	184.30102	166.41186	150.04523
7	233.13449	219.74175	202.07704	185.39979
8	268.39262	254.76205	237.30090	220.50243
9	303.25276	289.42280	272.13333	255.30668
10	337.78313	323.78638	306.63813	289.82299

Table 6.83. B + R by 1-D FEM for a square steel beam.

Frec.	C-C	C-S	C-R	C-F
1	19.63820	13.63702	5.39676	3.55323
2	42.35282	35.04834	23.92604	19.07639
3	65.84940	57.77239	46.41748	42.55906
4	88.48384	80.14162	69.02154	65.71651
5	110.62667	102.02298	91.13768	88.59575
6	132.25188	123.52185	112.81302	110.52020
7	153.62947	144.74375	134.16199	132.35927
8	174.73356	165.76569	155.27595	153.51820
9	195.71803	186.64088	176.21904	174.85062
10	216.54173	207.40616	197.03545	195.59430

Table 6.84. B + S by 1-D FEM for a square steel beam.

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