AN INDEPENDENT OR DEPENDENT SAMPLE OF LIFE DATA: DISTRIBUTIONS

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An independent or dependent sample of life data: Distributions and Estimation

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ABSTRACT

One of the biggest problems in reliability analysis is determining an appropriate distribution of life data. Therefore, this paper develops the estimation aspect of a family of life distributions obtained from spherical distributions. Additionally, a new family of life distributions is proposed for dependent life data, together with an optimization algorithm based on the simulated annealing method. This algorithm is very efficient for optimization purposes and does not require any manipulation of the log-likelihood functions for the distributions proposed in this study.

1. INTRODUCTION

A very important area in the analysis of parametric survival and reliability is the study of probability distributions in order to model the faults in a product and/or the lifetime of a product or entity. Below, we provide a brief historical overview of the evolution of the study, application and use of a wide variety of distributions that have been proposed for the modelling of life data, in the context of the theory of reliability, based on Barlow and Proschan (1965/1996) and Leitch (1995). One of the first areas of reliability analysis to be addressed with a certain degree of mathematical rigor was that of machine maintenance (Khintchine (1932) and Palm (1947)). The techniques used in the latter

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studies were inspired by others used previously by Erlang and Palm in problems of establishing dimensions for telephone switchboards. Initially, attempts to justify the Poisson distribution as being applicable to the number of calls to a switchboard were based on exponential distribution, as the distribution of times between the occurrences of an event. However, Erlang and Palm only put forward heuristic arguments to justify the Poisson distribution as the limit distribution of the number of calls to a switchboard. Ossoscov (1956) and subsequently Khintchine (1960) finally proved and established the necessary and sufficient conditions for this approach. The application of the Renewal Theory to problems of equipment replacement was discussed by Lotka (1939) and Campbell (1941), although it is Feller (1941) who is attributed with having developed the Renewal Theory as a mathematical discipline. The fatigue of materials is an issue related to the theory of extreme values, and has been studied by Weibull (1939), Gumbel (1935) and Epstein (1948), among others. In the early 1950s, some areas of reliability, such as life tests on electrical equipment within missiles and aircraft, were examined closely by engineers and statisticians working in the arms and aeronautics industries. This research was strongly supported by the Group on Reliability of Electronic Equipment (US Air Force). The popularity of exponential distribution in reliability analysis is largely due to the work of Davis (1952) and Epstein and Sobel (1953). However, after 1955, and thanks to the studies by Kao (1956, 1958) and Zelen-Dannemiller (1959), alternative lifetime models began to be seriously considered, for example Weibull's model. One reason for this was that many of the procedures adopted in life tests based on exponential distribution were shown to lack robustness. The reliability of systems with electromagnetic relays was the focus of work by Moore and Shannon (1956), who, encouraged by von Neumann's attempt, studied the reliability of certain operations of the human brain within complex biological organisms. In 1956, Weiss introduced the use of semi-Markovian processes to solve maintenance problems. The introduction of structure functions of coherent systems was inspired by the work of Birnbaum, Esary and Saunders (1961), which in turn was a generalization of the prior work of Moore and Shannon. Seeking to address the problem of vibrations arising from the construction of strips in commercial aircraft, Birnbaum-Saunders (1958) introduced a statistical model of lifetimes for structures under dynamic overload. This model made it possible to establish lifetimes in terms of the load, and proposed the use of the gamma distribution in some cases. The Birnbaum-Saunders distribution was derived from a model showing that faults are caused by the development and growth of a dominant crack (see Birnbaum and Saunders (1969a). Although this distribution is known as Birnbaum-Saunders, it was studied previously by Freudenthal and Shinozuka (1961).

In the 1970s and 1980s, special attention was paid to reliability problems associated with the security of nuclear reactors, among other problems of industrial security, and in resolving problems of networks of computers; research efforts were led by the Advanced Research Projects Agency (ARPA), the forerunner of the Internet and the World Wide Web. During this period, other probability distributions were also proposed for lifetime distributions, including lognormal, inverse gaussian and logistic distributions, see Nelson and Hahn (1972), Chhikara and Folks (1973) and Kalbfleish and Prentice (1980). In an alternative approach, other density functions, including log-gamma, extreme or Gumbel-values distributions and the truncated normal distribution (see Barlow and Proschan (1965/1996)) began to be used as life distributions, although these were not the focus of research in lifetime tests. During the 1990s, Mendel, inspired by physics and making use of differential geometry, outlined new directions in reliability research. Recently, the Birnbaum-Saunders distribution has been studied and generalized in different directions. Owen and Padgett (1999) proposed a generalisation of the Birnbaum-Saunders distribution from two to three parameters. Subsequently Díaz-García and Leiva-Sánchez (2005) suggested obtaining the Birnbaum-Saunders distribution from one of elliptical contours, rather than from the normal distribution, thus creating a whole family of lifetime distributions, in which there are multimodal distributions, those without moments, those with heavier or lighter tails, etc. (see Section 2). Most probabilistic models intended to describe lifetime data are chosen for one or more of the following reasons, Tobias (2004):

- There exists a physical or statistical argument that, theoretically, corresponds to the fault mechanism.
- A particular model has been used previously and successfully for an identical or very similar fault mechanism.
- The model is convenient, as it provides an empirically adequate fit to the lifetime data.

Whatever the method that is used to choose it, the model must be logical and must pass visual tests of fitness as well as statistical criteria. A common problem when few data are possessed is that many statistical models are so flexible that they seem to fit them very well, which is why arguments must exist to justify the use of a given distribution. For example, it has been reported that the extreme value-type argument justifies the use of the Weibull distribution, while the multiplicative degradation argument justifies the lognormal distribution and the fatigue argument justifies the use of the Birnbaum-Saunders distribution Tobias (2004). An additional problem in choosing a distribution is that, in general, the lifetime data in the sample are assumed to be independent. This assumption is not always justified, for example in an aquarium containing tropical fish, the lifetime of one fish is not independent of the lifetimes of the others in the aquarium. It is well known that when there is less competition for space and food, the lifetime of a fish is extended. Thus, when one fish dies, the others in the aquarium will undoubtedly live longer (under stable conditions). In this case, it is no longer possible to define the likelihood function as the product of the marginal conditions; it is now necessary to directly propose the joint density function of the sample. Another problem occurs, not only in estimating the parameters of lifetime distributions but in many other cases, such as non-linear regression and the estimation of variance components. This problem is the need to manipulate the log-likelihood function (algebraically or by reparametrizing) in order to resolve the likelihood equations, see Birnbaum and Saunders (1969b). With these methodologies, what is normally required is the algebraic or numerical calculation (depending on the case) of the first and second derivatives of the log-likelihood function. Additionally, we must analyse the log-likelihood function in greater detail when it is multimodal. These and other problems can be avoided or overcome by the application of alternative methods of optimization. Heuristic methods have recently played an important role in optimizing all kinds of functions arising in many areas of knowledge, especially that of statistical methodology. Among the methods of heuristic optimization, simulated annealing (SA) stands out for its simplicity and high efficiency Azencott (1992). The present paper includes the maximum likelihood estimators belonging to the generalized Birnbaum-Saunders family of distributions, see Díaz-García and Leiva-Sánchez (2005) It also proposes a new family of distributions for the case in which the lifetime data in the sample are not independent. The maximum likelihood estimators of its parameters are also found for this family. In both cases, to maximize the log-likelihood, we propose a procedure based on heuristic optimization and its combination with the quasi-Newton method. In carrying out this optimization, special attention is paid when discrete or continuous parameters exist. The results are applied to a data set that is available in the literature.

2. PRELIMINARY CONSIDERATIONS

We now present some basic, preliminary results for the development of the present paper.

Let us define the random variable:

$$S = \beta \left[\frac{\alpha}{2} Z + \sqrt{\left(\frac{\alpha}{2} Z \right)^2 + 1} \right]^2 \tag{1}$$

where $Z \sim N(0,1)$, $\alpha > 0$ and $\beta > 0$. The random variable S is said to have a Birnbaum-Saunders distribution, with the notation $S \sim \mathcal{BS}(\alpha, \beta)$. Furthermore, its density function is given by

$$f_S(s) = \frac{1}{(2\pi)^{1/2}} \exp\left[-\frac{1}{2\alpha^2}\left(\frac{s}{\beta} + \frac{\beta}{s} - 2\right)\right]\left(\frac{s^{-3/2}(s+\beta)}{2\alpha\beta^{1/2}}\right), \quad S > 0$$

where α is the shape parameter and β is the scale parameter and the median of the distribution, Birnbaum and Saunders (1969a).

The p-dimensional random vector $\mathbf{Y} = (Y_1, ..., Y_p)'$ is said to have an elliptical distribution of parameters $\boldsymbol{\mu} : p \times 1$ (localization vector) and a dispersion matrix $\boldsymbol{\Sigma} : p \times p$, $\boldsymbol{\Sigma} > 0$, if its density function is given by

$$f_{\mathbf{Y}}(\mathbf{y}) = c|\boldsymbol{\Sigma}|^{-1/2} h[(\mathbf{y} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\mu})], \quad \mathbf{y} \in \mathbb{R}^p, \tag{2}$$

where the function $g : \mathbb{R} \to [0, \infty)$ is termed the generator function, and is such that $\int_0^\infty u^{p-1} h(u^2) du < \infty$ and c is such that $f_{\mathbf{Y}}(\mathbf{y})$ is a density. This circumstance is denoted by $\mathbf{Y} \sim \mathcal{E}l_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}; g)$. When the vector Y has finite moments, then $E(\boldsymbol{Y}) = \boldsymbol{\mu}$ and $\text{Var}(\mathbf{Y}) = c_h \boldsymbol{\Sigma}$, where c_h is a positive constant, see for example Fang and Zhang (1990) or Fang et al. (1990). In the particular case in which $\boldsymbol{\mu} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \boldsymbol{I}_p$, we have the family of spherical distributions, denoted as $\mathbf{Y} \sim \mathcal{E}_p(\mathbf{0}, \mathbf{I}; h)$. These classes of distributions include, as particular cases, the Normal, t-Student, Pearson type VII, Logistic and Kotz distributions, among many others.

Now, by rewriting (1) as

$$T = \beta \left[\frac{\alpha}{2} U + \sqrt{\left(\frac{\alpha}{2} U\right)^2 + 1}\right]^2$$

and assuming that $U \sim \mathcal{E}_1(0, 1; h)$, then T has the density function

$$f_T(t) = c\ h\left(\frac{1}{\alpha^2}\left(\frac{t}{\beta} + \frac{\beta}{t} - 2\right)\right)\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right), \quad t > 0.$$

This is known as the generalized Birnbaum-Saunders distribution, denoted as $T \sim \mathcal{GBS}(\alpha, \beta; h)$, Díaz-García and Leiva-Sánchez (2005).

3. A NEW FAMILY OF DISTRIBUTIONS

In this section, we present an extension of the generalized Birnbaum-Saunders distribution to the multivariate case.

Theorem 1. *Assume that* $\mathbf{U} \sim \mathcal{E}_n(0, I; h)$ *and define the transform*

$$T_i = \beta \left(\frac{1}{2}\alpha U_i + \sqrt{\left(\frac{1}{2}\alpha U_i\right)^2 + 1}\right)^2, \quad i = 1, \ldots, n, \quad \alpha > 0, \quad \beta > 0.$$

Then, the distribution of $\mathbf{T} = (T_1, \ldots, T_n)'$ *is given by*

$$f_{\mathbf{T}}(\mathbf{t}) = c\, \frac{\prod_{i=1}^{n} t_i^{-3/2}(t_i + \beta)}{(2\alpha)^n (\beta)^{n/2}} h\left(\frac{1}{\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right)\right), \tag{3}$$

this being written as $\mathbf{T} \sim \mathcal{BSG}_n(\alpha, \beta; h)$.

Proof: Given that $\mathbf{U} \sim \mathcal{E}_n(0, I, ; h)$ the density of $\mathbf{U}$ is given by

$$f_{\mathbf{U}}(\mathbf{u}) = c\ h\left(\|\mathbf{u}\|^2\right)$$

Hence, observe that if

$$t_i = \beta\left(\frac{1}{2}\alpha u_i + \sqrt{\left(\frac{1}{2}\alpha u_i\right)^2 + 1}\right)^2 \qquad i = 1, \ldots, n,$$

we have

$$u_i = \frac{1}{\alpha}\left(\sqrt{\frac{t_i}{\beta}} - \sqrt{\frac{\beta}{t_i}}\right) \qquad i = 1, \ldots, n,$$

from which the Jacobian of the transform is given by

$$\left|\frac{\partial u_i}{\partial t_i}\right| = \frac{1}{(2\alpha\beta^{1/2})^n}\prod_{i=1}^{n} t_i^{-3/2}(t_i + \beta)$$

Furthermore, observe that

$$\|u\|^2 = \frac{1}{\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right)$$

from which we obtain the result we are seeking. ■

Given an n-dimensional random vector, it is well known that in the family of spherical multivariate distributions the only distribution in which the elements of the random vector are independent is the special case of the multivariate normal, see Fang and Zhang (1990, p. 72). Thus, if we are sure there is no independence in the sample, we must not use the generalized Birnbaum-Saunders distribution, based on the normal. This provides a guideline to enable us to take the density as the likelihood function (3) when we believe the lifetime data for the sample may not be independent. Explicit expressions for various classes of particular elliptical distributions are given in Appendix A.

Remark 1. *It is important to note that in the present paper we are proposing a distributional model and not a functional structure for the sample* $\mathbf{T}' = (t_1, \ldots, t_n)$*, see Bunke and Bunke (1986, Subsection 1.3.1). Thus, the stochastic dependence between the variables* $t_1, \ldots, t_n$ *is taken into account in the model by the proposition of a joint distribution for the sample (likelihood function) in which there is a stochastic dependence between the* t_i *elements of its arguments. It would be a different question if we were attempting to propose a functional model or structure (see Bunke and Bunke (1986, Subsection 1.3.1)) that sought to explain this stochastic dependence, either with just the* $t_1, \ldots, t_n$ *lifetime variables or considering also covariables, see Rieck and Nedelman (1991).*

4. OPTIMIZATION ALGORITHM

The technique known as Simulated Annealing was proposed as a means of maximizing the likelihood function, due to its simplicity and the fact that it does not require us to the manipulate the likelihood function or to calculate its derivatives. Further advantages are derived from its global optimization approach. This method was developed by Kirkpatrick, et al. (1983), who relied fundamentally on physical analogies for their theoretical results. Its simplicity has led to this technique being widely used, and different implementations have been studied. We present below a general SA algorithm, see Azencott (1992) and Siarry et al. (1997) among others.

A general algorithm for the implementation of SA:

- Step 0. Let $x_0 \in \mathbb{R}^n$ be a given start point, where n is the number of parameters to be estimated, and where $y_0 = x_0$ and $k = 0$.
- Step 1. Choose a random point $y_{k+1} = y_k + \Delta U[-1, 1]$, where U denotes the uniform distribution and Δ is the magnitude of the step to the instant k.
- Step 2. Obtain a uniform random number $p \in [0, 1]$ and determine that

$$x_{k+1} = \begin{cases} y_{k+1} & \text{si } p \leq \exp\left(\dfrac{f(x_k) - f(y_{k+1})}{t_k}\right) \\ x_k & \text{otherwise} \end{cases}$$

  where t_k is the temperature parameter in iteration k.
- Step 3. Repeat steps 1 and 2 N times.
- Step 4. Update $t_{k+1} = \rho t_k$, where $0 < \rho < 1$ is a cooling parameter and Δ.
- Step 5. Check the convergence and stopping criteria; if they are not met, take $k = k + 1$ and return to Step 1.

Some considerations to bear in mind when applying the SA method to maximize the log-likelihood function are listed below:

- Initial temperature. To establish the initial temperature, we recommend that the function should be evaluated at 100 randomly-generated points.
- Restriction of parameters. The algorithm enables us to introduce upper and lower bound restrictions for the parameters, accepting infinite values.
- Discrete and continuous optimization. With this algorithm, it is possible to combine optimization with discrete parameters (such as degrees of freedom) and with those on a continuous scale (for example, α, β).
- Maximum and minimum cooling factors of 0.9 and 0.1., respectively, are recommended.

Our implementation of the SA method takes into account whether the parameters to be estimated are discrete or continuous, addressing the question simultaneously when necessary. Finally, the SA algorithm is combined with the quasi-Newton method implemented in the S-plus statistical package. Basically, there are two main reasons for combining a heuristic method with another:

- **Accelerating convergence**: In general, heuristic methods are highly efficient at quickly locating a reasonably small area in which the global optimum is located. The idea is to let the heuristic method work until it obtains an initial point that is fairly close to the global optimum and then for this to be used as the initial one by another, faster convergence method.
- **Increasing precision**: As mentioned in the previous paragraph, heuristic methods quickly approach the global optimum, but usually require a lot of computing time to further increase precision. An alternative is to combine heuristics with another method, for example one making use of derivatives, and then increased precision is achieved much more quickly.

The algorithm implemented in SPLUS 2000 can be found at http://www.cimat.mx/~jrdguez/. It can be used, simply and easily, to adjust all the distribution models described in Appendix A, using the above-described techniques.

5. MODEL SELECTION

One criterion that is widely used for determining which model is most appropriate is the Schwart information criterion (SIC, see Spieglhaiter et al. (2002)), which is defined as

$$SIC = -2\log\left[P\left(Y|X, \text{Model}, \text{parameters}\right)\right] + \log\left(n\right)\left(\#\text{ of parameters of the model}\right) \tag{4}$$

This method, like many others of its type, is based on the fact that the likelihood logarithm can be associated to a method to fit the model to the data. The second summand in (4) is related to the number of parameters in the model, and the fact that the latter can be seen as a measure of its complexity. According to this criterion, the best model (i.e. the one that best fits the data) is the one obtaining the lowest SIC value. Vuong's test can be used to determine whether there is a significant difference between one model and another; this test is given by:

$$H_0 : E^0\left[\ln\left(\frac{f\left(y|\hat{\theta}\right)}{g\left(y|\hat{\gamma}\right)}\right)\right] = 0$$

which means that the two models are equivalent. The alternative hypotheses are

$$H_f : E^0\left[\ln\left(\frac{f\left(y|\hat{\theta}\right)}{g\left(y|\hat{\gamma}\right)}\right)\right] > 0 \qquad \textit{and} \qquad H_g : E^0\left[\ln\left(\frac{f\left(y|\hat{\theta}\right)}{g\left(y|\hat{\gamma}\right)}\right)\right] < 0$$

which would imply that the f model is better than the g model or that the g model is better than the f model, respectively.

Vuong proved, under very general conditions, that

$$\frac{1}{n}LR_n\left(\hat{\theta}, \hat{\gamma}\right) \overset{a.s}{\rightarrow} E^0\left[\ln\left(\frac{f\left(y|\hat{\theta}\right)}{g\left(y|\hat{\gamma}\right)}\right)\right]$$

and that

$$\text{under } H_0 : \frac{LR_n\left(\hat{\theta}, \hat{\gamma}\right)}{\sqrt{n}\hat{\omega}} \overset{D}{\rightarrow} N\left(0, 1\right)$$

$$\text{under } H_f : \frac{LR_n\left(\hat{\theta}, \hat{\gamma}\right)}{\sqrt{n}\hat{\omega}} \overset{D}{\rightarrow} +\infty$$

$$\text{under } H_g : \frac{LR_n\left(\hat{\theta}, \hat{\gamma}\right)}{\sqrt{n}\hat{\omega}} \overset{D}{\rightarrow} -\infty$$

where

$$LR_n\left(\hat{\theta}, \hat{\gamma}\right) \equiv \log\left[L_f\left(\hat{\theta}|y\right)\right] - \log\left[L_g\left(\hat{\gamma}|y\right)\right]$$

and

$$\hat{\omega}^2 \equiv \frac{1}{n}\sum_{i=1}^{n}\left[\ln\left(\frac{f\left(y_i|\hat{\theta}\right)}{g\left(y_i|\hat{\gamma}\right)}\right)\right]^2 - \left[\frac{1}{n}\sum_{i=1}^{n}\ln\left(\frac{f\left(y_i|\hat{\theta}\right)}{g\left(y_i|\hat{\gamma}\right)}\right)\right]^2$$

Finally, Vuong proposes a correction factor for the number of parameters in models f and g,

$$L\tilde{R}_n\left(\hat{\theta}, \hat{\gamma}\right) \equiv \log\left[L_f\left(\hat{\theta}|y\right)\right] - \log\left[L_g\left(\hat{\gamma}|y\right)\right] - \left[\left(\frac{p}{2}\right) - \left(\frac{q}{2}\right)\right]\ln\left(n\right)$$

6. EXAMPLE

To illustrate the methodology described in this paper, we estimated the parameters and the corresponding SIC for each of the distributions described in the previous section. The following data were taken from the original article in which Birnbaum and Saunders (1969) described a technique to estimate the parameters of their new distribution.

70	114	130	138	151	212
90	114	130	138	152	
96	116	131	139	155	
97	119	131	139	156	
99	120	131	141	157	
100	120	131	141	157	
103	120	131	142	157	
104	121	132	142	157	
104	121	132	142	158	
105	123	132	142	159	
107	124	133	142	162	
108	124	134	142	163	
108	124	134	144	163	
108	124	134	144	164	
109	124	134	145	166	
109	128	134	146	166	
112	128	136	148	168	
112	129	136	148	170	
113	139	137	149	174	
114	130	138	151	196	

Table 1: Number of cycles required to induce the breakup of a material subjected to a force of 31,000 psi.

Distribution	SIC	LogVer	NP	α	β	q	r	s
Special Case	922.649689	-456.709724	2	0.196856	132.001682			
Laplace	923.114023	-456.941891	2	0.129013	133.999998			
Normal	923.998994	-457.384377	2	0.170451	131.915372			
Pearson VII	925.434953	-455.794796	3	0.417363	132.624838	4.500533	1 (fixed)	
t	925.434959	-455.794799	3	0.147533	132.620640	8		
Bessel	925.600607	-455.877623	3	0.076453	132.911693	2	1 (fixed)	
Kotz	930.020891	-455.780205	4	0.184326	132.956198	1	1 (fixed)	0.691985
Logistic	933.294842	-462.032301	2	0.196606	130.542667			
Cauchy	947.601182	-469.185470	2	0.091718	134.286169			

Table 2: Fit of the distributions for the independent case, ordered according to the SIC criterion from lowest to highest. NP denotes the number of parameters considered in the optimization. In the Kotz distribution, it is assumed that $q \geq 1$.

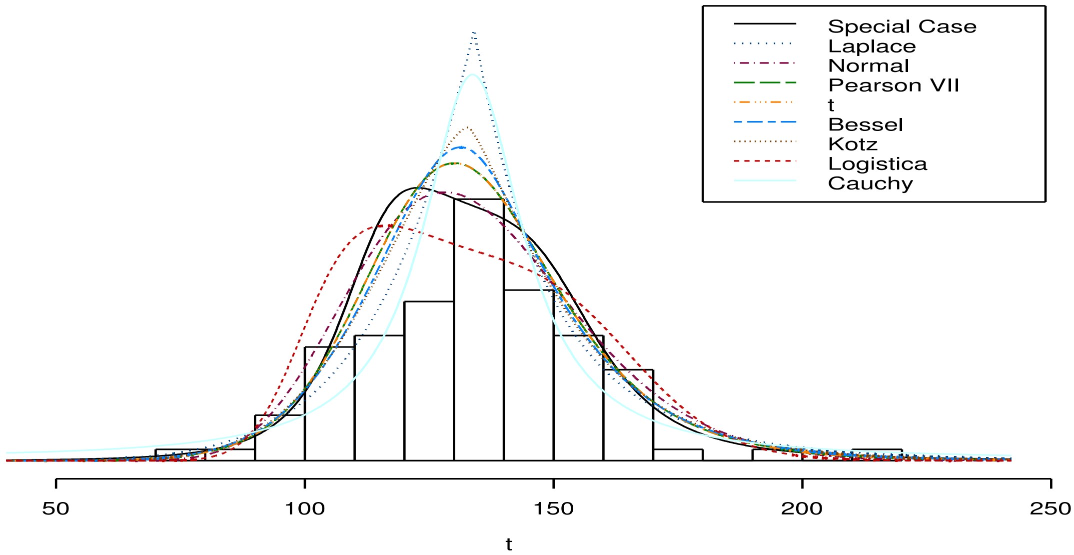

Figure 1: Density graphs of a generalized Birnbaum-Saunders life distribution, independent case.

Distribution	SIC	LogVer	NP	α	β	q	r	s
Logistic	923.998994	-457.384377	2	0.241054	131.915061			
Kotz	924.366168	-457.567964	2	0.077178	131.915101	1	1	0.691985
Bessel	924.676080	-457.722920	2	0.016715	131.915031	2	1	
Laplace	924.690491	-457.730125	2	0.016961	131.915066			
Pearson VII	926.536084	-458.652921	2	0.511384	131.915013	55.000533	1	
t	926.649424	-458.709591	2	0.170451	131.915276	8		
Special Case	927.061042	-458.915400	2	0.711149	131.915017			
Cauchy	928.927552	-459.848656	2	0.170451	131.915276			

Table 3: Fit of the distributions for the dependent case, ordered according to the SIC criterion from lowest to highest. NP denotes the number of parameters considered in the optimization. In the Kotz distribution, it is assumed that $q \geq 1$. The q, r and s parameters are assumed to be fixed in all the distributions in which they appear.

It is important to note that we are not attempting to compare the different fits achieved for situations of dependence and independence among the lifetime data of the sample. Of course, in an application, the procedure would be to use the SIC or Vuong's test to choose the most suitable distributional model from among those considering dependence or independence, as appropriate. Thus, for example, if we assume there is independence in the sample of lifetime data in our example, the following conclusions are reached: under the SIC the generalized Birnbaum-Saunders (GBS) distribution based on the Special Case distribution best fits the data. When Vuong's test is used, we find that the value in the case of the GBS based on the Special Case, compared to the GBS distribution based on the normal distribution, is 0.257. Comparison of this value with a standard normal distribution produces a p-value of 0.797, and so we cannot reject the null hypothesis that

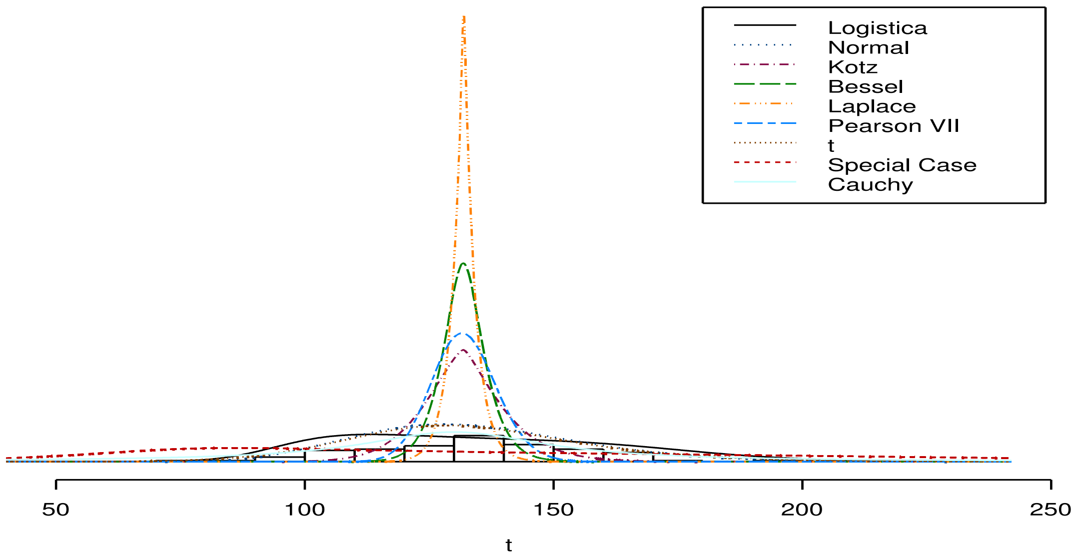

Figure 2: Density graphs of a generalized Birnbaum-Saunders life distribution, dependent case.

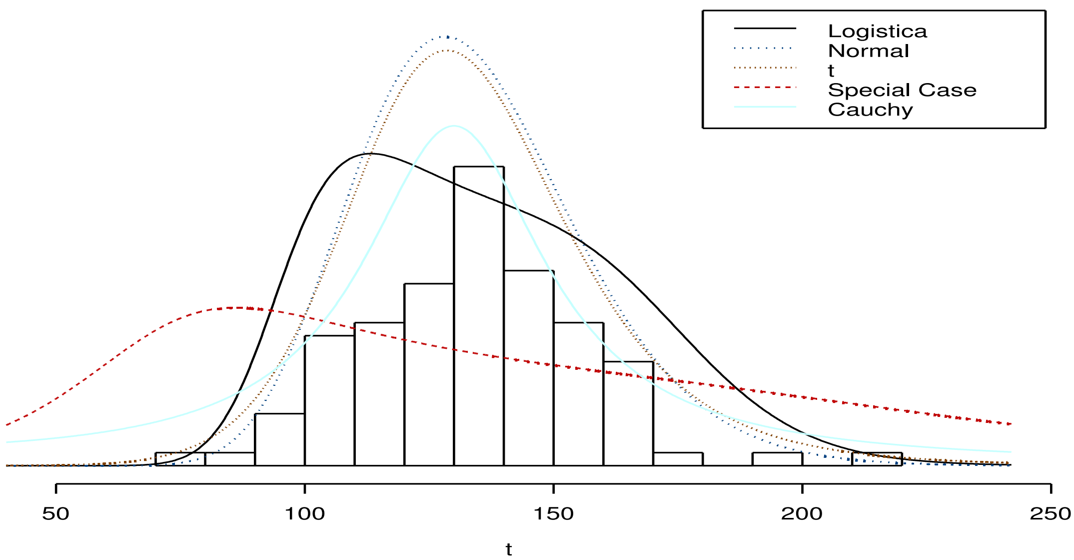

Figure 3: Density graphs of a generalized Birnbaum-Saunders life distribution, dependent case.

these two models are equivalent in fitting the data. In order to observe the capacity to discriminate between Vuong's-criterion models, we compared the GBS distribution based on the Special Case distribution and that based on the Cauchy distribution, obtaining a value of 2.914 for Vuong's criterion. As before, this value was compared with that of a standard, normal distribution and a p-value of 0.0035 was obtained. In this case, therefore, we reject the null hypothesis that the two models are equivalent as regards fitting the data. As a further example, if we now assume that there is dependence in the sample of lifetime data for the sample, then under the SIC the GBS distribution based on logistic distribution best fits the data. When Vuong's test is applied, there is no significant difference between the above model and the others. Figure 1 shows the densities obtained under the assumption of independence in the sample of lifetime data. It can be seen that the density that best fits the data is the GBS distribution based on the Special Case distribution, which is asymmetric. Figures 2 and 3 show the densities under the assumption, now, that there is dependence among the lifetime data. Figure 2 shows all the values of the different contributions, while in Figure 3 the densities with a very large mode value have been removed, to better show the densities that best describe the data. Note again that the density best fitting the data is an asymmetric distribution, in this case the GBS distribution based on a logistic distribution.

Note that, as mentioned in the Introduction, it is very important to consider the conditions of the process being studied. Assume, for example, that it has been empirically established that the distributional model that best describes the behaviour of the lifetime data is the GBS distribution based on the Cauchy distribution. Then, to decide between using this distribution or the one based on the normal one, apart from other possible considerations, it must be taken into account that the GBS distribution based on the Cauchy distribution does not have moments; this could have very important implications in the process being studied.

Remark 2. *Finally, note that the population being studied is a univariate one. From this population, we extract a sample with a size of* n, $\mathbf{t}' = (t_1, \ldots, t_n)$*; this has an* n*-variate distribution (likelihood function) given by:*

$$L(\alpha, \beta; t_1, \ldots, t_n) = \begin{cases} \prod_{i=1}^{n} f_{T_i}(t_i; \alpha, \beta), & \textit{independent case} \\ f_{\mathbf{T}}(\mathbf{t}; \alpha, \beta), & \textit{dependent case.} \end{cases}$$

Such densities are two alternative means of obtaining the estimators of the parameters, fundamentally α *and* β*, assuming independence or dependence, respectively, in the sample.*

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APPENDIX A.
GENERALIZED BIRNBAUM-SAUNDERS DISTRIBUTIONS

In the following, we derive explicit expressions for the density generated on the basis of certain distributions belonging to the family of spherical distributions. For cases presenting problems of parameter-identifiability, we determined which parameters should remain fixed for the optimization algorithm to have a suitable degree of convergence.

A1. Pearson type VII distribution

- **Independent case.**Let $U \sim \mathcal{E}(0,1;h)$

$$f_U(u) = \frac{\Gamma(q)}{(r\pi)^{1/2}\,\Gamma(q-1/2)}\left(1+\frac{u^2}{r}\right)^{-q}$$

$r>0$, $q>1/2$. Then $T \sim GBS(\alpha,\beta;g)$,

$$f_T(t) = \frac{\Gamma(q)}{(r\pi)^{1/2}\,\Gamma(q-1/2)}\left(1+\frac{1}{r\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-q}\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right)$$

with $\alpha,\beta,r>0$ and $q>1/2$. We find that the r parameters are not identifiable,

$$f_T(t) = \frac{\Gamma(q)}{(r\alpha^2\pi)^{1/2}\,\Gamma(q-1/2)}\left(1+\frac{1}{r\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-q}\left(\frac{t^{-3/2}(t+\beta)}{2\beta^{1/2}}\right)$$

and so it is useful to set the r parameter. Likelihood

$$L(t;\alpha,\beta,q,r) = \frac{\Gamma(q)^n}{(r\alpha^2\pi)^{n/2}\,\Gamma(q-1/2)^n\,2^n\beta^{n/2}}\prod_{i=1}^{n}\left\{\frac{(t_i+\beta)}{t_i^{3/2}}\left(1+\frac{1}{r\alpha^2}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{-q}\right\}$$

Log-likelihood

$$\begin{aligned} l(t;\alpha,\beta,q,r) = {} & n\left[\log(\Gamma(q)) - \log(2) - \log(\Gamma(q-1/2))\right] - \frac{n}{2}\left[\log\left(r\alpha^2\right)+\log(\pi)+\log(\beta)\right] \\ & - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i+\beta) - q\sum_{i=1}^{n}\log\left(1+\frac{1}{r\alpha^2}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right) \end{aligned}$$

- **Dependent case** Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0},\mathbf{I}_n;h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{\Gamma(q)}{(r\pi)^{n/2}\,\Gamma(q-n/2)}\left(1+\frac{\|\mathbf{u}\|^2}{r}\right)^{-q}$$

$r>0,\ q>n/2$. Then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha,\beta;h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\Gamma(q)}{(r\pi)^{n/2}\,\Gamma(q-n/2)}\left(1+\frac{1}{r\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{-q}\left(\frac{1}{(2\alpha)^n\,\beta^{n/2}}\prod_{i=1}^{n}\frac{(t_i+\beta)}{t_i^{3/2}}\right)$$

with $\alpha, \beta, r > 0$ and $q > n/2$. The $r\alpha^2$ parameters are not identifiable,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\Gamma(q)}{(r\alpha^2\pi)^{n/2}\,\Gamma(q-n/2)}\left(1+\frac{1}{r\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{-q}\left(\frac{1}{2^n\beta^{n/2}}\prod_{i=1}^{n}t_i^{-3/2}(t_i+\beta)\right)$$

and so it is useful to set the r parameter. Likelihood

$$L(\mathbf{t};\alpha,\beta,q,r) = \frac{\Gamma(q)}{(r\alpha^2\pi)^{n/2}\,\Gamma(q-n/2)}\left(1+\frac{1}{r\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{-q}\left(\frac{1}{2^n\beta^{n/2}}\prod_{i=1}^{n}t_i^{-3/2}(t_i+\beta)\right)$$

Log-likelihood

$$\begin{aligned} l(\mathbf{t};\alpha,\beta,q,r) = {} & \log(\Gamma(q)) - \log(\Gamma(q-n/2)) - n\log(2) - \frac{n}{2}\left[\log\left(r\alpha^2\right)+\log(\pi)+\log(\beta)\right] \\ & -\frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta) - q\log\left(1+\frac{1}{r\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right) \end{aligned}$$

A2. T Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0,1;h)$

$$f_U(u) = \frac{\Gamma((v+1)/2)}{(v\pi)^{1/2}\,\Gamma(v/2)}\left(1+\frac{u^2}{v}\right)^{-(v+1)/2}$$

then $T \sim GBS(\alpha,\beta;g)$,

$$f_T(t) = \frac{\Gamma((v+1)/2)}{(v\pi)^{1/2}\,\Gamma(v/2)}\left(1+\frac{1}{v\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{-(v+1)/2}\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right)$$

with $\alpha, \beta, v > 0$. Likelihood

$$L(t;\alpha,\beta,v) = \frac{\Gamma((v+1)/2)^n}{(v\pi)^{n/2}\,\Gamma(v/2)^n 2^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)\left(1+\frac{1}{v\alpha^2}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{-(v+1)/2}$$

Log-likelihood

$$\begin{aligned} l(t;\alpha,\beta,v) = {} & n\log\Gamma((v+1)/2) - \log(\Gamma(v/2) - \log(2) - \log(\alpha)) \\ & -\frac{n}{2}(\log(v)+\log(\pi)+\log(\beta)) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta) \\ & -\frac{v+1}{2}\sum_{i=1}^{n}\log\left(1+\frac{1}{v\alpha^2}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right) \end{aligned}$$

- **Dependent case.** Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}_n; h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{\Gamma((n+v)/2)}{(v\pi)^{n/2}\,\Gamma(v/2)} \left(1 + \frac{\|\mathbf{u}\|^2}{v}\right)^{-(n+v)/2}$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha, \beta; h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\Gamma((n+v)/2)}{(v\pi)^{n/2}\,\Gamma(v/2)} \left(1 + \frac{1}{v\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)^{-(n+v)/2} \left(\frac{1}{(2\alpha)^n\,\beta^{n/2}}\prod_{i=1}^{n} t_i^{-3/2}\,(t_i + \beta)\right)$$

with $\alpha, \beta, v > 0$. Likelihood

$$L(\mathbf{t}; \alpha, \beta, v) = \frac{\Gamma((n+v)/2)}{(v\pi)^{n/2}\,\Gamma(v/2)2^n\alpha^n\beta^{n/2}} \left(1 + \sum_{i=1}^{n}\frac{1}{v\alpha^2}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)^{-(n+v)/2} \prod_{i=1}^{n}\left(t_i^{-3/2}\,(t_i + \beta)\right)$$

Log-likelihood

$$\begin{aligned} l(\mathbf{t}; \alpha, \beta, v) = {} & \log\Gamma((n+v)/2)) - \log\Gamma(v/2) - n\left(\log(\alpha) + \log(2)\right) \\ & - \frac{n}{2}\left(\log(v) + \log(\pi) + \log(\beta)\right) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i + \beta) \\ & - \frac{n+v}{2}\log\left(1 + \frac{1}{v\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right) \end{aligned}$$

A3. Cauchy Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0, 1; h)$

$$f_U(u) = \frac{1}{\pi}\left(1 + u^2\right)^{-1}$$

then $T \sim GBS(\alpha, \beta; g)$,

$$f_T(t) = \frac{1}{\pi}\left(1 + \frac{1}{\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)^{-1}\left(\frac{t^{-3/2}\,(t + \beta)}{2\alpha\beta^{1/2}}\right)$$

with $\alpha, \beta, v > 0$. Likelihood

$$L(t; \alpha, \beta) = \frac{1}{\pi^n 2^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\left(t_i^{-3/2}\,(t_i + \beta)\right)\left(1 + \frac{1}{\alpha^2}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)^{-1}$$

Log-likelihood

$$\begin{aligned} l(t; \alpha, \beta) = {} & -n\left[\log(\pi) + \log(2) + \log(\alpha) + \frac{1}{2}\log(\beta)\right] - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i + \beta) \\ & - \sum_{i=1}^{n}\log\left(1 + \frac{1}{\alpha^2}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right) \end{aligned}$$

- **Dependent case.** Let$\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}_n; h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \left(1 + \|\mathbf{u}\|^2\right)^{-(n+1)/2}$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha, \beta; h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \left(1 + \frac{1}{\alpha^2} \sum_{i=1}^{n} \left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)^{-(n+1)/2} \left(\frac{1}{(2\alpha)^n \beta^{n/2}} \prod_{i=1}^{n} t_i^{-3/2} (t_i + \beta)\right)$$

with $\alpha, \beta, v > 0$. Likelihood

$$L(\mathbf{t}; \alpha, \beta) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \left(1 + \frac{1}{\alpha^2} \sum_{i=1}^{n} \left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)^{-(n+1)/2} \left(\frac{1}{(2\alpha)^n \beta^{n/2}} \prod_{i=1}^{n} t_i^{-3/2} (t_i + \beta)\right)$$

Log-likelihood

$$\begin{aligned} l(\mathbf{t}; \alpha, \beta) = \log\Gamma((n+1)/2) - \frac{n+1}{2}\log(\pi) - n\left[\log(2) + \log(\alpha) + \frac{1}{2}\log(\beta)\right] \\ - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i + \beta) - \frac{n+1}{2}\log\left(1 + \frac{1}{\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right) \end{aligned}$$

A4. Kotz Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0, 1; h)$

$$f_T(t) = \frac{s r^{(2q-1)/2s}}{\Gamma((2q-1)/2s)} u^{2(q-1)} \exp\left\{-r u^{2s}\right\}$$

then $T \sim GBS(\alpha, \beta; g)$,

$$f_T(t) = \frac{s r^{(2q-1)/2s}}{\Gamma((2q-1)/2s)} \frac{1}{\alpha^{2(q-1)}} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]^{q-1} \exp\left\{-\frac{r}{\alpha^{2s}}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]^s\right\} \left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right)$$

with $\alpha, \beta, r, s > 0$ and $2q + n > 2$. For a fixed s fijo we find that r/α^{2s} is not identifiable,

$$f_T(t) = \frac{s}{\Gamma((2q-1)/2s)} \left[\frac{r}{\alpha^{2s}}\right]^{(2q-1)/2s} \left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]^{q-1} \exp\left\{-\frac{r}{\alpha^{2s}}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]^s\right\} \left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right)$$

and so it is useful to set the r parameter. Likelihood

$$\begin{aligned} L(t; \alpha, \beta, q, r, s) = \frac{s^n}{\Gamma((2q-1)/2s)^n 2^n \beta^{n/2}} \left[\frac{r}{\alpha^{2s}}\right]^{n(2q-1)/2s} \\ \times \prod_{i=1}^{n}\left(\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]^{q-1} \exp\left\{-\frac{r}{\alpha^{2s}}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]^s\right\} \left(t_i^{-3/2}(t_i + \beta_i)\right)\right) \end{aligned}$$

Log-likelihood

$$l\left(t;\alpha,\beta,q,r,s\right)=n\left[\log\left(s\right)+\left(\frac{2q-1}{2s}\right)\log\left(\frac{r}{\alpha^{2s}}\right)-\log\Gamma((2q-1)/2s)-\log(2)-\frac{1}{2}\log\left(\beta\right)\right]$$
$$-\frac{3}{2}\sum_{i=1}^{n}\log\left(t_i\right)+\sum_{i=1}^{n}\log\left(t_i+\beta\right)+(q-1)\sum_{i=1}^{n}\log\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right)$$
$$-\frac{r}{\alpha^{2s}}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{s}$$

- **Dependent case.** Let $\mathbf{U}\sim\mathcal{E}_n(\mathbf{0},\mathbf{I}_n;h)$

$$f_{\mathbf{U}}(\mathbf{u})=\frac{sr^{(2q+n-2)/2s}\Gamma(n/2)}{\pi^{n/2}\Gamma((2q+n-2)/2s)}\left\|u\right\|^{2(q-1)}\exp\left\{-r\left\|\mathbf{u}\right\|^{2s}\right\}$$

then $\mathbf{T}\sim\mathcal{GBS}_n(\alpha,\beta;h)$,

$$f_{\mathbf{T}}(\mathbf{t})=\frac{sr^{(2q+n-2)/2s}\Gamma(n/2)}{\pi^{n/2}\Gamma((2q+n-2)/2s)}\frac{1}{\alpha^{2(q-1)}}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q-1}$$
$$\times\exp\left\{-\frac{r}{\alpha^{2s}}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{s}\right\}\left(\frac{1}{\left(2\alpha\right)^{n}\beta^{n/2}}\prod_{i=1}^{n}t_i^{-3/2}\left(t_i+\beta\right)\right)$$

with $\alpha,\beta,r,s>0$ and $2q+n>2$. For a fixed s we find that r is not identifiable,

$$f_{\mathbf{T}}(\mathbf{t})=\frac{sr^{(2q+n-2)/2s}\Gamma(n/2)}{\pi^{n/2}\Gamma((2q+n-2)/2s)}\left(\frac{r}{\alpha^{2s}}\right)^{(2q+n-2)/2s}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q-1}$$
$$\times\exp\left\{-\frac{r}{\alpha^{2s}}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{s}\right\}\left(\frac{1}{\left(2\alpha\right)^{n}\beta^{n/2}}\prod_{i=1}^{n}t_i^{-3/2}\left(t_i+\beta\right)\right)$$

and so it is useful to set the r parameter.

$$L(\mathbf{t};\alpha,\beta,q,r,s)=\frac{s\Gamma(n/2)}{\pi^{n/2}\Gamma((2q+n-2)/2s)}\left(\frac{r}{\alpha^{2s}}\right)^{(2q+n-2)/2s}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q-1}$$
$$\times\exp\left\{-\frac{r}{\alpha^{2s}}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{s}\right\}\left(\frac{1}{2^n\beta^{n/2}}\prod_{i=1}^{n}t_i^{-3/2}\left(t_i+\beta\right)\right)$$

$$l(\mathbf{t};\alpha,\beta,q,r,s)=\log\left(s\right)+\log\Gamma(n/2)+\frac{2q+n-2}{2s}\log\left(\frac{r}{\alpha^{2s}}\right)-\log\Gamma((2q+n-2)/2s)$$
$$-n\left[\log\left(2\right)+\frac{1}{2}\log\left(\pi\right)+\frac{1}{2}\log\left(\beta\right)\right]-\frac{3}{2}\sum_{i=1}^{n}\log\left(t_i\right)+\sum_{i=1}^{n}\log\left(t_i+\beta\right)$$
$$+(q-1)\log\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)-\frac{r}{\alpha^{2s}}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{s}$$

A5. Normal Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0,1;h)$

$$f_T(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right)$$

then $T \sim GBS(\alpha, \beta; g)$,

$$f_T(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\left(\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}\right)$$

with $\alpha, \beta > 0$. Likelihood

$$L(t;\alpha,\beta) = \frac{1}{\pi^{n/2}2^{3n/2}\alpha^n\beta^{n/2}} \exp\left(-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)$$

Log-likelihood

$$l(t;\alpha,\beta) = -n\left(\frac{3}{2}\log(2) + \log(\alpha) + \frac{1}{2}\log(\beta) + \frac{1}{2}\log(\pi)\right) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i+\beta)$$
$$-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]$$

- **Dependent case.** Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}_n; h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{1}{(2\pi)^{n/2}} \exp\left\{-\frac{1}{2}\|\mathbf{u}\|^2\right\}$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha, \beta; h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2\alpha^2}\left[\frac{t}{\beta} + \frac{\beta}{t} - 2\right]\right)\frac{1}{2^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)$$

with $\alpha, \beta > 0$.with $\alpha, \beta > 0$. Likelihood

$$L(\mathbf{t};\alpha,\beta) = \frac{1}{\pi^{n/2}2^{3n/2}\alpha^n\beta^{n/2}} \exp\left(-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]\right)\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)$$

Log-likelihood

$$l(\mathbf{t};\alpha,\beta) = -n\left(\frac{3}{2}\log(2) + \log(\alpha) + \frac{1}{2}\log(\beta) + \frac{1}{2}\log(\pi)\right) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i) + \sum_{i=1}^{n}\log(t_i+\beta)$$
$$-\frac{1}{2\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta} + \frac{\beta}{t_i} - 2\right]$$

Note that in the two cases the likelihood expressions are the same, as is to be expected given that this is one of the properties of the normal distribution.

A6. Laplace Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0,1;h)$

$$f_u(u) = \frac{1}{2}\exp(-|u|)$$

then $T \sim GBS(\alpha,\beta;g)$,

$$f_T(t) = \frac{1}{2}\exp\left(-\frac{1}{\alpha}\left(\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{1/2}\right)\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}$$

with $\alpha,\beta > 0$. Likelihood

$$\begin{aligned} L(t;\alpha,\beta) &= \frac{1}{2^n}\exp\left(-\frac{1}{\alpha}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{1/2}\right)\frac{1}{2^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right) \\ &= \frac{1}{2^{2n}\alpha^n\beta^{n/2}}\exp\left\{-\frac{1}{\alpha}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{1/2}\right\}\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right) \end{aligned}$$

Log-likelihood

$$\begin{aligned} l(t;\alpha,\beta) = -n\left(2\log(2)+\log(\alpha)+\frac{1}{2}\log(\beta)\right) - \frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta) \\ -\frac{1}{\alpha}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{1/2} \end{aligned}$$

- **Dependent case.** Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0},\mathbf{I}_n;h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{\Gamma\left(\frac{1}{2}n\right)}{2\pi^{n/2}\Gamma(n)}\exp(-\|\mathbf{u}\|)$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha,\beta;h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\Gamma\left(\frac{1}{2}n\right)}{2\pi^{n/2}\Gamma(n)}\exp\left(-\frac{1}{\alpha}\left[\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right]^{1/2}\right)\frac{1}{2^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)$$

with $\alpha,\beta > 0$. Likelihood

$$L(\mathbf{t};\alpha,\beta) = \frac{\Gamma\left(\frac{1}{2}n\right)}{\pi^{n/2}\Gamma(n)\,2^{n+1}\alpha^n\beta^{n/2}}\exp\left(-\frac{1}{\alpha}\left[\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right]^{1/2}\right)\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)$$

Log-likelihood

$$\begin{aligned} l(\mathbf{t};\alpha,\beta) = \log\left(\Gamma\left(\frac{1}{2}n\right)\right) - \log(\Gamma(n)) - (n+1)\log(2) - n\left(\log(\alpha)+\frac{1}{2}\log(\pi)+\frac{1}{2}\log(\beta)\right) \\ -\frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta)-\frac{1}{\alpha}\left[\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right]^{1/2} \end{aligned}$$

A7. Special Case Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0,1;h)$

$$f_u(u) = \frac{\sqrt{2}}{\pi}\left(1+u^4\right)^{-1}$$

then $T \sim GBS(\alpha,\beta;g)$,

$$\begin{aligned} f_T(t) &= \frac{\sqrt{2}}{\pi}\left(1+\frac{1}{\alpha^4}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]^2\right)^{-1}\frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}} \\ &= \frac{1}{2^{1/2}\pi\alpha\beta^{1/2}}\left(1+\frac{1}{\alpha^4}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]^2\right)^{-1} t^{-3/2}(t+\beta) \end{aligned}$$

with $\alpha,\beta>0$. Likelihood

$$L(t;\alpha,\beta) = \frac{1}{2^{n/2}\pi^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\left(1+\frac{1}{\alpha^4}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^2\right)^{-1} t_i^{-3/2}(t_i+\beta)$$

Log-likelihood

$$\begin{aligned} l(t;\alpha,\beta) = &-n\left(\log(\pi)+\log(\alpha)+\frac{1}{2}\log(2)+\frac{1}{2}\log(\beta)\right)-\frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta) \\ &-\sum_{i=1}^{n}\log\left(1+\frac{1}{\alpha^4}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^2\right) \end{aligned}$$

- **Dependent case.** Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0},\mathbf{I}_n;h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{2\Gamma(n/2)\Gamma((n+3)/4)}{\pi^{n/2}\Gamma(n/4)\Gamma(3/4)}\left(1+\|\mathbf{u}\|^4\right)^{-(n+3)/4}$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha,\beta;h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{2\Gamma(n/2)\Gamma((n+3)/4)}{\pi^{n/2}\Gamma(n/4)\Gamma(3/4)}\left(1+\frac{1}{\alpha^4}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^2\right)^{-(n+3)/4}\frac{\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)}{2^n\alpha^n\beta^{n/2}}$$

with $\alpha,\beta>0$. Likelihood

$$L(\mathbf{t};\alpha,\beta) = \frac{\Gamma(n/2)\Gamma((n+3)/4)\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)}{2^{n-1}\pi^{n/2}\Gamma(n/4)\Gamma(3/4)\alpha^n\beta^{n/2}}\left(1+\frac{1}{\alpha^4}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^2\right)^{-(n+3)/4}$$

Log-likelihood

$$l(\mathbf{t};\alpha,\beta)=\log\Gamma(n/2)+\log\Gamma((n+3)/4)-\left(\frac{n}{2}\right)\log(\pi)-n\left(\log(\alpha)+\frac{1}{2}\log(\beta)\right)$$
$$-(n-1)\log(2)-\log\Gamma(n/4)-\log\Gamma(3/4)-\frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta)$$
$$-\left(\frac{n+3}{4}\right)\log\left(1+\frac{1}{\alpha^4}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^2\right)$$

Remark 3. *It is important to note that the Special Case Distribution (independent case) is not a normal density and the process by which it is obtained is unexplained, Gupta and Varga (1993, p. 70). For the same reason, it is not possible to follow a procedure to obtain the corresponding multivariate density, on which the dependent case is based. Therefore, what is proposed is the actual density for the dependent case, simply observing that when* $n=1$ *the univariate density is obtained. However, note that this is not the only mltivariate density that can be proposed. If we make the exponent of* $(1+||U||^4)$ *equal to* $(n+1)/2$ *or to* $(3n+1)/4$ *and calculate the corresponding constant, such densities, too, for the case in which* $n=1$*, produce the univariate Special Case density, although this does not guarantee that such a density corresponds to the multivariate density of the univariate Special Case.*

A8. Logistic Distribution

- **Independent case.** Let $U\sim\mathcal{E}(0,1;h)$

$$f_u(u)=\frac{1}{I(z)}\frac{\exp\left(-u^2\right)}{\left[1+\exp\left(-u^2\right)\right]^2},\quad I(z)=\int_0^\infty z^{-1/2}\frac{\exp\left(-z\right)}{\left[1+\exp\left(-z\right)\right]^2}dz$$

then $T\sim GBS\left(\alpha,\beta;g\right)$,

$$f_T(t)=\frac{1}{I(z)}\frac{\exp\left(-\frac{1}{\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right)}{\left[1+\exp\left(-\frac{1}{\alpha^2}\left(\frac{t}{\beta}+\frac{\beta}{t}-2\right)\right)\right]^2}\frac{t^{-3/2}\left(t+\beta\right)}{2\alpha\beta^{1/2}}$$

with $\alpha,\beta>0$. Likelihood

$$L\left(t;\alpha,\beta\right)=\frac{1}{\left(I(z)\right)^n2^n\alpha^n\beta^{n/2}}\prod_{i=1}^{n}\frac{\exp\left(-\frac{1}{\alpha^2}\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right)\right)}{\left[1+\exp\left(-\frac{1}{\alpha^2}\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right)\right)\right]^2}t_i^{-3/2}\left(t_i+\beta\right)$$

Log-likelihood

$$l\left(t;\alpha,\beta\right)=-n\left(\log\left(I(z)\right)+\log\left(2\right)+\log\left(\alpha\right)+\frac{1}{2}\log\left(\beta\right)\right)$$
$$-\frac{3}{2}\sum_{i=1}^{n}\log\left(t_i\right)+\sum_{i=1}^{n}\log\left(t_i+\beta\right)-\frac{1}{\alpha^2}\sum_{i=1}^{n}\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right)$$
$$-2\sum_{i=1}^{n}\log\left(1+\exp\left(-\frac{1}{\alpha^2}\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right)\right)\right)$$

- **Dependent case.**Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}_n; h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{\Gamma(n/2)}{\pi^{n/2} I_n(z)} \frac{\exp\left(-\|\mathbf{u}\|^2\right)}{\left[1+\exp\left(-\|\mathbf{u}\|^2\right)\right]^2}, \quad I_n(z) = \int_0^\infty z^{\frac{n}{2}-1} \frac{\exp(-z)}{(1+\exp(-z))^2} dz$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha, \beta; h)$,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\Gamma(n/2)}{\pi^{n/2} I_n(z)} \frac{\exp\left(-\frac{1}{\alpha^2}\sum_{i=1}^n \left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)}{\left[1+\exp\left(-\frac{1}{\alpha^2}\sum_{i=1}^n \left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)\right]^2} \frac{\prod_{i=1}^n \left(t_i^{-3/2}(t_i+\beta)\right)}{2^n \alpha^n \beta^{n/2}}$$

with $\alpha, \beta > 0$. Likelihood

$$L(\mathbf{t}; \alpha, \beta) = \frac{\Gamma(n/2) \prod_{i=1}^n \left(t_i^{-3/2}(t_i+\beta)\right)}{\pi^{n/2} 2^n \alpha^n \beta^{n/2} I_n(z)} \frac{\exp\left(-\frac{1}{\alpha^2}\sum_{i=1}^n \left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)}{\left[1+\exp\left(-\frac{1}{\alpha^2}\sum_{i=1}^n \left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)\right]^2}$$

Log-likelihood

$$\begin{aligned} l(\mathbf{t}; \alpha, \beta) = {} & \log(\Gamma(n/2)) - \log(I_n(z)) \\ & - n\left(\log(2) + \log(\alpha) + \frac{1}{2}\log(\pi) + \frac{1}{2}\log(\beta)\right) \\ & - \frac{3}{2}\sum_{i=1}^n \log(t_i) + \sum_{i=1}^n \log(t_i+\beta) - \frac{1}{\alpha^2}\sum_{i=1}^n \left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right] \\ & - 2\log\left(1+\exp\left(-\frac{1}{\alpha^2}\sum_{i=1}^n \left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)\right) \end{aligned}$$

A9. Bessel Distribution

- **Independent case.** Let $U \sim \mathcal{E}(0, 1; h)$

$$f_u(u) = \frac{\left(u^2\right)^{q/2}}{2^q r^{q+1} \pi^{1/2} \Gamma\left(q+\frac{1}{2}\right)} K_q\left\{-\frac{1}{r}u\right\}$$

Where

$$K_q(z) = \frac{\pi}{2} \frac{I_{-q}(z) - I_q(z)}{\sin(q\pi)}, \ |\arg(z)| < \pi,$$

with q an integer, is the modified Bessel function of the third kind and

$$I_q(z) = \sum_{k=0}^\infty \frac{1}{k!\Gamma(k+q+1)} \left(\frac{z}{2}\right)^{q+2k}, \ |z| < \infty, \ |\arg(z)| < \pi.$$

Then $T \sim GBS(\alpha, \beta; g)$,

$$f_T(t) = \frac{\left(\frac{1}{\alpha^2}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{q/2}}{2^q r^{q+1} \pi^{1/2} \Gamma\left(q+\frac{1}{2}\right)} K_q\left\{\frac{1}{r\alpha}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]^{1/2}\right\} \frac{t^{-3/2}(t+\beta)}{2\alpha\beta^{1/2}}$$

with $\alpha, \beta, r > 0$ and $q > -1/2$. The $r\alpha$ parameters are not identifiable,

$$f_T(t) = \frac{\frac{1}{(r\alpha)^{q+1}}\left(\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]\right)^{q/2}}{2^q\pi^{1/2}\Gamma\left(q+\frac{1}{2}\right)} K_q\left\{\frac{1}{r\alpha}\left[\frac{t}{\beta}+\frac{\beta}{t}-2\right]^{1/2}\right\}\frac{t^{-3/2}(t+\beta)}{2\beta^{1/2}}$$

and so it is useful to set the r parameter. Likelihood

$$\begin{aligned}
L(t;\alpha,\beta,r,q) &= \prod_{i=1}^{n}\frac{\left(\frac{1}{\alpha^2}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q/2}}{2^q r^{q+1}\pi^{1/2}\Gamma\left(q+\frac{1}{2}\right)} K_q\left\{-\frac{1}{r\alpha}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{1/2}\right\}\frac{1}{2\alpha\beta^{1/2}}t_i^{-3/2}(t_i+\beta)\\
&= \frac{1}{2^{n(q+1)}(\alpha r)^{n(q+1)}\pi^{n/2}\Gamma\left(q+\frac{1}{2}\right)^n\beta^{n/2}}\\
&\qquad\times\prod_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{q/2}K_q\left\{\frac{1}{r\alpha}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{1/2}\right\}t_i^{-3/2}(t_i+\beta)
\end{aligned}$$

Log-likelihood

$$\begin{aligned}
l(t;\alpha,\beta,r,q) = -n&\left((q+1)\log(2)+(q+1)\log(r\alpha)+\log\left(\Gamma\left(q+\frac{1}{2}\right)\right)+\frac{1}{2}\log(\pi)+\frac{1}{2}\log(\beta)\right)\\
&-\frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta)+\frac{q}{2}\sum_{i=1}^{n}\log\left(\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right)\\
&+\sum_{i=1}^{n}\log\left(K_q\left\{\frac{1}{r\alpha}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]^{1/2}\right\}\right)
\end{aligned}$$

- **Dependent case.** Let $\mathbf{U} \sim \mathcal{E}_n(\mathbf{0}, \mathbf{I}_n; h)$

$$f_{\mathbf{U}}(\mathbf{u}) = \frac{\left(\frac{\|\mathbf{u}\|}{r}\right)^q}{2^{q+n-1}\pi^{n/2}r^n\Gamma\left(q+\frac{n}{2}\right)}\left[K_q\left(\frac{\|\mathbf{u}\|}{r}\right)\right]$$

then $\mathbf{T} \sim \mathcal{GBS}_n(\alpha, \beta; h)$,

$$\begin{aligned}
f_{\mathbf{T}}(\mathbf{t}) &= \frac{\left(\frac{1}{r^2\alpha^2}\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q/2}}{2^{q+n-1}\pi^{n/2}r^n\Gamma\left(q+\frac{n}{2}\right)}\left[K_q\left(\frac{1}{r\alpha}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{1/2}\right)\right]\\
&\qquad\times\frac{\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)}{2^n\alpha^n\beta^{n/2}}\\
&= \frac{\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q/2}}{2^{q+2n-1}\pi^{n/2}(r\alpha)^{n+q}\beta^{n/2}\Gamma\left(q+\frac{n}{2}\right)}\left[K_q\left(\frac{1}{r\alpha}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{1/2}\right)\right]\\
&\qquad\times\prod_{i=1}^{n}\left(t_i^{-3/2}(t_i+\beta)\right)
\end{aligned}$$

with $\alpha, \beta, r > 0$ and $q > -n/2$. The $r\alpha$ parameters are not identifiable,

$$f_{\mathbf{T}}(\mathbf{t}) = \frac{\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q/2}}{2^{q+2n-1}\pi^{n/2}\left(r\alpha\right)^{n+q}\beta^{n/2}\Gamma\left(q+\frac{n}{2}\right)}\left[K_q\left(\frac{1}{r\alpha}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{1/2}\right)\right] \times \prod_{i=1}^{n}\left(t_i^{-3/2}\left(t_i+\beta\right)\right)$$

and so it is useful to set the r parameter. Likelihood

$$L(\mathbf{t};\alpha,\beta,r,q) = \frac{1}{2^{q+2n-1}\pi^{n/2}\left(r\alpha\right)^{n+q}\beta^{n/2}\Gamma\left(q+\frac{n}{2}\right)} \times\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{q/2}\left[K_q\left(\frac{1}{r\alpha}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{1/2}\right)\right]\prod_{i=1}^{n}\left(t_i^{-3/2}\left(t_i+\beta\right)\right)$$

Log-likelihood

$$l(\mathbf{t};\alpha,\beta,r,q) = -\log\left(\Gamma\left(q+\frac{n}{2}\right)\right)-(q+2n-1)\log(2)-(n+q)\log(r\alpha)-\frac{n}{2}\left(\log(\pi)+\log(\beta)\right) -\frac{3}{2}\sum_{i=1}^{n}\log(t_i)+\sum_{i=1}^{n}\log(t_i+\beta)+\frac{q}{2}\log\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right) +\log\left(K_q\left\{\frac{1}{r\alpha}\left(\sum_{i=1}^{n}\left[\frac{t_i}{\beta}+\frac{\beta}{t_i}-2\right]\right)^{1/2}\right\}\right)$$