AN ANALOGUE TO FUNCTIONAL ANALYSIS IN DIALGEBRAS

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Abstract

It is the aim of this paper to give the first steps to establish a generalization of functional analysis for dialgebras. In particular we have described families of dialgebras each endowed with a norm and/or an involution. Such an extention could be very desirable, since as we can see in this paper it could shed light on the manner in which one can construct a lot of families of dialgebras with a bar-unit.

1 Introduction

Lately the study of algebraic and geometric structures on dialgebras have received much attention. The introduction and the first systematic investigation of the dialgebras was made by Loday in his related work with Leibniz algebras. However, to date the dialgebras have been studied outside the scope of functional analysis.

As it is well known the functional analysis becomes the study of (infinite-dimensional) vector space with some kind of metric or other structure, including ring structures (Banach Algebras and C^*-algebra for example). Appropriate generalizations of adjoint element, ideal and unit also belong to this area.

Throughout this paper, we defined an analogous of some of the basic structures of the functional analysis on dialgebras and we show that even in its elementary aspects, some of the known concepts and results acquire in dialgebras a diferent aspect.

We recall some definitions due to Loday

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Definition 1 *A dialgebra is a vector space V together with two bilinear operators, $\vdash$, $\dashv$, satisfying the following relations*

$$x \dashv (y \dashv z) = x \dashv (y \vdash z)\,,$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z)\,,$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z,$$

for all x, y and z of V. These operators are called respectively, right and left products.

It is well known that if a dialgebra is given then it gives rise to a Leibniz algebra which is obtained by defining the bracket as

$$[x, y] = x \dashv y - y \vdash x$$

see [4] for more detail. The Leibniz algebras are a generalization of Lie algebras, for which the antisymmetry condition of the bracket is dropped and only the Jacobi identity is retained.

At this point it seems proper to observe that the study of "deformations" of a Leibniz algebra structure has been considered in [2] where the notion of $R-$matrices and Yang-Baxter equations on Liebniz algebras were introduced and studied and was the initial motivation for the present paper.

We shall express that perhaps this paper establishes more problems that solutions, howover we expect that apart from it's intrinsic interest, these constructions can be relevant in mathematical physics (in particular for the construction of R-matrices on Liebniz algebras) and that in this context they lead to interesting generalizetions.

This paper is dedicated to I.M.Gelfand in occasion of his 90 th birthday.

2 Normed and Banach Dialgebras

Let us begin by a definition

Definition 2 *A normed dialgebra is a dialgebra $(\mathcal{U}, \vdash, \dashv)$ over the field $\mathbb{C}$ together with a norm $x \to \|x\|$, such that,*

$$\|x \vdash y\| \leq \|x\|\,\|y\|\,, \qquad \|x \dashv y\| \leq \|x\|\,\|y\| \qquad \forall \;\; x, y \in \mathcal{U} \qquad (1)$$

Note first that from (1) we conclude the continuity of these products with respect to both arguments. We recall that a bar-unit of a normed dialgebra $\mathcal{U}$, is an element $e \in \mathcal{U}$, such that $e \vdash x = x = x \dashv e$. From (1) it follows that if $\mathcal{U} \neq \{\theta\}$ and e is a bar-unit in $\mathcal{U}$ then $\|e\| \geq 1$.

Just as a normed algebra is defined by means of only one inequality that compares the norm of the product of two elements with the product of the norms of these, a normed dialgebra may be equivalently defined as a dialgebra $\mathcal{U}$ for which $\|\alpha(x \vdash y) + (1-\alpha)(x \dashv y)\| \leq \|x\| \|y\|$ for any $\alpha \in [0,1]$ and all $x, y \in \mathcal{U}$.

The following is an example of an important type of Banach dialgebra with which we shall be more concerned. Next we associate to each Hilbert space a structure of normed dialgebra.

Example 3 *Let $\mathcal{H}$ be a Hilbert space and $e \in \mathcal{H}$ with $\|e\| = 1$. We define the following two bilinear operators*

$$a \vdash b = \langle a, e \rangle b, \qquad\qquad a \dashv b = \langle b, e \rangle a,$$

it is clear that in general $a \vdash b \neq a \dashv b$. In fact, if a and b are linearly independent vectors and moreover $a, b \notin \{e\}^{\perp}$ we have $a \vdash b \neq a \dashv b$. Then $(\mathcal{H}, \vdash, \dashv)$ is a normed dialgebra. First, we check that $(\mathcal{H}, \vdash, \dashv)$ is a dialgebra. Let a, b and c be elements of $\mathcal{H}$

$$a \dashv (b \dashv c) = a \dashv (\langle c, e \rangle b) = \langle \langle c, e \rangle b, e \rangle a = \langle c, e \rangle \langle b, e \rangle a,$$

on the other hand

$$a \dashv (b \vdash c) = a \dashv (\langle b, e \rangle c) = \langle \langle b, e \rangle c, e \rangle a = \langle b, e \rangle \langle c, e \rangle a,$$

from the two last equations it now follows that $a \dashv (b \dashv c) = a \dashv (b \vdash c)$. Next we must prove that $(a \vdash b) \dashv c = a \vdash (b \dashv c)$

$$(a \vdash b) \dashv c = (\langle a, e \rangle b) \dashv c = \langle c, e \rangle \langle a, e \rangle b$$

and

$$a \vdash (b \dashv c) = a \vdash (\langle c, e \rangle b) = \langle a, e \rangle \langle c, e \rangle b$$

then, as was claimed, the equality holds. Finally we have

$$(a \dashv b) \vdash c = (\langle b, e \rangle a) \vdash c = \langle \langle b, e \rangle a, e \rangle c = \langle b, e \rangle \langle a, e \rangle c,$$

also we have

$$(a \vdash b) \vdash c = (\langle a, e \rangle b) \vdash c = \langle \langle a, e \rangle b, e \rangle c = \langle a, e \rangle \langle b, e \rangle c,$$

so $(a \dashv b) \vdash c = (a \vdash b) \vdash c$ and we see that $(\mathcal{H}, \vdash, \dashv)$ is a dialgebra. Now, because of the Cauchy-Schwartz inequality we have

$$\|a \vdash b\| = \|\langle a, e\rangle\, b\| = |\langle a, e\rangle|\, \|b\| \leq \|a\|\, \|b\|\,,$$

and

$$\|a \dashv b\| = \|\langle b, e\rangle\, a\| = |\langle b, e\rangle|\, \|a\| \leq \|b\|\, \|a\|\,,$$

that is $\mathcal{H}$ is a normed dialgebra. Notice that e is a bar-unit of $(\mathcal{H}, \vdash, \dashv)$. From now on we will denote this normed dialgebra by $\mathcal{H}(e)$.

In many areas of classic mathematics, an "algebra" is understood to have a unit. This is not so in functional analysis, where examples of algebras include space of continous functions vanishing at infinity such as $\mathcal{C}_0(\mathbb{R})$, group algebras such as $L^1(\mathbb{R})$ and various other normed algebras without units, howover there exist a method for "adding units".

In our case, we suspect that in general the study of normed dialgebras does not reduce to study normed dialgebras with bar-unit. In fact, let $(\mathcal{U}, \vdash, \dashv, \|.\|)$ a normed dialgebra, and $\mathcal{U}_I = \mathcal{U} \times \mathbb{C}$. In $\mathcal{U}_I$ we define the following two operators:

$$(x, \alpha) \vdash (y, \beta) = (\alpha y + \beta x + (x \vdash y)\,, \alpha\beta) \tag{2}$$

and

$$(x, \alpha) \dashv (y, \beta) = (\alpha y + \beta x + (x \dashv y)\,, \alpha\beta) \tag{3}$$

Now define the usual norm in $\mathcal{U}_I$, i.e.

$$\|(x, \alpha)\| = \|x\| + |\alpha|\,, \tag{4}$$

it is clear that $(\theta, 1)$ will be a bar-unit of $\mathcal{U}_I$. We wish to show that $\mathcal{U}_I$ is a normed dialgebra, howewer we have.

Calculation 1

$$\begin{aligned}(x, \alpha) &\dashv ((y, \beta) \dashv (z, \gamma)) = (x, \alpha) \dashv (\beta z + \gamma y + (y \dashv z)\,, \beta\gamma) \qquad (5)\\ &= (\alpha(\beta z + \gamma y + (y \dashv z)) + \beta\gamma x + (x \dashv (\beta z + \gamma y + (y \dashv z)))\,, \alpha\beta\gamma)\\ &= = (\alpha\beta z + \alpha\gamma y + \alpha(y \dashv z) + \beta\gamma x + \beta(x \dashv z) + \gamma(x \dashv y) +\\ &\quad (x \dashv (y \dashv z))\,, \alpha\beta\gamma),\end{aligned}$$

Calculation 2

$$\begin{aligned}(x, \alpha) &\dashv ((y, \beta) \vdash (z, \gamma)) = (x, \alpha) \dashv (\beta z + \gamma y + (y \vdash z)\,, \beta\gamma) \qquad (6)\\ &= (\alpha(\beta z + \gamma y + (y \vdash z)) + \beta\gamma x + (x \dashv (\beta z + \gamma y + (y \vdash z)))\,, \alpha\beta\gamma)\\ &= (\alpha\beta z + \alpha\gamma y + \alpha(y \vdash z) + \beta\gamma x + \beta(x \dashv z) + \gamma(x \dashv y) +\\ &\quad (x \dashv (y \vdash z))\,, \alpha\beta\gamma),\end{aligned}$$

thus

$$(x,\alpha) \dashv ((y,\beta) \dashv (z,\gamma)) = (x,\alpha) \dashv ((y,\beta) \vdash (z,\gamma))$$

if, and only if

$$(y \dashv z) = (y \vdash z)$$

hence, we have arrived to the classic associative case.

Alternatively we try to introduce operators of circle type $\odot$ and $\circledcirc$ in $\mathcal{U}$ by defining

$$x \odot y = x + y - (x \vdash y)\,,$$

and

$$x \circledcirc y = x + y - (x \dashv y)\,,$$

Due to the following two results

$$\begin{aligned} \theta \odot x &= \theta + x - (\theta \vdash x) = x, \\ \theta \circledcirc x &= \theta + x - (\theta \dashv x) = x, \end{aligned}$$

one could hope that $(\mathcal{U}, \circledcirc, \odot)$ was a normed dialgebras, but again the answer in negative in general. Since,

Calculation 1

$$\begin{aligned} x \circledcirc (y \circledcirc z) &= x \circledcirc (y + z - (y \dashv z)) \\ &= x + y + z - (y \dashv z) - (x \dashv (y + z - (y \dashv z))) \\ &= x + y + z - (y \dashv z) - (x \dashv y) - (x \dashv z) \\ &\quad + (x \dashv (y \dashv z))\,, \end{aligned} \tag{7}$$

and also we have

Calculation 2

$$\begin{aligned} x \circledcirc (y \odot z) &= x \circledcirc (y + z - (y \vdash z)) \\ &= x + y + z - (y \vdash z) - (x \dashv (y + z - (y \vdash z))) \\ &= x + y + z - (y \vdash z) - (x \dashv y) - (x \dashv z) \\ &\quad + (x \dashv (y \vdash z))\,, \end{aligned} \tag{8}$$

then, from (7) and (8) we conclude that the circle type oprators don't in general generate normed dialgebras with a bar-unit unless $y \dashv z = y \vdash z$.

Thus, we postpone a general discussion of "adding bar-unit" for another time. However, we shall find throughout that the presence of a bar-identity in a dialgebra makes the theory simpler and more interesting than is possible in its absence.

3 Inverse element in normed dialgebras

The definition of inverse element is very important in many areas of mathematics. In this section we formulate what we wish to call the inverse for an element in a normed dialgebra. We will prove some results related with this concept.

Definition 4 *An element x in a dialgebra $(\mathcal{U}, \vdash, \dashv)$ is said to be $(\vdash)-$regular $((\dashv)-$ regular$)$ with respect to a bar-unit e provided there exists $y \in \mathcal{U}$, such that $x \vdash y = (e-x) + (x \vdash e)$ $(y \dashv x = (e-x) + (e \dashv x))$. The element y is called a $(\vdash)$-inverse $((\dashv)$-inverse$)$ for x with respect to e. An element which is both $(\vdash)-$regular and $(\dashv)-$regular with respect to e, is called regular if it has a $(\vdash)-$inverse that is also a $(\dashv)-$inverse, both with respect to e.*

It is interesting to note that if $\vdash$ is equal to $\dashv$ then these definitions coincide with the usual ones. On the other hand, any bar-unit of a dialgebra is a regular element, whereas θ is neither $(\vdash)-$regular or $(\dashv)-$regular.

The next Theorem enables us to characterize the regular elements in $\mathcal{H}(e)$

Theorem 5 *Let x be an element of $\mathcal{H}(e)$ such that $\langle x, e \rangle \neq 0$ then x is regular*

Proof. According to Definition 4 we must prove that x is $(\vdash)-$regular and $(\dashv)-$regular and that it has a $(\vdash)-$inverse that is also a $(\dashv)-$inverse with respect to e. To begin, let y be a $(\vdash)-$inverse of x then we must have

$$x \vdash y = \langle x, e \rangle y = (e-x) + (x \vdash e) = (e-x) + \langle x, e \rangle e$$

it follows that

$$y = \frac{(e-x)}{\langle x, e \rangle} + e.$$

As we will show this vector is also a $(\dashv)-$inverse of x, in fact

$$\begin{aligned} y \quad \dashv \quad x &= \left(\frac{(e-x)}{\langle x, e \rangle} + e \right) \dashv x \\ &= \langle x, e \rangle \left(\frac{(e-x)}{\langle x, e \rangle} + e \right) \\ &= (e-x) + \langle x, e \rangle e \\ &= (e-x) + (e \dashv x), \end{aligned}$$

notice that in this example the inverse is unique. This proves the Theorem. ■

Now, we consider the dialgebra $M_2(\mathcal{U})$ of matrices of 2×2 (see [2]). It is easy to see that $\widehat{e}=\begin{pmatrix} e & \theta \\ \theta & e \end{pmatrix}$ is a bar-unit of $M_2(\mathcal{U})$ and if a and b are $(\vdash)-$regular elements of $\mathcal{U}$ then $A=\begin{pmatrix} a & \theta \\ \theta & b \end{pmatrix}$ is $(\vdash)-$regular in $M_2(\mathcal{U})$.

The space $M_2(\mathcal{U})$ will play an outstanding role in the building of $R-$matrices on dialgebras, for instance let $Q=\begin{pmatrix} \theta & e \\ -e & \theta \end{pmatrix}$ be the symplectic type matrix of 2×2, then we have the two well known embeddings $M_2(\mathcal{U})\to M_3(\mathcal{U})$, that is

$$Q^{12}=\begin{pmatrix} \theta & e & \theta \\ -e & \theta & \theta \\ \theta & \theta & \theta \end{pmatrix}; \quad Q^{23}=\begin{pmatrix} \theta & \theta & \theta \\ \theta & \theta & e \\ \theta & -e & \theta \end{pmatrix},$$

it is now a simple computation to see that Q satisfies the nonassociative Artin type identity

$$Q^{23}\vdash Q^{12}\dashv Q^{23}=Q^{12}\vdash Q^{23}\dashv Q^{12},$$

beginning from this equation we can define the quantum and classical Yang-Baxter equations and finally $R-$matrices (see [2] for more detail).

Definition 6 *A Banach dialgebra is a normed dialgebra* $(\mathcal{U},\vdash,\dashv,\|.\|)$ *such that* $(\mathcal{U},\|.\|)$ *is a Banach space.*

Example 7 $\mathcal{H}(e)$ *is a Banach dialgebra.*

A fact of fundamental importance is the following: an incomplete normed dialgebra can always be regarded as a dense subset in a Banach dialgebra.

The precise statement is as follows:

Theorem 8 *Let* $(\mathcal{U},\vdash,\dashv,\|.\|)$ *be an incomplete normed dialgebra. There exists a Banach dialgebra* $\left(\widehat{\mathcal{U}},\vdash,\dashv,\|.\|\right)$ *and a normed sub- dialgebra* $\widehat{\mathcal{V}}$ *dense in* $\widehat{\mathcal{U}}$ *such that* $\widehat{\mathcal{V}}$ *and* $\mathcal{U}$ *are isometric.*

Proof. As in the usual Cantor-Meray completion we consider the set $\widehat{\mathcal{U}}$ of all the equivalent classes of fundamental sequences of elements in $\mathcal{U}$ with the following standard identifications:

(a) We consider two fundamental sequences of elements of $\mathcal{U}$ identical if and only if, the norm of their difference tends towards 0.

(b) A sequence consisting of identical elements we identify with that element. It is easy to show that each equivalent class will contain only one sequence of this type. The set of all classes which contain a sequence consisting of identical elements, is the set $\widehat{\mathcal{V}}$.

(c) The norm of a fundamental sequence of elements in $\mathcal{U}$ is defined as the limit of the norms of the elements in that sequence. The norm of a class is the norm of any sequence in that class.

It is well known that $\widehat{\mathcal{U}}$ is complete, $\widehat{\mathcal{V}}$ is dense in $\widehat{\mathcal{U}}$ and $\mathcal{U}$ is isometric to $\widehat{\mathcal{V}}$. To complete our proof it is sufficient to show that in $\widehat{\mathcal{U}}$ can be defined two operators $\vdash$ and $\dashv$ such that $\widehat{\mathcal{U}}$ is a normed dialgebra. With this purpose we define

$$\begin{aligned}\{x_n\} \vdash \{y_n\} &= \{x_n \vdash y_n\}\,, \\ \{x_n\} \dashv \{y_n\} &= \{x_n \dashv y_n\}\,,\end{aligned}$$

for any two Cauchy sequences $\{x_n\}$ and $\{y_n\}$. This definition makes sense since the sequences $\{x_n \vdash y_n\}$ and $\{x_n \dashv y_n\}$ are Cauchy sequences. In fact there are then two positive constants C_x and C_y such that for any n $\|x_n\| \leq C_x$ and $\|y_n\| \leq C_y$, so we have

$$\begin{aligned}\|(x_n \vdash y_n) - (x_m \vdash y_m)\| &\leq \|(x_n \vdash y_n) - (x_m \vdash y_m) \pm (x_m \vdash y_n)\| \\ &\leq \|(x_n - x_m) \vdash y_n + x_m \vdash (y_n - y_m)\| \\ &\leq \|(x_n - x_m) \vdash y_n\| + \|x_m \vdash (y_n - y_m)\| \\ &\leq \|x_n - x_m\| \, \|y_n\| + \|x_m\| \, \|y_n - y_m\| \\ &\leq \|x_n - x_m\| \, C_y + C_x \, \|y_n - y_m\|\end{aligned}$$

from this estimation it follows that $\{x_n \vdash y_n\}$ is a fundamental sequence. This argument also just works to see that the sequence $\{x_n \dashv y_n\}$ is fundamental. The next step is to define these operators on $\widehat{\mathcal{U}}$. For $\widehat{x}$ and $\widehat{y}$ elements of $\widehat{\mathcal{U}}$ and if $\{x_n\} \in \widehat{x}$ and $\{y_n\} \in \widehat{y}$ we define

$$\widehat{x} \vdash \widehat{y} = \widehat{z}, \qquad \widehat{x} \dashv \widehat{y} = \widehat{w},$$

where $z = \{x_n \vdash y_n\}$ and $w = \{x_n \dashv y_n\}$, now, it is a simple matter to prove that these operators become $\widehat{\mathcal{U}}$ in a dialgebra. On the other hand

$$\begin{aligned}\|\widehat{x} \vdash \widehat{y}\| &= \lim_{n\to\infty} \|x_n \vdash y_n\| \leq \lim_{n\to\infty} (\|x_n\| \, \|y_n\|) \\ &= \left(\lim_{n\to\infty} \|x_n\|\right) \left(\lim_{n\to\infty} \|y_n\|\right) = \|\widehat{x}\| \, \|\widehat{y}\|\,,\end{aligned}$$

we further have that $\|\widehat{x} \dashv \widehat{y}\| \leq \|\widehat{x}\| \, \|\widehat{y}\|$. This proves that $\widehat{\mathcal{U}}$ is a Banach dialgebra and finishes the proof of the theorem. ∎

Corollary 9 *If e is a bar-unit for an incomplete normed dialgebra $\mathcal{U}$, then $\widehat{e}$ is a bar-unit in it's completion $\widehat{\mathcal{U}}$.*

Proposition 10 *Let $(\mathcal{U},\vdash,\dashv,\|.\|)$ be a Banach dialgebra with a bar-unit e, we define*

$$\begin{array}{l}(e-x)^{(\vdash,n)}=(e-x)\vdash(e-x)\vdash\cdots\vdash(e-x)\,,\\(e-x)^{(\dashv,n)}=(e-x)\dashv(e-x)\dashv\cdots\dashv(e-x)\,,\end{array}\tag{9}$$

where in each product of the right side of (9) we have n factors. Then

$$\left\|(e-x)^{(\vdash,n)}\right\|\leq\|(e-x)\|^{n}\,,\tag{10}$$

and

$$\left\|(e-x)^{(\dashv,n)}\right\|\leq\|(e-x)\|^{n}\,,\tag{11}$$

hence, if for $x\in\mathcal{U}$, holds that $\|(e-x)\|<1$, the infinite series $e+(e-x)+(e-x)^{(\vdash,2)}+\cdots+(e-x)^{(\vdash,n)}+\cdots$ and $e+(e-x)+(e-x)^{(\dashv,2)}+\cdots+(e-x)^{(\dashv,n)}+\cdots$ converges absolutely to elements of $\mathcal{U}$.

We have

Proposition 11 *Let $(\mathcal{U},\vdash,\dashv,\|.\|)$ be a Banach dialgebra and e a bar-unit. If $\|(e-x)\|<1$, then x is $(\vdash)-$regular and $(\dashv)-$regular with respect to e. In this case, it's inverses are given by the series*

$$\begin{array}{l}y_{\vdash}=e+(e-x)+(e-x)^{(\vdash,2)}+\cdots+(e-x)^{(\vdash,n)}+\cdots,\\y_{\dashv}=e+(e-x)+(e-x)^{(\dashv,2)}+\cdots+(e-x)^{(\dashv,n)}+\cdots.\end{array}$$

Proof. Since $\|(e-x)\|<1$ and $(\mathcal{U},\|.\|)$ is complete, the above series converges to an elements of $\mathcal{U}$. Have in mind the continuity of the two products of $\mathcal{U}$ with respect to both arguments and properties of absolutely convergent series in a Banach space

$$\begin{array}{rcl}(e-(e-x)) & \vdash & \left(e+(e-x)+(e-x)^{(\vdash,2)}+\cdots+(e-x)^{(\vdash,n)}+\cdots\right)\\ & = & e+(e-x)-((e-x)\vdash e)\\ & = & (e-x)+(x\vdash e)\,,\end{array}$$

in the same way, we have

$$\begin{array}{rcl}\left(e+(e-x)+(e-x)^{(\dashv,2)}+\cdots+(e-x)^{(\dashv,n)}+\cdots\right) & \dashv & (e-(e-x))\\ & = & 2e-x-e+(e\dashv x)\\ & = & (e-x)+(x\dashv e)\,.\end{array}$$

■

Remark 12 *The above series for the* $(\vdash)$ *−inverse and the* $(\dashv)$ *−inverse are generalizations of the classical Neuman series.*

Lemma 13 *If* x *is regular with respect to a bar-unit* e*, then*

$$x \vdash (e - x) = (e - x) \dashv x \tag{12}$$

Proof. Let x be regular, then there exists y, such that

$$x \vdash y = (e - x) + (x \vdash e),$$

and

$$y \dashv x = (e - x) + (e \dashv x),$$

from the first equation it follows that

$$(x \vdash y) \dashv x = x \vdash (y \dashv x) = ((e - x) + (x \vdash e)) \dashv x, \tag{13}$$

making use now of the second equation to replace $(y \dashv x)$ in (13) we obtain

$$x \vdash ((e - x) + (e \dashv x)) = ((e - x) + (x \vdash e)) \dashv x,$$

thus,

$$(x \vdash (e - x)) + (x \vdash (e \dashv x)) = ((e - x) \dashv x) + ((x \vdash e) \dashv x),$$

the last term in both sides of this equation is the same. This proves the Lemma. ■

Proposition 14 *If* x *satisfies* (12) *where* e *is a bar-unit, then for all* n

$$(e - x)^{(\vdash, n)} = (e - x)^{(\dashv, n)} \tag{14}$$

Proof. We proceed for induction. For $n = 2$ we have

$$\begin{aligned} (e - x)^{(\vdash,2)} &= (e - x) \vdash (e - x) \\ &= e \vdash (e - x) - x \vdash (e - x) \\ &= (e - x) - (e - x) \dashv x \\ &= (e - x) \dashv (e - x) \\ &= (e - x)^{(\dashv,2)}, \end{aligned}$$

we assume now that

$$(e-x)^{(\vdash,k)}=(e-x)^{(\dashv,k)},$$

where $k\geq 3$ then

$$\begin{aligned}
(e-x)^{(\vdash,k+1)} &= (e-x)^{(\vdash,k)}\vdash(e-x)\\
&= (e-x)^{(\dashv,k)}\vdash(e-x)\\
&= \Big((e-x)^{(\dashv,k-1)}\dashv(e-x)\Big)\vdash(e-x)\\
&= (e-x)^{(\dashv,k-1)}\dashv((e-x)\vdash(e-x))\\
&= (e-x)^{(\dashv,k-1)}\dashv((e-x)\dashv(e-x))\\
&= \Big((e-x)^{(\dashv,k-1)}\dashv(e-x)\Big)\dashv(e-x)\\
&= (e-x)^{(\dashv,k)}.
\end{aligned}$$

■

Theorem 15 *If x satisfies* (12)*, where e is a bar-unit and $\|e-x\|<1$, then x is regular with respect to e.*

Proof. The Theorem is immediate from Propositions 5 and 7. ■

4 Ideals in normed dialgebras

Below we will assume that $\mathcal{U}$ is a normed dialgebra with at least a bar-unit.

Definition 16 *A subset E of a normed dialgebra $\mathcal{U}$, is said to be a $(\vdash)-$ ideal provided it is a linear subspace such that $x\vdash y$, $y\vdash x\in E$ for all $x\in E$ and $y\in\mathcal{U}$. It is a $(\dashv)-$ ideal if the latter condition is replaced by $y\dashv x$, $x\dashv y\in E$ for all $x\in E$ and $y\in\mathcal{U}$. If E is both a $(\vdash)-$ ideal and a $(\dashv)-$ ideal, then it is called a two-sided ideal of $\mathcal{U}$. Any ideal of the same type, different from $\mathcal{U}$ is called proper.*

we have

Lemma 17 *Let $x\in\mathcal{U}$ be $(\vdash)-$regular with respect to a bar-unit e, then for all $z\in\mathcal{U}$*

$$(x\vdash y)\vdash z=z \tag{15}$$

where y is the $(\vdash)-$inverse of x.

Proof. Let $x \in \mathcal{U}$ be $(\vdash)-$regular with respect to a bar-unit e, as we have already known this means, that

$$x \vdash y = (e - x) + (x \vdash e),$$

and hence, for all $z \in \mathcal{U}$

$$\begin{aligned} (x \vdash y) \quad \vdash \quad & z = ((e - x) + (x \vdash e)) \vdash z \\ = \quad & (e - x) \vdash z + (x \vdash e) \vdash z \\ = \quad & (e \vdash z) - (x \vdash z) + (x \dashv e) \vdash z \\ = \quad & z. \end{aligned}$$

■

Corollary 18 *If x is $(\vdash)-$regular with respect to a bar-unit e, then it can't belong to a proper $(\vdash)-$ideal.*

Proof. Since, x is an element $(\vdash)-$regular with respect to a bar-unit e and E a $(\vdash)-$ideal, such that $x \in E$, we have that for all $z \in \mathcal{U}$, $z = (x \vdash y) \vdash z = x \vdash (y \vdash z) \in E$, where y is a $(\vdash)-$inverse of x with respect to a bar-unit e. ■

Two similar statements hold for $(\dashv)-$regular elements of $\mathcal{U}$ and proper $(\dashv)-$ideals, more exactly

Lemma 19 *If $x \in \mathcal{U}$ is $(\dashv)-$regular with respect to a bar-unit e, then for all $z \in \mathcal{U}$*

$$z \dashv (y \dashv x) = z \tag{16}$$

where y is the $(\dashv)-$inverse of x. In this case, x can't belong to a proper $(\dashv)-$ideal.

Note also that if x is not a $(\vdash)-$regular element with respect to a bar-unit e, then $E_x^{\vdash} = \{x \vdash z,\ z \vdash x \mid z \in \mathcal{U}\}$ is a proper $(\vdash)-$ ideal, in fact the element $(e - x) + (x \vdash e)$ cannot be in $E_x^{\vdash}$. Thus, for that $x \in \mathcal{U}$ to be $(\vdash)-$regular with respect to all bar-units of $\mathcal{U}$, it is necessary and sufficient that this element does not belong to any proper $(\vdash)-$ideal. It must be remarked that if we replace $(\vdash)-$regular elements by $(\dashv)-$regular elements the statement is the following: for that $x \in \mathcal{U}$ to be $(\dashv)-$regular with respect to all bar-units of $\mathcal{U}$ it is necessary and sufficient that this element does not belong to any proper $(\dashv)-$ideal.

Theorem 20 *Let $(\mathcal{U}, \vdash, \dashv, \|.\|)$ be a complex normed dialgebra and E a proper ideal then $\mathcal{U} / E$ is a normed dialgebra.*

Proof. The operations of vector space and the norm (the infimum norm) in $\mathcal{U}/E$ are defined as in the classic associative case. We introduce two products $\vdash$ and $\dashv$ in $\mathcal{U}/E$ and check that with respect to these products it is a dialgebra. For an element z of $\mathcal{U}$ we denote it's equivalence class by means of $[z]$, then we define $[x]\vdash[y]=[x\vdash y]$ and $[x]\dashv[y]=[x\dashv y]$. It is not difficult to see that

$$[x]\dashv([y]\dashv[z])=[x]\dashv([y]\vdash[z]),$$

$$([x]\vdash[y])\dashv[z]=[x]\vdash([y]\dashv[z]),$$

$$([x]\dashv[y])\vdash[z]=([x]\vdash[y])\vdash[z],$$

for instance

$$\begin{aligned}[x]\dashv([y]\dashv[z]) &= [x]\dashv[(y\dashv z)]\\ &= [x\dashv(y\dashv z)]\\ &= [x\dashv(y\vdash z)]\\ &= [x]\dashv[(y\vdash z)]\\ &= [x]\dashv([y]\vdash[z]).\end{aligned}$$

The rest of the axioms are proved in similar form. Now, $\mathcal{U}/I$ is also a normed dialgebra, in fact

$$\begin{aligned}\|([x]\vdash[y])\| &= \inf_{h\in I}\|(x\vdash y)+h\|\\ &\leq \inf_{g\in I,\ f\in I}\|(x+g)\vdash(y+f)\|\\ &\leq \inf_{g\in I,\ f\in I}\|x+g\|\,\|y+f\|\\ &\leq \left(\inf_{g\in I}\|x+g\|\right)\left(\inf_{f\in I}\|y+f\|\right)\\ &= \|[x]\|\,\|[y]\|.\end{aligned}$$

one can also prove that $\|([x]\dashv[y])\|\leq\|[x]\|\,\|[y]\|$. ■

Note that if e is a bar-unit of $\mathcal{U}$ then $[e]$ is a bar-unit of $\mathcal{U}/E$.

Definition 21 *Let $(\mathcal{U},\vdash,\dashv,\|.\|)$ be a Banach dialgebra. A complex linear functional φ on $\mathcal{U}$ which is not identically 0 is said to be multiplicative if:*

(1) $$\varphi(x\vdash y)=\varphi(x)\varphi(y)$$

(2) $$\varphi(x\dashv y)=\varphi(x)\varphi(y)$$

The set of all multiplicative linear functionals on $\mathcal{U}$ will be denoted by M. Let φ be a multiplicative linear functional then $\varphi(e) = 1$ for any bar-unit e in $\mathcal{U}$, in fact for some $z \in \mathcal{U}$, $\varphi(z) \neq 0$, since $\varphi(z) = \varphi(e \vdash z) = \varphi(e)\varphi(z)$ it follows that $\varphi(e) = 1$. If x is a $(\vdash)-$regular element of $\mathcal{U}$ with respect to the bar-unit e and y is a $(\vdash)-$inverse of x then for any φ in M we have that $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$. In fact, we know that $x \vdash y = (e - x) + (x \vdash e)$ hence $\varphi(x)\varphi(y) = \varphi(x \vdash y) = \varphi((e - x) + (x \vdash e)) = \varphi(e - x) + \varphi(x \vdash e) = \varphi(e) - \varphi(x) + \varphi(x)\varphi(e) = 1$, which implies $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$. It can be also easily verified that if x is a $(\dashv)-$regular element of $\mathcal{U}$ with respect to the bar-unit e and y is a $(\dashv)-$inverse of x then for any φ in M we have that $\varphi(x) \neq 0$ and $\varphi(y) \neq 0$.

We will show that the elements of M are bounded.

Proposition 22 *If $(\mathcal{U}, \vdash, \dashv, \|.\|)$ is a Banach dialgebra with a bar-unit e and $\varphi \in M$, then $\|\varphi\| = 1$.*

Proof. Since $\varphi(x - \varphi(x)e) = 0$, all elements x in $\mathcal{U}$ can be written in the form $x = z + \lambda e$ where $z \in \mathcal{U}$ such that $\varphi(z) = 0$ and $\lambda \in \mathbf{C}$, Thus

$$\sup_{x \neq \theta} \frac{|\varphi(x)|}{\|x\|} = \sup_{\substack{z \in \ker \varphi, \\ \lambda \neq 0}} \frac{|\varphi(z + \lambda e)|}{\|z + \lambda e\|} = \sup_{\substack{z \in \ker \varphi, \\ \lambda \neq 0}} \frac{|\lambda|}{\|z + \lambda e\|} = \sup_{w \in \ker \varphi} \frac{1}{\|e + w\|} = 1.$$

because $\|e + w\| < 1$ implies that w is for instance $(\vdash)-$regular by Proposition 5, which implies in turn that w is not in $\ker \varphi$. Therefore $\|\varphi\| = 1$ and the proof is complete. ■

Definition 23 *For the Banach dialgebra $(\mathcal{U}, \vdash, \dashv, \|.\|)$ we define the Gelfand transform as the function $\widehat{\cdot} : \mathcal{U} \to \mathcal{C}(M)$ given by $\widehat{x}(\varphi) = \varphi(x)$ for $\varphi \in M$.*

If $(\mathcal{U}, \vdash, \dashv, \|.\|)$ is a Banach dialgebra and $\widehat{\cdot}$ is the Gelfand transform on $\mathcal{U}$, then it is immediate that the following properties are hold: $\widehat{(x \vdash y)} = \widehat{(x)}.\widehat{(y)}$, $\widehat{(x \dashv y)} = \widehat{(x)}.\widehat{(y)}$ and $\widehat{\cdot}$ is a contractive mappings.

5 Dialgebras with an involution

We start this section whit a definition of involution in dialgebras.

Definition 24 *Let $(\mathcal{U}, \vdash, \dashv)$ be a complex dialgebra. A mapping $x \to x^*$ of $\mathcal{U}$ onto itself is called an involution of type I provided the following conditions are satisfied:*

$$\begin{array}{lc}
(i) & (x^*)^* = x, \\
(ii) & (x + y)^* = x^* + y^*, \\
(iii) & (x \vdash y)^* = y^* \dashv x^*, \\
(iv) & . \quad (\alpha x)^* = \overline{\alpha} x^*,
\end{array}$$

note that from (iii) it follows the following equality

$$(x \dashv y)^* = y^* \vdash x^*,$$

on the other hand, the mapping $x \to x^*$ is said to be an involution of type II if it satisfies all the properties $(i) - (iv)$ except (iii) which is substituted by the following condition:

$$(iii)' \qquad (x \vdash y)^* = y^* \vdash x^*;$$

finally when the condition (iii) is substituted for the following one

$$(iii)'' \qquad (x \dashv y)^* = y^* \dashv x^*.$$

we may say that $x \to x^*$ is an involution of type III.

A complex dialgebra with an involution of type I (respectively of type II or type III) is called a $*-$dialgebra of type I (respectively of type II or type III).

Let e be a bar-unit of $\mathcal{U}$ and $*$ an involution of type I in $\mathcal{U}$, then $e^* \vdash x = (x^* \dashv e)^* = (x^*)^* = x$ and $x \dashv e^* = (e \vdash x^*)^* = (x^*)^* = x$, thus e^* is also a bar-unit of $\mathcal{U}$. In this case we remark that in contrast to the usual case $x \vdash x^*$, $x \dashv x^*$, $x^* \vdash x$ and $x^* \dashv x$ are not in general selfadjoint. However, the elements $((x \vdash x^*) \pm (x \dashv x^*))$ and $((x^* \vdash x) \pm (x^* \dashv x))$ are selfadjoint for all $x \in \mathcal{U}$.

We have

Proposition 25 *Let e be a bar-unit of $\mathcal{U}$ and $*$ an involution of type I defined on $\mathcal{U}$. If x is $(\vdash)$-regular for e, then x^* is $(\dashv)$-regular for e^*.*

Proof. since x is $(\vdash)$-regular, there exists $y \in \mathcal{U}$ such that

$$x \vdash y = (e - x) + (x \vdash e)$$

then

$$(x \vdash y)^* = (e - x)^* + (x \vdash e)^*$$

therefore

$$y^* \dashv x^* = (e^* - x^*) + (e^* \dashv x^*).$$

■

A similar statement can, of course, be made for $(\dashv)$-regular elements, that is $x \to x^*$ transforms $(\dashv)$-regular elements of $\mathcal{U}$ with respect to e into $(\vdash)$-regular with respect to e^*.

Example 26 *Let $e \in \mathbb{R}^n$ such that $\|e\| = 1$. Let us consider the normed dialgebra $\mathbb{C}^n(e)$ (see Example 3) then it is a $*-$dialgebra of type I with the usual involution $z \to \overline{z}$. We only examine that the condition (iii) of the Definition 19 holds, since the rest of the conditions are evident. If $x, y \in \mathbb{C}^n$ then*

$$(x \vdash y)^* = \overline{\langle x, e\rangle y} = \langle e, x\rangle\, \overline{y},$$

and

$$y^* \dashv x^* = \langle \overline{x}, e\rangle\, \overline{y}$$

have in mind that $e \in \mathbb{R}^n$ we conclude that $(x \vdash y)^* = y^* \dashv x^*$.

Example 27 *Assume that $(\mathcal{U}, \vdash, \dashv)$ is a dialgebra with an involution $*$ of type I, then in the dialgebra $M_n(\mathcal{U})$ of matrices of $n \times n$ (see [2]) is given an involution of the same type defined in the following way: if $A = (a_{ij})$*

$$(A^*)_{ij} = a^*_{ji},$$

we only check the properties (iii), if $X = (x_{ij})$ and $Y = (y_{ij})$ we have

$$\begin{aligned} \left((X \vdash Y)^*\right)_{ij} &= \left((X \vdash Y)_{ji}\right)^* = \left(\sum_k (x_{jk} \vdash y_{ki})\right)^* \\ &= \sum_k (x_{jk} \vdash y_{ki})^* \\ &= \sum_k \left((y_{ki})^* \dashv (x_{jk})^*\right), \end{aligned}$$

on the other hand

$$\begin{aligned} (Y^* \dashv X^*)_{ij} &= \sum_k \left(y^*_{ik} \dashv x^*_{kj}\right) \\ &= \sum_k \left((y_{ki})^* \dashv (x_{jk})^*\right), \end{aligned}$$

therefore $(X \vdash Y)^* = Y^* \dashv X^*$.

Definition 28 *A normed involutive dialgebra of type I (respectively of type II or type III) is a normed $*-$dialgebra $\mathcal{U}$ of type I (respectively of type II or type III) such that $\|x^*\| = \|x\|$ for each $x \in \mathcal{U}$. If in addition, $\mathcal{U}$ is complete then $\mathcal{U}$ is called an involutive Banach dialgebra of type I (respectively of type II or type III). A C^*-dialgebra of type I (respectively of type II or type III) is an involutive Banach dialgebra $\mathcal{U}$ of type I (respectively of type II or type III) such that $\|x^* \vdash x\| = \|x\|^2 = \|x \dashv x^*\|$.*

Example 29 *Let $e \in \mathbb{R}^n$ such that $\|e\| = 1$, then $\mathbb{C}^n(e)$ is an involutive Banach dialgebra of type I, but it will not be a C^*-dialgebra.*

It is clear that in a normed involutive dialgebra of type I (respectively of type II or type III) the involution is continuous.

Since $(x^* \dashv x)^* = (x^* \vdash x)$, from this definition we see that in a normed involutive dialgebra of type I

$$\|x\|^2 = \|(x^* \vdash x)\| = \left\|(x^* \dashv x)^*\right\| = \|(x^* \dashv x)\|,$$

and also because $(x^* \vdash x)^* = (x^* \dashv x)$ then it may be seen that $\|x\|^2 = \|x \vdash x^*\|$.

Let $\mathcal{U}$ an involutive Banach dialgebra of type I such that

$$\|x\|^2 \leq \|x^* \vdash x\|, \quad \|x\|^2 \leq \|x \dashv x^*\|,$$

it follows of the first inequality that $\|x\|^2 \leq \|x^*\| \|x\|$, hence that $\|x\| \leq \|x^*\|$ and interchanging x and x^* we see that $\|x^*\| = \|x\|$. The above suppositions then imply that

$$\|x\|^2 \leq \|x^* \vdash x\| \leq \|x^*\| \|x\| = \|x\|^2$$

and

$$\|x\|^2 \leq \|x \dashv x^*\| \leq \|x\| \|x^*\| = \|x\|^2$$

so that $\mathcal{U}$ is a C^*-dialgebra of type I.

Let $\mathcal{U}$ be a C^*-dialgebra of type I, then for each x in $\mathcal{U}$

$$\|x\| = \sup_{\|y\| \leq 1} \|x \dashv y\| = \sup_{\|y\| \leq 1} \|y \vdash x\|.$$

In fact, let y be such that $\|y\| \leq 1$ we have $\|x \dashv y\| \leq \|x\|$ and also $\|y \vdash x\| \leq \|x\|$. To prove the inequalities $\|x\| \leq \sup_{\|y\| \leq 1} \|x \dashv y\|$ and $\|x\| \leq \sup_{\|y\| \leq 1} \|y \vdash x\|$ we can assume that $\|x\| = 1$, then $\|x^*\| = 1$, therefore

$$\sup_{\|y\| \leq 1} \|x \dashv y\| \geq \|x \dashv x^*\| = \|x\|^2 = 1$$

and also

$$\sup_{\|y\| \leq 1} \|y \vdash x\| \geq \|x^* \vdash x\| = \|x\|^2 = 1.$$

Let $\mathcal{U}$ be a C^*-dialgebra of type I with a bar-unit e, since e^* is also a bar-unit and we have

$$\|e\|^2 = \|e^* \vdash e\| = \|e\|$$

so that $\|e\| = 1$ or 0. Thus, we see that unless $\mathcal{U} = \{\theta\}$ $\|e\| = 1$. Therefore any bar-unit in $\mathcal{U}$ has norm one.

We proceed now to study ideals in a $*-$dialgebra of type I.

Theorem 30 *If E is a $(\vdash)-$ideal $((\dashv)-$ideal$)$ in a $*-$dialgebra $\mathcal{U}$ of type I. the set E^* of all adjoints z^* of elements z of E is a $(\dashv)-$ideal $((\vdash)-$ideal$)$.*

Proof. Since E^* is evidently a subspace, it is necessary only to prove-when E is a $(\vdash)-$ideal-that E^* contains $x^* \dashv y$ and $y \dashv x^*$ for all $x \in E$ and all $y \in \mathcal{U}$. But this follows at once the following relations $x^* \dashv y = (y^* \vdash x)^*$, $y \dashv x^* = (x \vdash y^*)^*$ and the fact that $(y^* \vdash x)$ and $(x \vdash y^*)$ belong to E. When E is a $(\dashv)-$ideal a similar argument is valid. ■

Corollary 31 *A sufficient condition that a $(\vdash)-$ideal $((\dashv)-$ideal$)$ E be two-sided ideal is that $E = E^*$.*

6 Operator theory in a normed dialgebra

Most of linear algebra involves the study of mapping between linear spaces which preserve the linear structure, that is, linear mapping, such should also be the case in the study of normed dialgebras. In this section some properties of operators defined in a normed dialgebra $\mathcal{U} \neq \{\theta\}$ are studied. In particular, a dialgebra of bounded mapping which are defined on the whole $\mathcal{U}$, is constructed.

Let $\mathcal{U}$ be a normed dialgebra with a bar-unit e. Let $L(\mathcal{U})$ denote the set of all bounded linear mapping acting on $\mathcal{U}$ that are defined on the whole $\mathcal{U}$, that is, $A : \mathcal{U} \to \mathcal{U}$ and $D(A) = \mathcal{U}$ for all $A \in L(\mathcal{U})$. Let A and B be elements of $L(\mathcal{U})$. Then we define $A \vdash B$ and $A \dashv B$ in the following form

$$(A \vdash B)(u) = Ae \vdash Bu, \qquad (A \dashv B)(u) = Au \dashv Be, \tag{17}$$

clearly $A \vdash B$ and $A \dashv B$ are linear mapping. Observe that the definition (17) depends on e. We denote to the space $L(\mathcal{U})$ with the operators $\vdash$ and $\dashv$ defined by means of (17) as $L(\mathcal{U}, e)$

Proposition 32 *The operators $A \vdash B$ and $A \dashv B$ are elements of $L(\mathcal{U}, e)$.*

Proof. Is obvious that these operators are defined on the whole $\mathcal{U}$. Thus, it remains to see that they are bounded. Indeed

$$\|(A \vdash B)(u)\| = \|Ae \vdash Bu\| \leq \|Ae\| \|Bu\| \leq \|A\| \|B\| \|u\|, \tag{18}$$

in the same way may be proved that $A \dashv B$ is bounded and that

$$\|(A \dashv B)(u)\| \leq \|A\| \, \|B\| \, \|u\| \, . \tag{19}$$

■

Theorem 33 *$(L(\mathcal{U}, e), \vdash, \dashv)$ is a normed dialgebra.*

Proof. From (18) and (19) it follows that $\|A \vdash B\| \leq \|A\| \, \|B\|$ and also $\|A \dashv B\| \leq \|A\| \, \|B\|$. To prove the proposition it is sufficient to see that $(L(\mathcal{U}), \vdash, \dashv)$ is a dialgebra. By (17), we have for A, B and C in $L(\mathcal{U})$

$$\begin{aligned} (A \dashv (B \dashv C))(u) &= Au \dashv (B \dashv C)(e) \\ &= Au \dashv (Be \dashv Ce), \end{aligned}$$

on the other hand

$$\begin{aligned} (A \dashv (B \vdash C))(u) &= Au \dashv (B \vdash C)(e) \\ &= Au \dashv (Be \vdash Ce), \end{aligned}$$

since $\mathcal{U}$ is a dialgebra then $Au \dashv (Be \dashv Ce) = Au \dashv (Be \vdash Ce)$. Therefore we have $A \dashv (B \dashv C) = (A \dashv (B \vdash C))$.

In a similar way one can show that $(A \vdash B) \dashv C = A \vdash (B \dashv C)$. In fact

$$\begin{aligned} ((A \vdash B) \dashv C)(u) &= (A \vdash B)(u) \dashv Ce \\ &= (Ae \vdash Bu) \dashv Ce, \end{aligned}$$

and we check

$$\begin{aligned} (A \vdash (B \dashv C))(u) &= Ae \vdash (B \dashv C)(u) \\ &= Ae \vdash (Bu \dashv Ce), \end{aligned}$$

using now the fact that $\mathcal{U}$ is a dialgebra we have $((A \vdash B) \dashv C) = (A \vdash (B \dashv C))$. The reader easily examines that $(A \dashv B) \vdash C = (A \vdash B) \vdash C$. This concludes our proof. ■

Let I be the identity operator on $\mathcal{U}$, then I is a bar-unit of $(L(\mathcal{U}, e), \vdash, \dashv)$. Observe that then $L(\mathcal{U}, e)$ is a Liebniz algebra. The bracket in $(L(\mathcal{U}, e), \vdash, \dashv)$ is defined for A, B in $L(\mathcal{U}, e)$ as

$$[A, B] = (A \dashv B) - (B \vdash A) \tag{20}$$

Now, we consider the space $L(\mathcal{U})$ when $\mathcal{U}$ is C^*-dialgebra of type I. Let T be a linear operator on $\mathcal{U}$ defined on a dense linear domain $D(T)$. We define the adjoint operator T^* in the following form

$$D(T) = \left\{x \in \mathcal{U} \mid \exists\, y \text{ such that } \left((Tu)^* \vdash x = u^* \dashv y\right) \forall u \in D(T)\right\}$$

and we set $T^*x = y$. One can easily check that T^* is a closed linear mapping. Note that even if T is bounded there is no guarantee that $D(T^*)$ is dense in $\mathcal{U}$. However, if $T \in L(\mathcal{U})$ then we can show that T^* is bounded. Let e be a bar-unit of $\mathcal{U}$ and $L_0(\mathcal{U}, e) = \{T \in L(\mathcal{U}, e) \mid T^* \in L(\mathcal{U}, e)\}$.

Theorem 34 *Let $\mathcal{U}$ be a C^*-dialgebra of type I and e a bar-unit of $\mathcal{U}$ such that $e^* = e$, then $L_0(\mathcal{U}, e)$ is a $*-$dialgebra of type II with respect to the mapping $T \to T^*$ where $T \in L_0(\mathcal{U}, e)$.*

Proof. It is easy to verify that the adjoint of elements in $L_0(\mathcal{U}, e)$ obeys the properties (i), (ii) and (iv) of the definition 17. Now let A, B and C be elements of $L_0(\mathcal{U}, e)$ then we have for any u and x in $\mathcal{U}$

$$\begin{aligned}
((A \vdash B)(u))^* &\vdash x = (Ae \vdash Bu)^* \vdash x \\
&= \left((Bu)^* \dashv (Ae)^*\right) \vdash x \\
&= \left((Bu)^* \vdash (Ae)^*\right) \vdash x \\
&= (Bu)^* \vdash \left((Ae)^* \vdash x\right) \\
&= (Bu)^* \vdash (e \dashv A^*x) \\
&= \left((Bu)^* \vdash e\right) \dashv A^*x \\
&= (u^* \dashv B^*e) \dashv A^*x \\
&= u^* \dashv (B^*e \dashv A^*x) \\
&= u^* \dashv (B^*e \vdash A^*x) \\
&= u^* \dashv (B^* \vdash A^*)(x)
\end{aligned}$$

on the other hand $((A \vdash B)(u))^* \vdash x = u^* \dashv (A \vdash B)^* x$ for any u and x in $\mathcal{U}$. Thus

$$(A \vdash B)^* = (B^* \vdash A^*).$$

Therefore $L_0(\mathcal{U}, e)$ has all the required properties in Definition 17 for an involution of type II. ■

We continue to investigate the normed dialgebra $L(\mathcal{H}(e), e)$.

Theorem 35 *$L(\mathcal{H}(e), e)$ is an involutive Banach dialgebra of type I, with respect to the usual definition of adjoint for a bounded linear operator of $\mathcal{H}$ into $\mathcal{H}$.*

Proof. Let A and B be two bounded linear operators of $\mathcal{H}$ into $\mathcal{H}$ and $x, y \in \mathcal{H}$ then we have

$$\begin{aligned}
\langle (A \vdash B)\, x, y\rangle &= \langle (Ae \vdash Bx)\, , y\rangle \\
&= \langle \langle Ae, e\rangle\, Bx, y\rangle \\
&= \langle Ae, e\rangle\, \langle Bx, y\rangle \\
&= \langle e, A^* e\rangle\, \langle x, B^* y\rangle \\
&= \left\langle x, \overline{\langle e, A^* e\rangle} B^* y\right\rangle \\
&= \langle x, \langle A^* e, e\rangle\, B^* y\rangle \\
&= \langle x, B^* y \dashv A^* e\rangle = \langle x, (B^* \dashv A^*)\, y\rangle
\end{aligned}$$

from this it follows that $(A \vdash B)^* = B^* \dashv A^*$. The rest of conditions for a involutive Banach dialgebra of type I clearly hold. This proves the Theorem. ∎

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