DOUBLY NONCENTRAL MATRIX VARIATE BETA DISTRIBUTION

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Abstract

In this paper, we determine the symmetrised density of a nonsingular doubly noncentral matrix variate beta type I and II distributions under different definitions.

Key words and phrases: Random matrices, noncentral distribution, doubly noncentral distribution, matrix variate beta.

1 Introduction

In the univariate case, the doubly non-central beta type II distribution (also termed doubly non-central $\mathcal{F}$ distribution) has been studied by Searle (1971, p. 53) and Tiku (1965). This distribution has been utilized to find power functions for the analysis of variance tests in the presence of an interaction for the two-way model layout with one observation per cell (see Bulgren (1971)). It has also been used in engineering problems in the context of information theory to calculate the error probability for a particular binary signalling system in which the receiver tries to learn the state of a multiple parallel link noise perturbed channel (see Price (1962)). Doubly non-central distributions have also been applied to problems in communications, in signals captured through radar, and pattern recognition where quadratic forms on Normal data are involved (see, for example, Turin (1959), Kailath (1961), Sebestyen (1961) and Wishner (1962)).

In the multivariate case, the matrix variate beta type I and II distributions for the central, non-central and doubly non-central cases have been studied by different authors from diverse approaches, see Olkin and Rubin (1964), Khatri (1970), Chikuse

(1980), Muirhead (1982), Cadet (1996), Gupta and Nagar (2000), Díaz-García and Gutiérrez-Jáimez (2001), among many others. In particular, doubly non-central distributions play a very important role in testing the power of hypotheses in the context of multivariate analysis, such as canonical correlation analysis and general linear hypothesis in MANOVA, see Muirhead (1982) and Srivastava (1968). Moreover, the univariate problems mentioned above, in the context of the theory of information and communication have recently been studied in the multivariate case, and doubly non-central matrix variate distributions have again featured in these studies, see Ting *et al.* (2004) and Ratnarajah and Vaillancourt (2005), among many others.

In general, the use of non-central, doubly non-central and, especially, beta-type distributions has not been developed as much as could be desired, due particularly to the fact that these distributions depend on hypergeometric functions whith matrix argument, zonal or invariant polynomials. Until very recently, such functions were quite complicated to evaluate. Studies have recently appeared describing algorithms that are very efficient in their calculations, both of zonal polynomials and of hypergeometric functions with a matrix argument. These algorithms enable a broader and more efficient use of non-central distributions in general, see Gutiérrez *et al.* (2000), Sáez (2004), Demmel and Koev (2004), Koev (2004), Koev and Demmel (2004) and Dimitriu *et al.* (2005).

In statistical literature, as well as the classification of the beta distribution as beta type I and type II (see Gupta and Nagar (2000) and Srivastava and Khatri (1979)), two alternative definitions have been proposed for each of the latter. Let us refer initially to the beta type I distribution. If A and B have a central Wishart distribution, i.e. $A \sim \mathcal{W}_m(r, I)$ and $B \sim \mathcal{W}_m(s, I)$ independently, then the beta matrix U can be defined as

$$U = \left\{ \begin{array}{ll} (A+B)^{-1/2}A((A+B)^{-1/2})', & \text{Definition 1 or,} \\ A^{1/2}(A+B)^{-1}(A^{1/2})', & \text{Definition 2,} \end{array} \right. \tag{1}$$

where $C^{1/2}(C^{1/2})' = C$ is a reasonable non-singular factorisation of C, see Gupta and Nagar (2000), Srivastava and Khatri (1979) and Muirhead (1982). It is apparent that under Definitions 1 and 2, its density function is given and denoted as

$$\mathcal{B}I_m(U; r/2, s/2) = \frac{1}{\beta_m[r/2, s/2]} |U|^{(r-m-1)/2} |I_m - U|^{(s-m-1)/2}, \tag{2}$$

$0 < U < I_m$, denoting as $U \sim \mathcal{B}I_m(r/2, s/2)$, with $r \geq m$ and $s \geq m$; where $\beta_m[r/2, s/2]$ denotes the multivariate beta function defined by

$$\beta_m[b, a] = \int_{0<S<I_m} |S|^{a-(m+1)/2} |I_m - S|^{b-(m+1)/2} (dS) = \frac{\Gamma_m[a]\Gamma_m[b]}{\Gamma_m[a+b]},$$

where $\Gamma_m[a]$ denotes the multivariate gamma function and is defined as

$$\Gamma_m[a] = \int_{R>0} \operatorname{etr}(-R) |R|^{a-(m+1)/2} (dR),$$

$\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{etr}(\cdot) \equiv \exp(\operatorname{tr}(\cdot))$.

An alternative definition for the beta type I matrix was proposed by Srivastava and Khatri (1979, pp. 94-95), Srivastava (1968), Muirhead (1982, pp. 451-452) and Gupta and Nagar (2000); it is formulated as follows:

Let $B \sim \mathcal{W}_m(s, I)$ and let us state $A = Y'Y$ donde $Y \sim \mathcal{N}_{r\times m}(0, I_r \otimes I_m)$, $m > r$, independently of B. Then $U = Y(Y'Y + B)^{-1}Y' = Y(A + B)^{-1}Y'$. Moreover, $U \sim \mathcal{B}I_r(m/2, (s + r - m)/2)$.

However, note that in the central and non-central cases, the density, the properties and the associated distributions can be obtained from the definitions in (1) by replacing m by r, r by m and s by $s + r - m$, i.e., by making the substitutions

$$m \to r, \quad r \to m, \quad s \to s + r - m, \tag{3}$$

see Srivastava and Khatri (1979, p. 96) or Muirhead (1982, eq. (7), p. 455). For this reason, we shall focus our attention on the definitions stated in (1).

In an analogous way for the beta type II distributions, the following definitions have been proposed:

$$V = \left\{ \begin{array}{ll} B^{-1/2}A(B^{-1/2})', & \text{Definition 1,} \\ A^{1/2}B^{-1}(A^{1/2})', & \text{Definition 2,} \\ Y^{1/2}B^{-1}Y', & \text{Definition 3.} \end{array} \right. \tag{4}$$

The distribution is denoted by $V \sim \mathcal{B}II_m(r/2, s/2)$. In a similar way to the case of the beta type I distribution, the results under Definition 3 can be obtained from the results of Definition 2 and applying the transforms (3), see James (1964) and Muirhead (1982, pp.451-455).

In this case, the central beta type II density under definitions 1 and 2 is denoted and defined as

$$\mathcal{B}II_m(V; r/2, s/2) = \frac{1}{\beta[r/2, s/2]}|V|^{(r-m-1)/2}|I + V|^{-(r+s)/2}, \quad V > 0.$$

When these ideas are extended to the doubly non-central case, i.e. when $A \sim \mathcal{W}_m(r, I, \Omega_1)$ and $B \sim \mathcal{W}_m(s, I, \Omega_2)$, strictly speaking, we have not found the densities of the beta types I and II distributions under Definitions 1 or 2. Rather, for the case of the beta type II distribution, Chikuse (1980) found the distribution of $\tilde{V} = \tilde{B}^{-1/2}\tilde{A}(\tilde{B}^{-1/2})'$ where $\tilde{A} = H'AH$ y $\tilde{B} = H'BH$, $H \in \mathcal{O}(m)$, with $\mathcal{O}(m) = \{H \in \Re^{m\times m} | HH' = H'H = I_m\}$. It is straightforward to show that the procedure proposed by Chikuse (1980) is equivalent to finding the symmetrised density defined by Greenacre (1973), Greenacre (1973), see also Roux (1975).

In this paper we find the symmetrised density function of the doubly non-central matrix variate beta types I and II under the three definition proposed in the literature. Moreover, we find the densities corresponding to the eigenvalues of the beta distribution types I and II. It is immediately apparent that the central and non-central distributions are found as particular cases of the distributions being studied. We propose this as a solution to the problem of determining the non-central beta densities, as described by Constantine (1963), Khatri (1970) and reconsidered in Farrell (1985, p. 191) and Gupta and Nagar (2000), see also Díaz-García and Gutiérrez-Jáimez (2006).

2 Preliminar results

Given a function $f(X)$, $X: m \times m$, $X > 0$, Greenacre (1973) (see also Roux (1975)) proposes the following definition:

$$f_s(X) = \int_{\mathcal{O}(m)} f(HXH')(dH), \quad H \in \mathcal{O}(m) \tag{5}$$

where $\mathcal{O}(m) = \{H \in \Re^{m\times m} | HH' = H'H = I_m\}$ and (dH) denotes the normalised invariant measure on $\mathcal{O}(m)$ (Muirhead, 1982, p. 72). This function $f_s(X)$ is termed the symmetrised function. The approach we adopt is to apply this idea of Greenacre (1973) to find the densities of the symmetrised doubly non-central beta distributions. To do so, let us consider the following:

Theorem 2.1. *Let $X > 0$, $E > 0$ matrices $m \times m$, $a \geq (m-1)/2$, $b \geq (m-1)/2$ and*

$$g(X) = \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) C_\kappa\left(\Theta E^{1/2} R(X) (E^{1/2})'\right) \\ \times \; C_\lambda\left(\Xi E^{1/2} S(X) (E^{1/2})'\right)(dE)$$

where $Q(X) > 0$, $R(X) > 0$ and $S(X) > 0$ are $m \times m$ matrix functions of matrix X such that, $Q(HXH') = HQ(X)H'$, $H \in \mathcal{O}(m)$, with the same property for $R(X)$ and $S(X)$; $C_\kappa(M)$ is the zonal polynomial of M corresponding to the partition $\kappa = (k_1, \dots, k_m)$ of k with $\sum_{i=1}^m k_i = k$ and $C_\lambda(N)$ is the zonal polynomial of N corresponding to the partition $\lambda = (l_1, \dots, l_m)$ of l with $\sum_{i=1}^m l_i = l$. Then

$$g_s(X) = \sum_{\phi \in \kappa\cdot\lambda} \frac{\Gamma_m[a+b](a+k)_\phi}{|Q(X)|} \frac{C_\phi^{\kappa,\lambda}(\Theta, \Xi) C_\phi^{\kappa,\lambda}(R(X)Q(X)^{-1}, S(X)Q(X)^{-1})}{C_\phi(I)},$$

where $Q(X)^{-1}$ denotes the inverse of matrix $Q(X)$ (not the inverse function of $Q(\cdot)$), $C_\phi^{\kappa,\lambda}(\cdot)$ is the invariant polynomial with two matrix arguments, $(t)_\tau$ is the generalised hypergeometric coefficient or product of Pochhammer symbols.

Proof. We have

$$g(X) = \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) C_\kappa\left(\Theta E^{1/2} R(X) (E^{1/2})'\right) \\ \times \; C_\lambda\left(\Xi E^{1/2} S(X) (E^{1/2})'\right)(dE).$$

Consider the symmetrised function g and the transform $E = HEH'$, noting that $(dE) = (dHEH')$. Then

$$g_s(X) = \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) \int_{\mathcal{O}(m)} C_\kappa\left(\Theta H E^{1/2} R(X) (E^{1/2})' H'\right) \\ \times \; C_\lambda\left(\Xi H E^{1/2} S(X) (E^{1/2})' H'\right)(dH)(dE),$$

from Davis (1980, equation (4.13)) (see also Chikuse (1980, equation (2.2))) and thus

$$g_s(X) = \sum_{\phi\in\kappa\cdot\lambda} \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}(-Q(X)E)$$
$$\times \frac{C_\phi^{\kappa,\lambda}(\Theta,\Delta)C_\phi^{\kappa,\lambda}(R(X)E, S(X)E)}{C_\phi(I)}(dE).$$

Now, from Davis (1980, pp. 297-298)

$$g_s(X) = \sum_{\phi\in\kappa\cdot\lambda} \frac{\Gamma[(a+b),\phi]_m}{|Q(X)|^{a+b}} \frac{C_\phi^{\kappa,\lambda}(\Theta,\Delta)C_\phi^{\kappa,\lambda}(R(X)Q(X)^{-1}, S(X)Q(X)^{-1})}{C_\phi(I)},$$

where $\Gamma_m[(a+b),\phi] = (a+b)_\phi\Gamma_m[(a+b)]$, see Constantine (1963). $\square$

3 Doubly noncentral beta type I distribution

Theorem 3.1. *Suppose that U has a doubly non-central matrix variate beta type I under the definition 1, which is denoted as $U \sim \mathcal{B}I_1(r/2, s/2, \Omega_1, \Omega_2)$. Then, using the notation for the operator sum as in Davis (1980) we have that its symmetrised density function is*

$$f_s(U) = \mathcal{B}I_m(U; r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}(\Omega_1+\Omega_2)\right)$$
$$\times \sum_{\kappa,\lambda;\ \phi}^{\infty} \frac{\frac{1}{2}(r+s)_\phi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\ l!} \frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2)C_\phi^{\kappa,\lambda}(U,(I-U))}{C_\phi(I)}, \quad 0 < U < I.$$

Proof. By independence, the joint density of A and B is

$$f_{A,B}(A,B) = c|A|^{(r-m-1)/2}|B|^{(s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}(A+B)\right)$$
$$\times\ {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 A\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 B\right), \quad (6)$$

where

$$c = \frac{\operatorname{etr}\left(-\frac{1}{2}(\Omega_1+\Omega_2)\right)}{2^{m(r+s)/2}\Gamma_m[r/2]\Gamma_m[s/2]}. \quad (7)$$

By performing the transforms $C = A+B$ with $(dA)\wedge(dB) = (dA)\wedge(dC)$ and then the transform $A = C^{1/2}U(C^{1/2})'$ with $(dA)\wedge(dC) = |C|^{(m+1)/2}(dC)\wedge(dU)$, we find that the joint density of C and U is given by

$$f_{C,U}(C,U) = c|U|^{(r-m-1)/2}|I-U|^{(s-m-1)/2}|C|^{(r+s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}C\right)$$
$$\times\ {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 C^{1/2}U(C^{1/2})'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 C^{1/2}(I-U)(C^{1/2})'\right),$$

From which, by expanding the hypergeometric functions in infinite series of zonal polynomials and integrating with respect to C, the desired result now follows with the assistance of Theorem 2.1. Note that in this case $Q(\cdot) = \frac{1}{2}I$, $R(\cdot) = U$ and $S(\cdot) = (I-U)$ in Theorem 2.1. $\square$

Similarly, under Definition 2, we have:

Theorem 3.2. *Suppose that U has a doubly non-central matrix variate beta type I distribution under Definition 2, which shall be denoted as $U \sim \mathcal{B}I_2(r/2, s/2, \Omega_1, \Omega_2)$. Then, its symmetrised density function is the same as in Theorem 3.1.*

Proof. By independence, the joint density of A and B is given by (6). Let $C = A+B$ with $(dA) \wedge (dB) = (dA) \wedge (dC)$ and consider the transform $C = (A^{1/2})'U^{-1}A^{1/2}$ with $(dA) \wedge (dC) = |A|^{(m+1)/2}|U|^{-(m+1)}(dA) \wedge (dU)$. Then, the joint density of A and U is given by

$$f_{A,U}(A,U) = c|U|^{-(s+m+1)/2}|I-U|^{(s-m-1)/2}|A|^{(r+s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}AU^{-1}\right) \times {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 A)'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 A^{1/2}(I-U)U^{-1}(A^{1/2})'\right).$$

The result follows from integrating with respect to A, taking $Q(\cdot) = \frac{1}{2}U^{-1}$, $R(\cdot) = I$ and $S(\cdot) = (I-U)U^{-1}$ and $C = A$ in Theorem 2. □

Corollary 3.1. *Let $U \sim \mathcal{B}I_j(B)(s/2, r/2, \Omega_1, \Omega_2)$, $j = 1$ or 2, then the joint density function of the eignvalues $\Lambda = \operatorname{diag}(u_1, \dots, u_m)$, $1 > u_1 > \cdots > u_m > 0$ of U is*

$$f(u_1, \dots, u_m) = \frac{\pi^{m^2/2} \operatorname{etr}\left(-\frac{1}{2}(\Omega_1 + \Omega_2)\right)}{\Gamma_m[m/2]\beta_m[r/2, s/2]} \prod_{i=1}^m \left\{u_1^{(r-m-1)/2}(1-u_i)^{(s-m-1)/2}\right\} \times \prod_{i<j}^m (u_i - u_j) \sum_{\kappa,\lambda;\phi}^\infty \frac{\frac{1}{2}(r+s)_\phi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\ l!} \frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2) C_\phi^{\kappa,\lambda}(\Lambda, (I-\Lambda))}{C_\phi(I)},$$

Proof. The proof follows immediately by applying the Theorem 3.2.17 in Muirhead (1982, p. 104) to the beta type I density in Theorem 3.1, using the equation (3.12) in Chikuse (1980). □

4 Doubly noncentral beta type II distribution

Theorem 4.1. *Suppose that $F > 0$ has a doubly non-central matrix variate beta type II distribution under Definition 1, denoted as $F \sim \mathcal{B}II_1(r/2, s/2, \Omega_1, \Omega_2)$. Then, using the notation for the operator sum as in Davis (1980), we have that its symmetrised density function is*

$$g_s(F) = \mathcal{B}II_m(F; r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}(\Omega_1 + \Omega_2)\right) \times \sum_{\kappa,\lambda;\phi}^\infty \frac{\frac{1}{2}(r+s)_\phi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\ l!} \frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2) C_\phi^{\kappa,\lambda}((I+F)^{-1}F, (I+F)^{-1})}{C_\phi(I)}.$$

Proof. The joint density function of A and B is given by (6). Transforming $F = B^{-1/2}A(B^{-1/2})'$ and noting that $(dA) \wedge (dB) = |B|^{(m+1)/2}(dB) \wedge (dF)$, the joint density of B and F is

$$g_{F,B}(F,B) = c|F|^{(r-m-1)/2}|B|^{(r+s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}B^{1/2}(I+F)(B^{1/2})'\right) \times {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 B^{1/2}F(B^{1/2})'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 B)'\right),$$

From which, by expanding the hypergeometric functions in infinite series of zonal polynomials and integrating with respect to B, and then taking $Q(\cdot) = \frac{1}{2}(I+F)$, $R(\cdot) = F$ and $S(\cdot) = I$ in Theorem 1, the result is obtained. □

From Definition 2, we have:

Theorem 4.2. *Suppose that F has a doubly non-central matrix variate beta type II distribution under Definition 2, which shall be denoted as $F \sim \mathcal{B}I_2(r/2, s/2, \Omega_1, \Omega_2)$. Then, its symmetrised density functions is the same as in Theorem 4.1.*

Proof. By independence the joint density function of A and B is given by (6). Now, we make the change of variable $F = (A^{1/2})'B^{-1}A^{1/2}$ observing that $(dA) \wedge (dF) = |A|^{(m+1)/2}|F|^{-(m+1)}(dA) \wedge (dF)$. The joint density of A, F is then

$$g_{A,F}(A, F) = c|F|^{-(s+m+1)/2}|A|^{(r+s-m-1)/2} \operatorname{etr}\left(-\tfrac{1}{2}A(I + F^{-1})\right) \times {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 A)'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 A^{1/2}F^{-1}(A^{1/2})'\right).$$

Integrating with respect to A using the Theorem 3.1 with $Q(\cdot) = \frac{1}{2}(I + F^{-1})$, $R(\cdot) = I$ and $S(\cdot) = F^{-1}$ gives the stated marginal density for F. □

Corollary 4.1. *Let $F \sim \mathcal{B}II_j(B)(s/2, r/2, \Omega_1, \Omega_2)$, $j = 1$ or 2, then the joint density function of the eignvalues $\Upsilon = \operatorname{diag}(f_1, \dots, f_m)$, $f_1 > \cdots > f_m > 0$ of F is*

$$g(f_1, \dots, f_m) = \frac{\pi^{m^2/2} \operatorname{etr}\left(-\frac{1}{2}(\Omega_1 + \Omega_2)\right)}{\Gamma_m[m/2]\beta_m[r/2, s/2]} \prod_{i=1}^{m} \left\{ f_1^{(r-m-1)/2}(1 + f_i)^{-(s+r)/2} \right\} \times \prod_{i<j}^{m} (f_i - f_j) \sum_{\kappa,\lambda;\phi}^{\infty} \frac{\frac{1}{2}(r+s)_\phi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\ l!} \frac{C_\phi^{\kappa,\lambda}((I+\Upsilon)^{-1}\Upsilon, (I-\Upsilon)^{-1})}{\left[C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2)\right]^{-1} C_\phi(I)},$$

Proof. The proof follows immediately by applying Theorem 3.2.17 in Muirhead (1982, p. 104) to the beta type II density in Theorem 3.1, using equation (3.12) in Chikuse (1980). □

Remark 4.1. *Note that when $\Omega_1 = \Omega_2 = 0$ in Theorem 3.1, the central matrix variate beta type I symmetrised or nonsymmetrised distribution is obtained (in this case, the two coincide).*

Similarly, note that when $\Omega_1 = 0$, we obtain the symmetrised noncentral matrix variate beta type I(A) distribution, see Greenacre (1973) and Gupta and Nagar (2000, p. 188), given by

$$f_s(U) = \mathcal{B}I_m(U; r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2, (I-U)\right). \tag{8}$$

However, from (8) it is possible to propose an expression for the nonsymmetrised density of U; this is done by inversely applying the definition of symmetrised density given by Greenacre (1973). This way observing that

$$\int_{H \in \mathcal{O}(m)} {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2 H(I-U)H'\right)(dH) = {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2, (I-U)\right)$$

we obtain

$$f_U(U) = \mathcal{B}I_m(U; r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2(I-U)\right). \tag{9}$$

Of course, densities (8) and (9) are still invariant under definitions 1 and 2. Note, moreover, that from (9) we have indirectly reached a solution to the integral proposed by Constantine (1963), Khatri (1970) and reformulated by Farrell (1985, p. 191) and Gupta and Nagar (2000, pp. 188-189); see also Díaz-García and Gutiérrez-Jáimez (2006).

Analogous particular results are straightforwardly obtained for the noncentral matrix variate type I(B) distribution from Theorem 3.1 and for the noncentral matrix variate type II(A) and II(B) distributions from Theorem 4.1, see Gupta and Nagar (2000) and Greenacre (1973). Finally, note that the densities of the eigenvalues of the central and noncentral beta type I and II distributions in all their variates are found as particular cases of Corollaries 3.1 and 4.1.

5 Conclusions

In this paper, we show that the densities of doubly non-central matrix variate beta type I distributions, obtained under Definitions 1 and 2, coincide. An analogous result is obtained for the case of the doubly non-central beta type II distribution, and therefore we need not concern ourselves with which definition to adopt, as either will serve our purpose. Note, furthermore, that both in the case of the beta type I distribution and in that of type II, when we take $\Omega_1 = \Omega_2 = 0$, the corresponding central distributions are obtained (these being symmetrised or non-symmetrised, as in this case, they coincide). In addition, when we assume $\Omega_1 = 0$, we obtain beta type I(A) and II(A) non-central distributions (symmetrised and non-symmetrised), see Gupta and Nagar (2000, pp. 188 and 190) or Greenacre (1973). Otherwise, if $\Omega_2 = 0$, the distributions obtained are beta type I(B) and beta type II(B) symmetrised and non-symmetrised, see Gupta and Nagar (2000, p. 189-192) or Greenacre (1973). In the case of the non-central distributions obtained, intrinsically the problem presented by Constantine (1963) and by Khatri (1970), and reconsidered in Farrell (1985, p. 191) and Gupta and Nagar (2000, pp. 188-189), see also Daz-Garca and Gutirrez-Jimez (2006) is resolved.

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