DIALGEBRAS AND A CLASS OF MATRIX "COQUECIGRUES"

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Abstract. Starting with the Leibniz algebra defined by a φ-dialgebra, we construct examples of "coquecigrues," in the sense of Loday; that is to say, manifolds whose tangent structure at a distinguished point coincides with that of the Leibniz algebra. We discuss some possible implications and generalizations of this construction.

Introduction.

Around 1990 J. L. Loday (see *e.g.*, [L1]) introduced the notion of a Leibniz algebra, a generalization of a Lie algebra where the skew-symmetry of the bracket is supressed; and although his initial (and main) motivation was the homology theory that can be defined on them, it was soon realized that Leibniz algebras were useful in a variety of contexts.

More to the point of this work, in the paper quoted, Loday also posed "Lie's third problem for Leibniz algebras." That is, given a (say) finite dimensional Leibniz algebra, to find a manifold with an algebraic operation, whose tangent structure at some distinguished point would inherit the structure of the Leibniz algebra. Since no such objects (besides the "trivial" case of Lie groups) were explicitly known, they were jokingly dubbed by him "coquecigrues" (which translates to something like "nonsense"); and their construction has proven to be quite an elusive task indeed (see *e.g.* [KW]).

On the other hand, a few years later in [L2], Loday also introduced the notion of *dialgebra*—which is in turn a generalization of associative algebra, but possessing two operations—and showed the existence of a functor relating Leibniz algebras to dialgebras, analogous to the functor existing between Lie algebras and associative algebras. Quite recently, Loday's definition of dialgebra was taken as a basis to define *digroups* (*cf.* [F],

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[K], [L]), where the key elements are the introduction of an appropriate notion of neutral element and inversion.

Following this line of reasoning, in this paper I will construct some very explicit examples of manifolds, with the algebraic structure of a digroup, that are not Lie groups, but that have the essential properties required for a "coquecigrue." Moreover, we shall see that they have a rather nice geometrical structure.

Certainly, I do not claim to have solved the general problem posed by Loday: my point is rather, as we shall see, that the Leibniz algebra structure by itself might not in general uniquely determine the type of integral manifold. Hence, most likely there are different classes of "coquecigrues" (and so the ones discussed here would be just one of them); I hope nevertheless that the construction given here sheds some additional light into a possible general structure of these intriguing objects.

I. φ-dialgebras and coquecigrues.

Let us begin by recalling that a dialgebra is a vector space, V, together with two bilinear, associative operations, $\dashv$, $\vdash$, satisfying the relations

$$x \dashv (y \dashv z) = x \dashv (y \vdash z)$$
$$(x \vdash y) \dashv z = x \vdash (y \dashv z)$$
$$(x \dashv y) \vdash z = (x \vdash y) \vdash z.$$

And a well known fact is that a dialgebra canonically defines a Leibniz algebra, with bracket

$$[x, y] = x \dashv y - y \vdash x.$$

Also, recall that a non-trivial bar unit in a dialgebra is an element, e, satisfying

$$e \vdash x = x = x \dashv e; \quad \forall x \in V.$$

Here non-trivial just means that the corresponding relations from the pointer side (*i.e.*, $e \dashv x = x = x \vdash e$) do not necessarily hold (in which case it is also well known that the two operations coincide, and the dialgebra is simply an associative algebra with unit).

The set of bar units is called the *halo* of the dialgebra, and shall be denoted here by $hl(V)$. When it exists, it is an affine subspace of the dialgebra: indeed, since by bilinearity the operations in V satisfy

$$0 \vdash x = x \vdash 0 = 0 \dashv x = x \dashv 0 = 0,$$

if we set

$$N_{\vdash} = \{x | x \vdash y = 0 \quad \forall y\} \; ; \quad {}_{\dashv}N = \{x | y \dashv x = 0 \quad \forall y\},$$

and e is a non-trivial bar unit, then the halo is the affine space modelled after the subspace $N_{\vdash} \cap {}_{\dashv}N$, and passing through e.

The important example for us, considered also in [1], is the following: Let V be any vector space and fix $\varphi \in V'$ (the algebraic dual). Then one can define a dialgebra structure on V by setting

$$x \dashv y = \varphi(y)x \; ; \quad x \vdash y = \varphi(x)y.$$

Verification of the dialgebra axioms is straightforward; for instance, to check the first axiom we have for the LHS

$$x \dashv (y \dashv z) = x \dashv (\varphi(z)y) = x\varphi(z)\varphi(y),$$

while for the RHS we have

$$(x \dashv y) \dashv z = (x\varphi(y)) \dashv z = x\varphi(y)\varphi(z),$$

etc. We shall call such a dialgebra a φ-dialgebra, and sometimes denote it by V_{φ}.

However, the main reason why φ-dialgebras are of interest to us is that it is easy to exhibit their non-trivial bar units; more precisely, we have the following lemma:

Lemma 1. *Let V be any vector space, and fix $\varphi \in V'$, $\varphi \neq 0$. Then V_{φ} is a dialgebra, with non-trivial bar units. Moreover, its halo is an affine space modelled after the subspace* $\ker \varphi$.

Proof. Since $\varphi \neq 0$, from the equation $x \dashv e = x$, for all $x \in V_{\varphi}$, we get that e is a bar unit in V_{φ} iff $\varphi(e) = 1$. So, if x_0 is any element in V_{φ} such that $\varphi(x_0) \neq 0$, $x_0/\varphi(x_0)$ is a bar unit.

But moreover, if e is any fixed bar unit, it is clear that another element e' will be a bar unit iff $\varphi(e - e') = 0$. In other words, $N_{\vdash} = {}_{\dashv}N = \ker \varphi$ in this case, and hence the bar units in V_{φ} form an affine hyperplane modelled after $\ker \varphi$, as stated. $\square$

φ-dialgebras give rise to trivial Leibniz algebras, because—as one easily checks—the bracket vanishes identically; in particular, these Leibniz algebras are Lie algebras. Nevertheless, from the point of view of "integration of the linear structure," they are not really trivial. Indeed—and mostly as a motivation for what follows—let us discuss the following particular example:

Let $V = \mathbb{R}^n$ ($\cong V^*$) denote Euclidean n-space, with the usual interior (dot) product, and fix $e \in S^{n-1}$.

The choice of e defines a φ-dialgebra structure in V putting $\varphi(x) = e \cdot x$, hence also a Leibniz algebra structure (abelian in this case). Now, by Lemma 1, the fixed element e is a bar-unit for this dialgebra, and, since $e \in S^{n-1}$, projection along the subspace $\ker \varphi$ is orthogonal (thus, e is somewhat special, being the bar unit of minimal length, but this is not essential).

Now fix this bar unit e and say that $x \in V$ is (pointer) invertible (relative to e) if there exists a unique $y \in V$ such that $y \dashv x = x \vdash y = e$.

Remark. When abstracted, this notion of invertibility leads precisely to the definition of digroup, in the sense of [L], [F], and [K].

At any rate, for a φ-dialgebra this simply means $\varphi(x)y = e$; hence, applying φ to this equation we see that this is the same as $\varphi(x)\varphi(y) = 1$, and so x is invertible iff $x \notin \ker \varphi$, with inverse

$$x^{-1} = \frac{1}{\varphi(x)} e.$$

The set of invertible elements is therefore the open subset $V^\times = V \setminus \ker \varphi$.

Thus, inversion is a well defined operation for those $x \notin \ker \varphi$, and actually it has a nice geometrical interpretation: to invert an element x we take its projection onto the space spanned by e, and then, on this line, we take the inverse of this point in the sense of classical geometry (see figure 1).

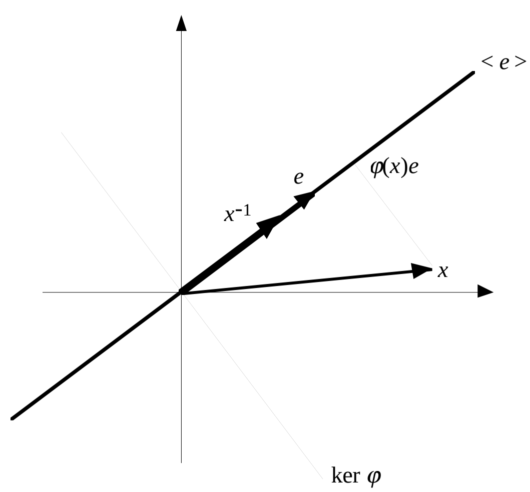

Figure 1. Inversion of an element in $V^\times$

On the other hand, notice that inversion is certainly not an involution in $V^\times$, since x and y will have the same inverse if (and only if) $x - y \in \ker\varphi$. In fact, $y = x^{-1} \implies x = y^{-1}$ iff $x = \varphi(x)e$: thus, the set of invertible elements is not a group, but the subset consisting of the line spanned by e is actually a group, isomorphic to the non-zero real numbers.

The key step is now the observation that we can make sense of a conjugation by elements of $V^\times$, by considering the action:

$$(x, y) \mapsto x \vdash y \dashv x^{-1}.$$

(One *a priori* reason why this is the right combination of the dialgebra opertions to define an "adjoint" action comes from the second axiom for Leibniz algebras, which guarantees that the left and right "translations" so defined commute.)

Actually, for this particular example this action is trivial, since

$$x \vdash y \dashv x^{-1} = \varphi(x)\varphi\left(\frac{e}{\varphi(x)}\right) y = y.$$

The point, however, is the following:

Take an element $a \in V$, and consider a curve $x(t)$ in $V^\times$, such that $x(0) = e$, and $x'(0) = a$, and let $y \in V$ be any other vector; then we can compute:

$$\begin{aligned}\frac{d}{dt}|_{t=0}\left(x(t) \vdash y \dashv x(t)^{-1}\right) &= \left(\varphi(x(t))'y\varphi(x^{-1}(t)) + \varphi(x(t))y(\varphi(x^{-1}(t))'\right)|_{t=0} \\ &= \varphi(a)y\varphi((e) + \varphi(e)y\varphi\left(\frac{\varphi(a)e}{\varphi(e)^2}\right) \\ &= \varphi(a)y - y\varphi(a) \\ &= y \dashv a - a \vdash y = [y, a].\end{aligned}$$

Although this might not seem very interesting at first, since, y being fixed, the derivative is of course zero, this is all right, because also the Leibniz algebra V is Abelian. Thus, what we have seen is that the tangent space to $V^\times$ at e can be identified to the Leibniz algebra V_φ, as required by Loday's program, and so, following his suggestion, we might call $V^\times$ a φ-coquecigrue.

Remark. This example already shows that the Leibniz algebra structure alone is not enough to determine the coquecigrue, even locally, since in this case the Leibniz algebra is a Lie algebra, but the coquecigrue just constructed is not a Lie group.

(Also, obviously what was said applies equally well for instance to any Hilbert space—by Riesz's theorem; but to avoid in what follows the technical difficulties involved in the definition of vector fields, tensor products, etc. in infinite dimensions, I will nevertheless stay in the finite dimensional case.)

II. The Matrix Case.

Using the example of the previous section as a guide, let us now construct a more general and interesting class of coquecigrues.

For this, we recall first that if V is a dialgebra, it is well known that the space $M_k = Mat(k, V)$, of square $k \times k$ matrices with entries in V, is also a dialgebra, with the operations defined entry-wise, and again denoted $\dashv, \vdash$ (*cf.* [L2]). And if V has a non-trivial bar unit e, M_k also has a non-trivial bar unit, namely

$$E = \begin{pmatrix} e & 0 & \dots & 0 \\ 0 & e & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & e \end{pmatrix}.$$

Moreover, since as a vector space

$$M_k \cong Mat(k, \mathbb{R}) \otimes V \cong V \otimes Mat(k, \mathbb{R}),$$

M_k is a $Mat(k, \mathbb{R})$-bimodule; *i.e.* the product of scalars with vectors in V extends to give actions of the algebra $Mat(k, \mathbb{R})$ on M_k, both on the right and the left, in an obvious way:

$$(X, A) \mapsto X \otimes A, \text{ where } (X \otimes A)_{ij} = \sum_k x_{ik} a_{kj}, \ A \in Mat(k, \mathbb{R}), \ X \subset M_k,$$

and similarly for the product on the left side (and we shall usually omit the symbol $\otimes$).

Now, fix a φ-dialgebra V_φ—which we could assume is given as in the example of the previous section—and an integer k, and consider the corresponding space M_k. As noted, M_k is also a dialgebra with the distinguished bar unit E, and moreover, the linear functional, φ defines a linear map $M_k \to Mat(k, \mathbb{R})$, which we still denote φ, sending $X = (x_{ij})$ to $\varphi(X) = (\varphi(x_{ij}))$.

Let us state a few properties of this space:

Lemma 2. *Let V_φ be a φ-dialgebra, and M_k the associated dialgebra of square $k \times k$ matrices. The following properties hold* *i)* $\varphi(E) = Id \in Mat(k, \mathbb{R})$

ii) $AE = EA$, *for all* $A \in Mat(k, \mathbb{R})$

iii) $\varphi(\varphi(X)Y) = \varphi(X)\varphi(Y) = \varphi(X\varphi(Y))$, *for all* $X, Y \in M_k$.

iv) $X \dashv Y = X\varphi(Y)$; $X \vdash Y = \varphi(X)Y$ *for all* $X, Y \in M_k$.

Proof. This is again quite straightforward, so let us just verify *iii*). If $X = (x_{ij})$ and $Y = (y_{ij})$, then $(\varphi(X)Y)_{ij} = \sum_k \varphi(x_{ik})y_{kj}$. Therefore,

$$(\varphi(\varphi(X)Y))_{ij} = \varphi\left(\sum_k \varphi(x_{ik})y_{kj}\right) = \sum_k \varphi(x_{ik})\varphi(y_{kj}) = (\varphi(X)\varphi(Y))_{ij}$$

as desired. □

Item *iv)* in the lemma gives us a somewhat more usable description of the dialgebra operations in M_k; and, since matrix multiplication is not commutative, this immediately shows that the resulting Leibniz algebra structure in M_k is certainly non-Abelian; indeed, it is not even a Lie algebra, because

$$[X, Y] = X\varphi(Y) - \varphi(Y)X \neq -(Y\varphi(X) - \varphi(X)Y) = -[Y, X].$$

Nevertheless we can repeat the constructions of the 1-dimensional case:

Definition. *Given* $X \in M_k$ *we say that it has a pointer inverse relative to* E *if there is a unique* $Y \in M_k$ *such that* $Y \dashv X = E$; $X \vdash Y = E$.

For simplicity we shall simply say that such an X is invertible. As in the example, we have the following explicit characterization of inverses:

Lemma 3. $X \in M_k$ *is invertible iff* $\varphi(X) \in GL(k, \mathbb{R})$, *and its inverse is* $\varphi(X)^{-1}E$.

Proof. By *iv)* in the previous lemma, the condition for invertibility becomes $Y\varphi(X) = E$; $\varphi(X)Y = E$, and applying φ to these equalities we get as a necessary condition for X to be invertible that

$$\varphi(Y)\varphi(X) = \varphi(X)\varphi(Y) = \text{Id}.$$

It follows in particular that $\varphi(X) \in GL(k, \mathbb{R})$ and that we must choose Y to equal both $\varphi(X)^{-1}E$, and $E\varphi(X)^{-1}$. Both quantities coincide however, because of *ii*) in Lemma 2, and therefore, in the open subset $M_k^\times = \varphi^{-1}(GL(k,\mathbb{R}))$ of M_k, pointer inversion is well defined. $\square$

Also, notice that lemmas 2 and 3 imply that $\varphi(X^{-1}) = \varphi(X)^{-1}$.

Thus $M_k^\times$ is a digroup, and again, $M_k^\times$ acts on the dialgebra M_k by an adjoint action:

$$(X,Y) \mapsto Ad_X Y = X \vdash Y \dashv X^{-1} = \varphi(X)Y\varphi(X)^{-1}; \quad X \in M_k^\times,\ Y \in M_k.$$

Lemma 4. *The adjoint action defined above is a left $\vdash$ action, in the sense that*

$$Ad_X(Ad_Y Z) = Ad_{X\vdash Y} Z = Ad_{\varphi(X)Y} Z; \quad X, Y \in M_k^\times,\ Z \in M_k.$$

Proof. It suffices to verify that $(X \vdash Y)^{-1} = Y^{-1} \dashv X^{-1}$, which is rather clear from the characterisation of inverses given in the previous lemma; but it also follows from the dialgebra axioms, and the properties of pointer inverses: On one side this is direct

$$\begin{aligned}(X \vdash Y) \vdash (Y^{-1} \dashv X^{-1}) &= X \vdash (Y \vdash (Y^{-1} \vdash X^{-1})) \\ &= X \vdash ((Y \vdash Y^{-1}) \vdash X^{-1}) \\ &= X \vdash (E \vdash X^{-1}) = E\end{aligned}$$

while on the other we need to use the first axiom for dialgebras once:

$$\begin{aligned}(Y^{-1} \dashv X^{-1}) \dashv (X \vdash Y) &= Y^{-1} \dashv (X^{-1} \dashv (X \vdash Y)) \\ &= Y^{-1} \dashv (X^{-1} \dashv (X \dashv Y)) \\ &= Y^{-1} \dashv (E \dashv Y) \\ &= Y^{-1} \dashv Y = E.\end{aligned}$$

$\square$

Now, since M_k is a finite dimensional space, we can directly compute derivatives to see that

$$\left(X^{-1}(t)\right)' = -\varphi(X(t))^{-1}\varphi(X'(t))E\varphi(X(t))^{-1}.$$

(Recall that E commutes with scalar matrices, from Lemma 2.)

Thus, if $Y \in M_k$ and $X(t)$ is a curve in $M_k^\times$ such that $X(0) = E$ and $X'(0) = A \in M_k$, we have

$$\begin{aligned} \frac{d}{dt}|_{t=0} \left(X(t) \vdash Y \dashv X(t)^{-1}\right) &= \left(\varphi(X(t))' Y \varphi(X^{-1}(t)) + \varphi(X(t)) Y (\varphi(X^{-1}(t))'\right)|_{t=0} \\ &= \varphi(A) Y \varphi((E) + \varphi(E) Y \varphi\left(\varphi(E)\varphi(A) E \varphi(E)\right) \\ &= \varphi(A) Y - Y \varphi(A) \\ &= Y \dashv A - A \vdash Y = [Y, A]. \end{aligned}$$

This time the result of taking derivatives is certainly not trivial, and we have indeed proven the following result:

Theorem 1. *The tangent space to $M_k^\times$ at the point E is endowed with the Leibniz algebra structure induced from the dialgebra structure of M_k; thus, the digroup $M_k^\times$ is a coquecigrue, in the sense of Loday.*

We shall sometimes call it a matrix φ-coquecigrue. Let us now analyze its geometric structure.

First, consider the set

$$GL(k, \mathbb{R}) \otimes \{E\} = \{AE; A \in GL(k, \mathbb{R})\}.$$

It is actually a homomorphic copy of the Lie group $GL(k, \mathbb{R})$ included in the coguecigrue: Obviously, φ restricted to this subset is a diffeomorphism onto $GL(k, \mathbb{R})$; but moreover, for $A, B \in GL(k, \mathbb{R})$ an easy computation shows that

$$(AE) \dashv (BE) = (AE) \vdash (BE) = ABE,$$

so that φ restricted to this subset is also a group isomorphism, regardless of which digroup operation we choose in the dialgebra.

Moreover, $X^{-1} = Y^{-1}$ iff $\varphi(X - Y) = 0$. Therefore, again we see that the matrix φ-coquecigrue fibers over $GL(k, \mathbb{R}) \otimes \{E\}$, with fiber the space $\ker \varphi \subset M_k$. Actually we can say more:

Proposition 1. *M_k has the structure of a trivial vector bundle over $GL(k, \mathbb{R})$, with fiber* $\ker \varphi$.

Proof. We have already shown that $M_k^\times$ is a vector bundle, but we can actually set up explicitly a global diffeomorphism $M_k^\times \to GL(k, \mathbb{R}) \times \ker \varphi$:

$$X \mapsto (\varphi(X), X - \varphi(X)E),$$

which gives a global trivialization, since it is linear on the fibers. □

This gives a rather neat description of the geometric structure of the coquecigrue, illustrated in figure 2 for the case $k = 1$.

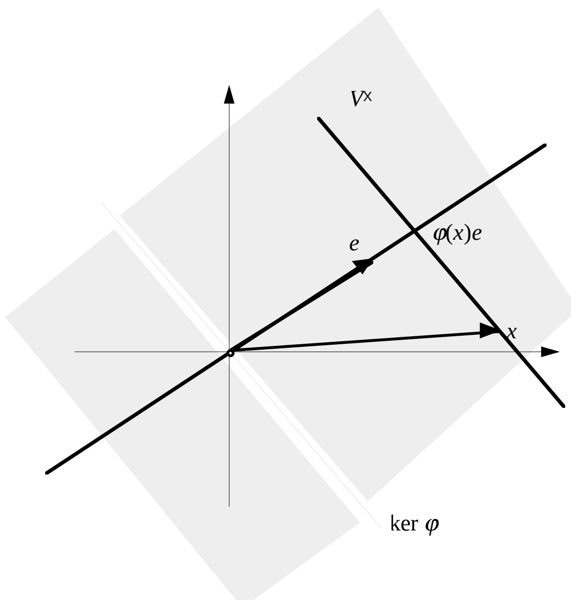

Figure 2. Fibering of $M_k^\times$ over $GL(k, \mathbb{R})$.

But let us now have a closer look at the relationship to the algebraic structure of the dialgebra; the key point here is that multiplication of X and X^{-1}, in the "reverse" order, gives a "transverse structure" to the fibered structure given in the previous proposition, namely:

Lemma 5. *Both $X \dashv X^{-1} = X\varphi(X)^{-1}$ and $X^{-1} \vdash X = \varphi(X)^{-1}X$ belong to $hl(M_k)$. In particular, they are transverse to* $\ker\varphi$.

Proof. This is a quite straightforward computation, so let us just check one of the conditions for $X \dashv X^{-1}$ to be a bar unit:

$$(X \dashv X^{-1}) \vdash Y = (X \vdash X^{-1}) \vdash Y = \varphi(X\varphi(X^{-1})Y = \operatorname{Id} Y = Y,$$

from Lemma 2. □

Notice that in general $X \dashv X^{-1} \neq X^{-1} \vdash X$; we could therefore choose any of them to define the transverse structure (and the results will be essentially the same); for reasons that will be clear soon, we choose the former. In any case, the important consequence of the existence of this transverse structure is that $M_k^\times$ should not be viewed as vector bundle; rather, we have the following:

Proposition 2. *The map*

$$R_A(X) = X + \varphi(X)A \ , \ X \in M_k^\times \ , \ A \in \ker\varphi,$$

is a right action of $\ker\varphi$*, as an abelian group, on* $M_k^\times$*, that turns it into a* $\ker\varphi$*-principal bundle over* $GL(k,\mathbb{R})$*.*

Proof. Notice that the projection is simply the map φ, so the action obviously preserves the fibers.

To see that this is indeed a right action, we compute

$$\begin{aligned} R_B(R_A(X)) &= X + \varphi(X)A + \varphi(X + \varphi(X)A)B \\ &= X + \varphi(X)(A+B) + \varphi(\varphi(X)A)B \\ &= X + \varphi(X)(A+B) = R_{A+B}(X). \end{aligned}$$

Finally, to check equivariance, we first notice that since $A \in \ker\varphi$, $(R_A(X))^{-1} = X^{-1}$, and then modify the trivialization of Proposition 1, defining $\psi(X) = \varphi(X)^{-1}X - E$. Then (although ψ is no longer linear in X), we have

$$\begin{aligned} \psi(R_A(X)) &= \varphi((R_A(X)^{-1})R_A(X) - E \\ &= \varphi(X)^{-1}(X + \varphi(X)A) - E \\ &= \varphi(X)^{-1}X - E + A = \psi(X) + A. \end{aligned}$$

proving the equivariance of the action in the global trivialization $(\varphi(X), \psi(X))$. $\square$

We now combine this proposition and the transverse structure of Lemma 5 to obtain our main result:

Theorem 2. *For each* $X \in M_k^\times$ *consider the space*

$$H_X = \{XA, A \in Mat(k,\mathbb{R})\ \},$$

regarded as a vector subspace of $T_X M_k^\times$ *in the natural way. Then* $X \mapsto H_X$ *defines an equivariant horizontal distribution for the action* R *defined in Proposition 2; i.e., a connection.*

The horizontal component of a tangent vector $Y \in T_X M_k^\times$ *is given by*

$$h(Y) = X\varphi(X)^{-1}\varphi(Y).$$

Therefore, the associated $\ker\varphi$*-valued connection 1-form is given by*

$$\omega = dX - X\varphi(X)^{-1}\varphi(dX).$$

Before proving the theorem, it is perhaps convenient to clarify what we mean by dX and $\varphi(dX)$ in the last expression: At any given point $Z \in M_k$, by dX_Z we simply mean the M_k-valued linear form on $T_Z M_k$ that to a tangent vector $Y \in T_Z M_k \cong M_k$ associates Y itself; $\varphi(dX)$ is then the $Mat(k, \mathbb{R})$-valued form that associates to this tangent vector the matrix $\varphi(Y)$ (recall that M_k is a $Mat(k, \mathbb{R})$-module).

Proof of Theorem 2. The equivariance of the distribution H under the action R means that $R_{B*}H_X = H_{R_B X}$. Now, if $XA \in H_X$, since R_B is linear in X we have,

$$R_{B*}(XA) = (R_B X)A;$$

thus, H is equivariant.

We still have to show that $H_X \oplus \ker\varphi \cong T_X M_k^\times \cong M_k$; but if $Y \in M_k$ is any vector, we obviously have

$$Y = (X\varphi(X)^{-1}\varphi(Y)) + (Y - X\varphi(X)^{-1}\varphi(Y)).$$

Now, by construction, the first summand on the RHS belongs to H_X, while the second satisfies

$$\varphi\left(Y - X\varphi(X)^{-1}\varphi(Y)\right) = \varphi(Y) - \varphi(X)\varphi(X)^{-1}\varphi(Y) = 0,$$

so it belongs to $\ker\varphi$.

Using Lemma 5, this proves the desired direct sum decomposition of $T_X M_k^\times$, and also explicitly exhibits the horizontal part of a vector as $h(Y) = X\varphi(X)^{-1}\varphi(Y)$, proving the second assertion.

The expression for the connection form is now almost immediate: Since $\ker\varphi$ is an abelian group, its Lie algebra is identified to itself; therefeore, from the definiton of dX and $\varphi(dX)$

$$\omega(Y) = Y - X\varphi(X)^{-1}\varphi(Y),$$

and so, from what has just been shown, ω takes its values on the Lie algebra of the structure group of the principal bundle. Hence, all that would remain to be shown is that $H_X = \ker\omega_X$, which again is clear from the definitions of H_X and ω. □

Remarks. Theorem 2 shows that these matrix coquecigrues have a geometrical structure reminiscent of, but not exactly identical to, the one discussed in [KW].

Also, the expression for $h(Y)$ given in the theorem shows exactly how $X\varphi(X)^{-1}$ determines the transverse structure to the fibers.

And again, one has a nice picture of the connection in the case $k = 1$: The horizontal subspace at a point X is the space complementary to $\ker\varphi$ and passing trhough X (see figure 3).

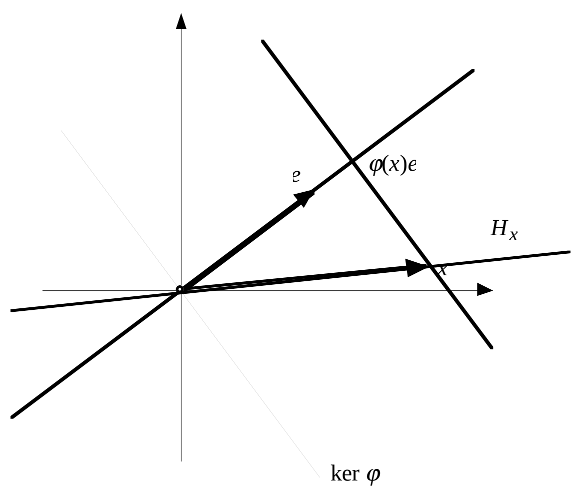

Figure 3. Horizontal space of the connection at a point X

Finally, we can also compute the curvature of ω:

Theorem 3. *The connection ω of Theorem 2 is flat.*

Proof. By definition, the curvature of the connection ω is

$$D\omega(Y, Z) = d\omega(h(Y), h(Z)).$$

Now, since

$$d\left(\varphi(X)^{-1}\right) = -\varphi(X)^{-1} d\varphi(X)\varphi(X)^{-1} = -\varphi(X)^{-1}\varphi(dX)\varphi(X)^{-1},$$

we have

$$\begin{aligned} d\omega &= - dX \wedge \varphi(X)^{-1}\varphi(dX) - X d(\varphi(X)^{-1}) \wedge \varphi(dX) \\ &= - dX \wedge \varphi(X)^{-1}\varphi(dX) + X\varphi(X)^{-1}\varphi(dX)\varphi(X)^{-1} \wedge \varphi(dX). \end{aligned}$$

Hence,

$$\begin{aligned} D\omega(Y,Z) &= -X\varphi(X)^{-1}\varphi(X)\varphi(Y)\varphi(X)^{-1}\varphi(X)\varphi(Z) \\ &\quad + X\varphi(X)\varphi(Z)\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Y) \\ &\quad + X\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Y)\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Z) \\ &\quad - X\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Z)\varphi(X)^{-1}\varphi(X)\varphi(X)^{-1}\varphi(Y) \\ &= -X\varphi(X)^{-1}\varphi(Y)\varphi(X)\varphi(Z) + X\varphi(X)^{-1}\varphi(Z)\varphi(X)\varphi(Y) \\ &\quad + X\varphi(X)^{-1}\varphi(Y)\varphi(X)\varphi(Z) - X\varphi(X)^{-1}\varphi(Z)\varphi(X)\varphi(Y) \\ &= 0. \end{aligned}$$

□

III. Some final Remarks.

Although I have considered here only matrix φ-dialgebras, this is mostly because of the ease with which bar units and inverses can be handled, and it is clear that at least some parts of the previous constructions can be generalized. For this purpose, it is convenient to rewrite the relevant conditions in terms exclusively of the dialgebra operations, as was done for instance in proving Lemma 4.

Thus for instance, the action of $\ker\varphi$ on $M_k^\times$ is given by

$$R_A X = X + X \vdash A.$$

Similarly, the horizantal projection of a vector is

$$h(Y) = (X \dashv X^{-1}) \dashv Y,$$

and hence the connection form can be written as

$$\omega = dX - (X \dashv X^{-1}) \dashv dX,$$

etc.

Nevertheless, the results proven here do not in general carry over to the abstract dialgebra context without some additional hypotheses—that moreover usually depend on the specific point at hand (*e.g.*, Theorem 1, where it is necessary that the invertible elements form an open subset), and so this generalization is not straightforward; the full extent of this possibility will be discussed in a forthcoming work.

On the other hand, the structure of these matrix coquecigrues raises the question of constructing the algebraic operations of a dialgebra (or a digroup) starting from the geometric structure, namely: is some kind of converse of Theorem 2 true? As already seen in the analysis done in [KW], this is probably not an easy matter either.

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