SINGULAR NONCENTRAL AND DOUBLY NONCENTRAL MATRIX VARIATE BETA DISTRIBUTIONS

José Antonio Díaz García & Ramón Gutiérrez-Jáimez

Comunicación Técnica No I-06-09/27-04-2006
(PE/CIMAT)

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José A. Díaz-García

Universidad Autónoma Agraria Antonio Narro
Department of Statistics and Computation
25315 Buenavista, Saltillo
Coahuila, México

Ramón Gutiérrez Jáimez

University of Granada
Department of Statistics and O.R
Granada 18071, Spain.

Abstract

In this paper, we determine the symmetrised density of doubly noncentral singular matrix variate beta type I and II distributions under different definitions. As particular cases we obtain the noncentral singular matrix variate beta type I and II distributions and the corresponding joint density of the nonnull eigenvalues. In addition, we propose an alternative approach to find the corresponding nonsymmetrised densities. From the latter, we solve the integral proposed by Constantine (1963) and Khatri (1970) and reconsidered in Farrell (1985, p. 191), see also Díaz-García and Gutiérrez-Jáimez (2006a), for the singular and nonsingular cases.

Key words: Random matrices, doubly noncentral distribution, noncentral distribution, matrix variate beta, singular distribution, Hausdorff measure.
PACS: 62E15.

1 Introduction

In the multivariate case, the matrix variate beta type I and II distributions for central, noncentral and doubly noncentral cases have been studied by different authors from diverse approaches, see Olkin and Rubin (1964), Khatri

Email addresses: jadiaz@uaaan.mx (José A. Díaz-García), rgjaimez@goliat.ugr.es (Ramón Gutiérrez Jáimez).

(1970), Muirhead (1982), Cadet (1996), Gupta and Nagar (2000), Díaz-García and Gutiérrez-Jáimez (2001) and Chikuse (1980), among many others. These distributions play a very important role in different areas of multivariate analysis such as canonical correlation analysis and the general linear hypothesis in MANOVA, see Muirhead (1982) and Srivastava (1968). Furthermore, we examine the role of beta type distributions in the context of shape theory, see Goodall and Mardia (1992).

In general, noncentral and doubly noncentral distributions are expressed in terms of zonal polynomials, hypergeometric functions with one or two matrix arguments, or invariant polynomials. Until very recently, no efficient algorithms were available for calculating both zonal polynomials and hypergeometric functions with a matrix argument, but now there are some algorithms that facilitate a very efficient evaluation of such functions, thus enabling better use of these noncentral distributions, see Gutiérrez *et al.* (2000), Sáez (2004), Demmel and Koev (2004), Koev (2004), Koev and Demmel (2004) and Dimitriu *et al.* (2005).

In the context of distributions of singular matrices, various studies have been made of matrix normal, Wishart, Pseudo-Wishart and singular elliptic distributions, as well as some applications in the field of time series, shape theory and Bayesian statistics, see Khatri (1968), Uhlig (1994), Díaz-García and Gutiérrez (1997), Díaz-García *et al.* (1997), Díaz-García and González-Farías (1999), Díaz-García and Gutiérrez-Jáimez (2005),Díaz-García and González-Farías (2005a), Díaz-García and González-Farías (2005b), among other works. Furthermore, recent studies have been made of the role of singular distributions in the context of information theory, see Ratnarajah and Vaillancourt (2005) and Ratnarajah and Vaillancourt (2005), among others.

In particular, the study of noncentral beta type I and II distributions has been sidelined, to a certain extent, because the final expressions of the densities depend on an integral that has not been resolved in an explicit way, see Constantine (1963) and Khatri (1970), reconsidered in Farrell (1985, p. 191) and Gupta and Nagar (2000, pp. 188-189), see also Díaz-García and Gutiérrez-Jáimez (2006a). In order to address this problem by means of a different approach, three alternative definitions were proposed for each type of beta; another definition was suggested for the density, termed the symmetrised density, see Srivastava (1968), Srivastava and Khatri (1979), Gupta and Nagar (2000), Greenacre (1973) and Roux (1975). The case of singular beta type I and II distributions has received much less attention and the central case has only been approached under one of the possible definitions, see Uhlig (1994) and Díaz-García *et al.* (1997), although previously Khatri (1970) proposed an alternative means of addressing the problem.

In this paper, we extend the definitions of matrix variate beta type I and II distributions to the singular case. We establish a general result that enables us to find the respective densities in the doubly noncentral case, see Section 2. In Section 3, we find the singular doubly noncentral symmetrised density for the matrix variate beta type I distribution and the corresponding joint distribution of the eigenvalues. This section also describes the type A and B noncentral singular symmetrised densities, and proposes an expression for the corresponding nonsymmetrised densities. The section concludes by proposing the noncentral density of the eigenvalues. All the results proposed in Section 3 are found for the case of the matrix variate beta type II distribution in Section 4.

2 Preliminary results

In the nonsingular case, as well as the classification of the beta distribution as beta type I and type II (see Gupta and Nagar (2000) and Srivastava and Khatri (1979)), two alternative definitions have been proposed for each of these, see see Muirhead (1982), Srivastava (1968) and Díaz-García and Gutiérrez-Jáimez (2001). By extending these definitions to the singular case and proposing, initially, these generalisations for the matrix variate beta type I distribution, we find the following: if A and B have a Pseudo-Wishart and Wishart distribution, respectively, i.e. $A \sim \mathcal{PW}_m(r, I)$ and $B \sim \mathcal{W}_m(s, I)$ are independent, then the singular beta matrix U can be defined as

$$U = \begin{cases} (A+B)^{-1/2}A((A+B)^{-1/2})', & \text{Definition 1 or,} \\ A^{1/2}(A+B)^{-1}(A^{1/2})', & \text{Definition 2,} \end{cases} \tag{1}$$

where $C^{1/2}(C^{1/2})' = C$ is a reasonable nonsingular factorization of C, see Gupta and Nagar (2000), Srivastava and Khatri (1979) and Muirhead (1982). Under definition 1 its density function is given and denoted as (see Díaz-García and Gutiérrez (1997))

$$\mathcal{B}I_m(U; q, r/2, s/2) = c|L|^{(r-m-1)/2}|I_m - U|^{(s-m-1)/2}(dU), \quad 0 \leq U < I_m \tag{2}$$

denoting as $U \sim \mathcal{B}I_m(q, r/2, s/2)$, $s \geq m$; where $U = H_1LH_1'$, with $H_1 \in \mathcal{V}_{q,m}$; $\mathcal{V}_{q,m} = \{H_1 \in \Re^{m\times q}|H_1'H_1 = I_q\}$ denotes the Stiefel manifold; $L = \operatorname{diag}(l_1, \dots, l_q)$, $1 > l_1 > \cdots > l_q > 0$; $q = m$ (nonsingular case) or $q = r < m$ (singular case);

$$c = \frac{\pi^{(-mr+rq)/2}\Gamma_m[(r+s)/2]}{\Gamma_q[r/2]\Gamma_m[s/2]} \tag{3}$$

and (dU) denotes the Hausdorff measure on $(mq - q(q-1)/2)$-dimensional manifold of rank-q positive semidefinite $m \times m$ matrices U with q distinct nonnull eigenvalues, given by (see Uhlig (1994) and Díaz-García and Gutiérrez (1997))

$$(dU) = 2^{-q} \prod_{i=1}^{q} l_i^{m-q} \prod_{i<j} (l_i - l_j) \left(\bigwedge_{i=1}^{q} dl_i \right) \wedge (H_1' dH_1), \tag{4}$$

where $(H_1' dH_1)$ denotes the invariant measure on $\mathcal{V}_{q,m}$ and where finality, $\Gamma_m[a]$ denotes the multivariate gamma function and is defined as

$$\Gamma_m[a] = \int_{R>0} \operatorname{etr}(-R)|R|^{a-(m+1)/2}(dR),$$

$\operatorname{Re}(a) > (m-1)/2$ and $\operatorname{etr}(\cdot) \equiv \exp(\operatorname{tr}(\cdot))$.

An alternative definition of the matrix variate beta type I was proposed by Khatri (1970) see also Srivastava and Khatri (1979, pp. 94-95), Srivastava (1968), Muirhead (1982, pp. 451-452) and Gupta and Nagar (2000); this is given as follows: assume $B \sim \mathcal{W}_m(s, I)$ and write $A = Y'Y$ where $Y \sim \mathcal{N}_{r\times m}(0, I_r \otimes I_m)$, $m > r$, independently of B. Then $U_1 = Y(Y'Y + B)^{-1}Y' = Y(A+B)^{-1}Y'$ and indeed $U_1 \sim \mathcal{B}I_r(m/2, (s+r-m)/2)$. However, note that in the central case, its properties and associated distributions can be obtained from Definition (1) by replacing m by r, r by m and s by $s+r-m$, i.e., by making the substitutions

$$m \rightarrow r, \quad r \rightarrow m, \quad s \rightarrow s+r-m, \tag{5}$$

see Srivastava and Khatri (1979, p. 96) or Muirhead (1982, eq. (7), p. 455). Note that in this definition the singular case is being considered, as $r < m$; however, in this case, the density is found with respect to Lebesgue's measure (dU_1), which is defined over the space of dimension r of the positive defined matrices $U_1 : r \times r$.

In an analogous fashion, in the singular case the following definitions can be proposed for the matrix variate beta type II distribution:

$$F = \begin{cases} B^{-1/2}A(B^{-1/2})', & \text{Definition 1,} \\ A^{1/2}B^{-1}(A^{1/2})', & \text{Definition 2,} \\ Y^{1/2}B^{-1}Y', & \text{Definition 3,} \end{cases} \tag{6}$$

which in the singular case is denoted by $F \sim \mathcal{B}II_m(q, r/2, s/2)$.

If $F = H_1 G H_1'$, with $H_1 \in \mathcal{V}_{q,m}$ and $G = \operatorname{diag}(g_1, \dots, g_q)$; $g_1 > \cdots > g_q > 0$, in this case the central matrix variate beta type II distribution under Definition 1 is denoted by and defined as (see Díaz-García and Gutiérrez (1997))

$$\mathcal{B}II_m(F; q, r/2, s/2) = c|G|^{(r-m-1)/2}|I + F|^{-(r+s)/2}(dF), \quad F \geq 0. \tag{7}$$

where c is given by (14) and (dV) is given in analogous form to (4).

When these ideas are extended to the noncentral case, i.e. when $A \sim \mathcal{PW}_m(r, I, \Omega)$ and $B \sim \mathcal{W}_m(s, I)$, in the non-singular case there appears a further classification in the definitions of the matrix variate beta type I and II distributions, see Greenacre (1973) and Gupta and Nagar (2000), which can be extended to the singular case. Thus, for the matrix variate beta type I distribution, we have:

$$U = \begin{cases} (A+B)^{-1/2}A((A+B)^{-1/2})', & \text{denoting as} \quad \mathcal{B}I_1(A)_m(q, r/2, s/2, \Omega) \\ (A+B)^{-1/2}B((A+B)^{-1/2})', & \text{denoting as} \quad \mathcal{B}I_1(B)_m(q, s/2, r/2, \Omega) \end{cases} \tag{8}$$

under Definition 1; or

$$U = \begin{cases} A^{1/2}(A+B)^{-1}(A^{1/2})', & \text{denoting as} \quad \mathcal{B}I_2(A)_m(q, r/2, s/2, \Omega) \\ B^{1/2}(A+B)^{-1}(B^{1/2})', & \text{denoting as} \quad \mathcal{B}I_2(B)_m(q, s/2, r/2, \Omega) \end{cases} \tag{9}$$

under Definition 2.

For the matrix variate beta type II distribution, we have:

$$F = \begin{cases} B^{-1/2}A(B^{-1/2})', \text{ denoting as} \quad \mathcal{B}II_1(A)_m(q, r/2, s/2, \Omega) \\ A^{-1/2}B(A^{-1/2})', \text{ denoting as} \quad \mathcal{B}II_1(B)_m(q, s/2, r/2, \Omega) \end{cases} \tag{10}$$

under Definition 1; or

$$F = \begin{cases} A^{1/2}B^{-1}(A^{1/2})', \text{ denoting as} \quad \mathcal{B}II_2(A)_m(q, r/2, s/2, \Omega) \\ B^{1/2}A^{-1}(B^{1/2})', \text{ denoting as} \quad \mathcal{B}II_2(B)_m(q, s/2, r/2, \Omega) \end{cases} \tag{11}$$

under Definition 2. Both classes of distributions, types A and B, play a fundamental role in various areas of statistics, for example in the W, U and other criteria proposed by Wilks (1932).

Let us extend these ideas to the doubly noncentral case, i.e. when $A \sim \mathcal{PW}_m(r, I, \Omega_1)$ and $B \sim \mathcal{W}_m(s, I, \Omega_2)$, strictly speaking not even in the nonsingular case have the densities been found for the beta type I and II distributions under definitions 1 and 2. Instead, for the case of the matrix variate beta type II distribution, Chikuse (1980) found the distribution of $\tilde{F} = \tilde{B}^{-1/2}\tilde{A}(\tilde{B}^{-1/2})'$, where $\tilde{A} = H'AH$ and $\tilde{B} = H'BH$, $H \in \mathcal{O}(m)$, with $\mathcal{O}(m) = \{H \in \Re^{m\times m} | HH' = H'H = I_m\}$; the procedure for this, as proposed by Chikuse (1980) is equivalent to finding the symmetrised density defined by Greenacre (1973), see also Roux (1975).

It can be seen that both the central and the noncentral density functions (type A or B) in the beta type I and II distributions can be obtained easily from the doubly noncentral densities. This method is adopted in the remainder of this paper.

Given a function $f(X)$, $X : m \times m$, $X > 0$, Greenacre (1973), (see also Roux (1975)) proposes the following definition:

$$f_s(X) = \int_{\mathcal{O}(m)} f(HXH')(dH), \quad H \in \mathcal{O}(m) \tag{12}$$

where $\mathcal{O}(m) = \{H \in \Re^{m\times m} | HH' = H'H = I_m\}$ and (dH) denotes the normalised invariant measure on $\mathcal{O}(m)$ (Muirhead, 1982, p. 72). This function $f_s(X)$ is called the symmetrised function.

Our approach is to apply this idea of Greenacre's (1973) to find the densities of the symmetrised doubly noncentral matrix variate beta distributions and then to apply Greenacre's idea (1973) again, but in an inverse way, to propose the corresponding nonsymmetrised densities in the cases of the noncentral distributions. To this purpose, let us consider the following result:

Theorem 1 *Let $X \geq 0$, $E > 0$ matrices $m \times m$, $a + b \geq (m-1)/2$ and*

$$g(X) = \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) C_\kappa\left(\Theta E^{1/2} R(X)(E^{1/2})'\right)$$

$$\times \; C_\lambda\left(\Xi E^{1/2} S(X)(E^{1/2})'\right)(dE)$$

where $Q(X) > 0$, $R(X) \geq 0$ and $S(X) \geq 0$ are $m \times m$ matrix functions of matrix X such that, $Q(HXH') = HQ(X)H'$, $H \in \mathcal{O}(m)$, with the same property for $R(X)$ and $S(X)$; $C_\kappa(M)$ is the zonal polynomial of M corresponding to the partition $\kappa = (k_1, \dots, k_m)$ of k with $\sum_{i=1}^m k_i = k$ and $C_\lambda(N)$ is the zonal polynomial of N corresponding to the partition $\lambda = (l_1, \dots, l_m)$ of l with

$\sum_{i=1}^{m} l_i = l$. *Then*

$$g_s(X) = \sum_{\phi \in \kappa\cdot\lambda} \frac{\Gamma_m[a+b](a+k)_\phi}{|Q(X)|} \frac{C_\phi^{\kappa,\lambda}(\Theta, \Xi) C_\phi^{\kappa,\lambda}(R(X)Q(X)^{-1}, S(X)Q(X)^{-1})}{C_\phi(I)},$$

where $Q(X)^{-1}$ denotes the inverse of matrix $Q(X)$ (not the inverse function of $Q(\cdot)$), $C_\phi^{\kappa,\lambda}(\cdot)$ is the invariant polynomial with two matrix arguments, and $(t)_\phi$ is the generalised hypergeometric coefficient or product of Pochhammer symbols.

Proof. i) Given that

$$g(X) = \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) C_\kappa \left(\Theta E^{1/2} R(X)(E^{1/2})'\right) \times C_\lambda \left(\Xi E^{1/2} S(X)(E^{1/2})'\right) (dE),$$

let us consider the symmetrised function g and the transformation $E = HEH'$, noting that $(dE) = (dHEH')$; then

$$g_s(X) = \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) \int_{\mathcal{O}(m)} C_\kappa \left(\Theta H E^{1/2} R(X)(E^{1/2})' H'\right) \times C_\lambda \left(\Xi H E^{1/2} S(X)(E^{1/2})' H'\right) (dH)(dE),$$

from Davis (1980, equation (4.13)) (see also Chikuse (1980, equation (2.2))). Then, we have

$$g_s(X) = \sum_{\phi \in \kappa\cdot\lambda} \int_{E>0} |E|^{a+b-(m+1)/2} \operatorname{etr}\left(-Q(X)E\right) \times \frac{C_\phi^{\kappa,\lambda}(\Theta, \Delta) C_\phi^{\kappa,\lambda}(R(X)E, S(X)E)}{C_\phi(I)} (dE).$$

Now, from Davis (1980, pp. 297-298)

$$g_s(X) = \sum_{\phi \in \kappa\cdot\lambda} \frac{\Gamma[(a+b), \phi]_m}{|Q(X)|^{a+b}} \frac{C_\phi^{\kappa,\lambda}(\Theta, \Delta) C_\phi^{\kappa,\lambda}(R(X)Q(X)^{-1}, S(X)Q(X)^{-1})}{C_\phi(I)},$$

where $\Gamma_m[(a+b), \phi] = (a+b)_\phi \Gamma_m[(a+b)]$, see Constantine (1963). □

3 Doubly noncentral beta type I distribution

Theorem 2 *Suppose that U has a doubly noncentral matrix singular variate beta type I under the definition 1, denotes its as $U \sim \mathcal{B}I_{1m}(q, r/2, s/2, \Omega_1, \Omega_2)$. Then using the notation for the operator sum as in Davis (1980) we have that its* ***symmetrised*** *density function is*

$$dF_s(U) = \mathcal{B}I_m(U; q, r/2, s/2)\operatorname{etr}\left(-\tfrac{1}{2}(\Omega_1+\Omega_2)\right)$$

$$\times \sum_{\kappa,\lambda;\ \phi}^{\infty} \frac{\left(\frac{1}{2}(r+s)\right)_\phi}{\left(\frac{1}{2}r\right)_\kappa\left(\frac{1}{2}s\right)_\lambda k!\ l!}\frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1,\frac{1}{2}\Omega_2)C_\phi^{\kappa,\lambda}(U,(I-U))}{C_\phi(I)}(dU),$$

with $0 \leq U < I$.

Proof. Let $A = M_1 K M_1'$, $M_1 \in \mathcal{V}_{q,m}$; $K = \operatorname{diag}(k_1,\dots,k_q)$, $k_1 > \dots > k_q > 0$, then by independence, the joint density of A and B is (see Díaz-García *et al.* (1997) and Díaz-García and González-Farías (2005b)),

$$dF_{A,B}(A,B) = d|K|^{(r-m-1)/2}|B|^{(s-m-1)/2}\operatorname{etr}\left(-\tfrac{1}{2}(A+B)\right)$$

$$\times\ {}_0F_1\left(\tfrac{1}{2}r;\tfrac{1}{4}\Omega_1 A\right){}_0F_1\left(\tfrac{1}{2}s;\tfrac{1}{4}\Omega_1 B\right)(dA)(dB), \qquad (13)$$

where

$$d = \frac{\pi^{\left(-mr+r^2\right)/2}\operatorname{etr}\left(-\frac{1}{2}\left(\Omega_1+\Omega_2\right)\right)}{2^{m(r+s)/2}\Gamma_r[r/2]\Gamma_m[s/2]}. \qquad (14)$$

By performing the transforms $C = A + B$ with $(dA)\wedge(dB) = (dA)\wedge(dC)$ and then the transform $A = C^{1/2}U(C^{1/2})'$ with $U = H_1\Lambda H_1'$ where $H_1 \in \mathcal{V}_{q,m}$; $\Lambda = \operatorname{diag}(l_1,\dots,l_q)$, $l_1 > \cdots > l_q > 0$ and

$$(dA)\wedge(dC) = |K|^{(m+1-r)}|\Lambda|^{-(m+1-r)/2}|C|^{r/2}(dC)\wedge(dU),$$

see Díaz-García and Gutiérrez (1997), we find that the joint density of C and U is given by

$$dF_{C,U}(C,U) = d|\Lambda|^{(r-m-1)/2}|I-U|^{(s-m-1)/2}|C|^{(r+s-m-1)/2}\operatorname{etr}\left(-\tfrac{1}{2}C\right)$$

$$\times\ {}_0F_1\left(\tfrac{1}{2}r;\tfrac{1}{4}\Omega_1 C^{1/2}U(C^{1/2})'\right){}_0F_1\left(\tfrac{1}{2}s;\tfrac{1}{4}\Omega_1 C^{1/2}(I-U)(C^{1/2})'\right),$$

from which, by expanding the hypergeometric functions in infinite series of zonal polynomials and taking $Q(\cdot) = \frac{1}{2}I$, $R(\cdot) = U$ and $S(\cdot) = (I-U)$ from Theorem 1, we obtain the required result. □

Remark 3 *Note that, in fact, the density proposed in Theorem 2 is still a function of* r*, and not of* q*. The expression in terms of* q *is obtained by consolidating into a single expression the doubly noncentral nonsingular densities* ($q = m$) *and the singular density* ($q = r$)*, thus obtaining the expected result.*

Remark 4 *Under Definition 2, and proceeding as in Theorem 2, we find that the joint density of* A *and* B *is given by (13), and taking into account the change of variable* $C = A + B$ *with* $(dA) \wedge (dB) = (dA) \wedge (dC)$*, we then have:*

$$dF_{A,C}(A,C) = d|K|^{(r-m-1)/2}|C-A|^{(s-m-1)/2}\operatorname{etr}\left(-\tfrac{1}{2}C\right)$$

$$\times\ {}_0F_1\left(\tfrac{1}{2}r;\tfrac{1}{4}\Omega_1 A\right){}_0F_1\left(\tfrac{1}{2}s;\tfrac{1}{4}\Omega_1(C-A)\right)(dA)(dC).$$

The next step in establishing the density of U *under Definition 2 is to perform the transform* $U = (A^{1/2})'C^{-1}A^{1/2}$*. In the nonsingular case, the change of variable is carried out from* C *to* U*, but in the singular case this is not possible, because* $C > 0$ *but* $U \geq 0$*. Therefore, in the singular case, the change of variable must be from* $A \geq 0$ *to* $U \geq 0$*. However, the volume element* $(dA) \wedge (dC) =?(dU) \wedge (dC)$ *is not known.*

This distribution in the nonsingular case has been studied by Díaz-García and Gutiérrez-Jáimez (2006b), in which it is shown that the corresponding densities under Definitions 1 and 2 coincide. Assuming that this is also so in the singular case, this fact might be made use of in the search for the necessary Jacobian.

Corollary 5 *Let* $U \sim \mathcal{B}I_{1m}(q, s/2, r/2, \Omega_1, \Omega_2)$*, then the joint density function of the eigenvalues* $\Lambda = \operatorname{diag}(u_1,\dots,u_m)$*,* $1 > u_1 > \cdots > u_q > 0$ *of* U *is*

$$f(u_1,\dots,u_q) = \frac{\pi^{mq/2}\mathcal{B}I_m(\Lambda;q,r/2,s/2)\operatorname{etr}\left(-\frac{1}{2}(\Omega_1+\Omega_2)\right)\prod_{i=1}^{q}u_i^{m-q}}{\Gamma_q[m/2]}$$

$$\times\prod_{i<j}^{q}(u_i-u_j)\sum_{\kappa,\lambda;\,\phi}^{\infty}\frac{\left(\frac{1}{2}(r+s)\right)_\phi}{\left(\frac{1}{2}r\right)_\kappa\left(\frac{1}{2}s\right)_\lambda k!\ l!}\frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1,\frac{1}{2}\Omega_2)C_\phi^{\kappa,\lambda}(\Lambda,(I-\Lambda))}{C_\phi(I)}.$$

Proof. The proof follows immediately by applying the Lemma 2.1 in Díaz-García and González-Farías (2005a) to the beta type I density in Theorem 2 and making use of the fact that $C_\phi^{\kappa,\lambda}(AB,CD) = C_\phi^{\kappa,\lambda}(BA,DC)$, see Chikuse (1980) and Davis (1980). □

As particular cases, let us now examine the noncentral types A and B cases, together with the central case.

Corollary 6 *With respect to Theorem 2:*

i) *If $\Omega_1 = 0$, i.e. $A \sim \mathcal{PW}_m(r, I)$, then we obtain the noncentral* ***singular*** *matrix variate beta type I(A) distribution denoted as*

$$U \sim \mathcal{B}I_1(A)_m(q, r/2, s/2, \Omega_2),$$

and its ***symmetrised*** *density function is given by*

$$dF_s(U) = \mathcal{B}I_m(U; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) \times {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2, (I-U)\right)(dU)$$

ii) *alternatively, its* ***nonsymmetrised*** *density function is given by*

$$dF_U(U) = \mathcal{B}I_m(U; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) \times {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2(I-U)\right)(dU)$$

with $0 \leq U < I$, and where ${}_1F_1^{(m)}(\cdot)$ and ${}_1F_1(\cdot)$ are the hypergeometric function with two and one matrices arguments, respectively, see Muirhead (1982, definitions 7.3.1 and 7.3.2, pp. 258-259).

Proof.

i) Follows immediately from Theorem 2.
ii) Follows from result i), applying (12) inversely, and Theorem 7.3.3 in Muirhead (1982). □

Similarly:

Corollary 7 *If in Theorem 2:*

i) *$\Omega_2 = 0$, i.e. $B \sim \mathcal{W}_m(s, I)$, then we obtain the noncentral* ***singular*** *matrix variate beta type I(B) distribution denoted as $U \sim \mathcal{B}I_1(B)_m(q, r/2, s/2, \Omega_1)$, for which its* ***symmetrised*** *density function is*

$$dF_s(U) = \mathcal{B}I_m(U; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_1\right) {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_1, U\right)(dU)$$

ii) *and its* ***nonsymmetrised*** *density function is given by*

$$dF_U(U) = \mathcal{B}I_m(U; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_1\right) {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_1 U\right)(dU)$$

with $0 \leq U < I$.

Proof. The proof is analogous to that given in Corollary 6. □

Now if $\Omega_1 = \Omega_2 = 0$ we obtain the central **singular** matrix variate beta type I distribution for which the **symmetrised** and **nonsymmetrised** density function are the same, and are given by (2).

By an analogous procedure to that described in Corollaries 6 and 7, or from Corollary 5, we obtain expressions for the distributions of the nonnull eigenvalues of the matrix U in every case. Note that the distributions of the eigenvalues may be obtained from the symmetrised or nonsymmetrised distributions, see Greenacre (1973) and Roux (1975): thus, for example, by taking $\Omega_2 = 0$ in Corollary 5, we have the density of the eigenvalues when $U \sim \mathcal{B}I_1(B)_m(q, r/2, s/2, \Omega_1)$, thus obtaining

$$f(u_1, \dots, u_q) = \frac{\pi^{mq/2}\mathcal{B}I_m(\Lambda; q, r/2, s/2) \operatorname{etr}\left(-\frac{1}{2}\Omega_1\right) \prod_{i=1}^{q} u_i^{m-q}}{\Gamma_q[m/2]}$$

$$\times \prod_{i<j}^{q} (u_i - u_j)\, {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_1, \Lambda\right),$$

the expression for which in the nonsingular case was obtained by Constantine (1963), by taking from the above expression $q = m$.

Remark 8 *Intrinsically in Corollaries 6 and 7, the problem presented by Constantine (1963) and by Khatri (1970), and reconsidered in Farrell (1985, p. 191) and and Gupta and Nagar (2000, pp. 188-189), see also Díaz-García and Gutiérrez-Jáimez (2006a), is resolved for the singular case (and, naturally, for the nonsingular case, too). It is important to note that all the results in the nonsingular case are obtained as particular cases of those presented in the present paper, simply taking $q = m$.*

4 Doubly noncentral beta type II distribution

Theorem 9 *Suppose that $F > 0$ has a doubly noncentral singular matrix variate beta type II under the definition 1, denotes its as*

$$F \sim \mathcal{B}II_{1m}(q, r/2, s/2, \Omega_1, \Omega_2).$$

*Then using the notation for the operator sum as in Davis (1980) we have that its **symmetrised** density function is*

$$dG_s(F) = \mathcal{B}II_m(F; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}(\Omega_1 + \Omega_2)\right)$$

$$\times \sum_{\kappa,\lambda;\phi}^{\infty} \frac{\frac{1}{2}(r+s)_\phi}{\left(\frac{1}{2}r\right)_\kappa \left(\frac{1}{2}s\right)_\lambda k!\ l!} \frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2) C_\phi^{\kappa,\lambda}((I+F)^{-1}F, (I+F)^{-1})}{C_\phi(I)} (dF),$$

Proof. The joint density function of A and B is given by (13). Transforming $F = B^{-1/2}A(B^{-1/2})'$, we note that

$$(dA) \wedge (dB) = |K|^{(m+1-r)/2}|\Upsilon|^{-(m+1)/2}|B|^{r/2}(dF) \wedge (dB),$$

where $F = G_1\Upsilon G_1$, $G_1 \in \mathcal{V}_{r,m}$ and $\Upsilon = \operatorname{diag}(f_1, \dots f_r)$, $f_1 > \cdots > f_r > 0$. The joint density of B and F is

$$g_{F,B}(F,B) = c|\Upsilon|^{(r-m-1)/2}|B|^{(r+s-m-1)/2}\operatorname{etr}\left(-\tfrac{1}{2}B^{1/2}(I+F)(B^{1/2})'\right)$$

$$\times \ {}_0F_1\left(\tfrac{1}{2}r; \tfrac{1}{4}\Omega_1 B^{1/2}F(B^{1/2})'\right) {}_0F_1\left(\tfrac{1}{2}s; \tfrac{1}{4}\Omega_1 B\right),$$

from which by expanding the hypergeometric functions in infinite series of zonal polynomials and integrating with respect to B and taking $Q(\cdot) = \frac{1}{2}(I + F)$, $R(\cdot) = F$ and $S(\cdot) = I$ in Theorem 1, we obtain the required result. □

Under Definition 2, a situation analogous to that described in Remark 4 is obtained.

Corollary 10 *Let $F \sim \mathcal{B}II_{1m}(q, s/2, r/2, \Omega_1, \Omega_2)$, then the joint density function of the eigenvalues $\Upsilon = \operatorname{diag}(f_1, \dots, f_m)$, $f_1 > \cdots > f_m > 0$ of F is*

$$g(f_1, \dots, f_q) = \frac{\pi^{mq/2}\mathcal{B}II_m(\Upsilon; q, r/2, s/2)\operatorname{etr}\left(-\frac{1}{2}(\Omega_1+\Omega_2)\right)\displaystyle\prod_{i=1}^{q} f_i^{m-q}}{\Gamma_q[m/2]}$$

$$\times \prod_{i<j}^{q}(f_i - f_j)\sum_{\kappa,\lambda;\ \phi}^{\infty}\frac{\left(\frac{1}{2}(r+s)\right)_\phi}{\left(\frac{1}{2}r\right)_\kappa\left(\frac{1}{2}s\right)_\lambda}\frac{C_\phi^{\kappa,\lambda}(\frac{1}{2}\Omega_1, \frac{1}{2}\Omega_2)C_\phi^{\kappa,\lambda}(\Upsilon(I+\Upsilon)^{-1}, (I+\Upsilon)^{-1})}{k!\ l! C_\phi(I)}.$$

Proof. The proof follows immediately by applying the Lemma 2.1 in Díaz-García and González-Farías (2005a) to the beta type II density in Theorem 9, using the fact that $C_\phi^{\kappa,\lambda}(AB, CD) = C_\phi^{\kappa,\lambda}(BA, DC)$, see Chikuse (1980) and Davis (1980). □

We now obtain as particular cases the noncentral cases type A and B distributions, and the central case.

Corollary 11 *Under the conditions of Theorem 9:*

i) *if $\Omega_1 = 0$, i.e. $A \sim \mathcal{PW}_m(r, I)$, then we obtain the noncentral **singular** matrix variate beta type II(A) distribution denoted as*

$$F \sim \mathcal{B}II_1(A)_m(q, r/2, s/2, \Omega_2),$$

*the **symmetrised** density function of which is given by*

$$dG_s(F) = \mathcal{B}II_m(F; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) \times {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2, (I+F)^{-1}\right)(dF)$$

ii) *and its* ***nonsymmetrised*** *density function is*

$$dG_F(F) = \mathcal{B}II_m(F; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_2\right) \times {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_2(I+F)^{-1}\right)(dF)$$

with $0 \leq F$.

Proof.

i) Follows immediately from Theorem 2.
ii) Follows from Result i) by applying (12) in inverse fashion , and from Theorem 7.3.3 in Muirhead (1982). □

Similarly:

Corollary 12 *When in Theorem 2:*

i) $\Omega_2 = 0$*, i.e.* $B \sim \mathcal{W}_m(s, I)$*, then we obtain the noncentral* ***singular*** *matrix variate beta type II(B) distribution denoted as* $F \sim \mathcal{B}II_1(B)_m(q, r/2, s/2, \Omega_1)$*, for which its* ***symmetrised*** *density function is*

$$dG_s(F) = \mathcal{B}II_m(F; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_1\right) \times {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_1, (I+F)^{-1}F\right)(dF)$$

ii) *and its* ***nonsymmetrised*** *density function is given by*

$$dG_F(F) = \mathcal{B}II_m(F; q, r/2, s/2) \operatorname{etr}\left(-\tfrac{1}{2}\Omega_1\right) \times {}_1F_1\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_1(I+F)^{-1}F\right)(dF)$$

with $0 \leq F$.

Proof. The proof is analogous to that given in Corollary 11. □

Similarly to the beta type I case, if $\Omega_1 = \Omega_2 = 0$ we obtain the central singular matrix variate beta type II distribution for which the symmetrised or nonsymmetrised density function coincide, and which is given by (7). Moreover, from Corollaries 10 or 11 and 12, is possible to determine the distributions of the nonnull eigenvalues of matrix F in type A or B noncentral cases, or in central cases, and to obtain these distributions from the symmetrised or nonsymmetrised densities. In particular, if $\Omega_2 = 0$ in Corollary 10, then the density of the eigenvalues when $F \sim \mathcal{B}II_1(B)_m(q, r/2, s/2, \Omega_1)$ is given by

$$g(f_1,\dots,f_q) = \frac{\pi^{mq/2}\mathcal{B}II_m(\Upsilon; q, r/2, s/2)\operatorname{etr}\left(-\frac{1}{2}\Omega_1\right)\displaystyle\prod_{i=1}^{q} f_i^{m-q}}{\Gamma_q[m/2]}$$

$$\times \prod_{i<j}^{q}(f_i - f_j)\, {}_1F_1^{(m)}\left(\tfrac{1}{2}(r+s); \tfrac{1}{2}s; \tfrac{1}{2}\Omega_1, (I+\Upsilon)^{-1}\Upsilon\right).$$

This density was obtained by James (1964) in the nonsingular case ($q = m$) under Definition 2, see also Muirhead (1982, Theorems 10.4.2, p. 450).

5 Conclusions

In this study we determine the symmetrised densities of the *doubly noncentral singular* matrix variate beta type I and II distributions. From these, we find the corresponding joint densities of their eigenvalues. As particular results, we find the type A and B noncentral and central symmetrised densities and propose a means of obtaining the corresponding nonsymmetrised densities by applying in inverse fashion the definition of the symmetrised density proposed by Greenacre (1973), from which, implicitly, we resolve the integral proposed by Constantine (1963), Khatri (1970) and reconsidered in Farrell (1985, p. 191), see also Díaz-García and Gutiérrez-Jáimez (2006a), in the singular case, and in the nonsingular one, of course, by simply taking $q = m$.

Acknowledgment

This research work was partially supported by IDI-Spain, grant MTM2005-09209, and CONACYT-Mexico, Research Grant No. 45974-F. This paper was written during J. A. Díaz- García's stay as visiting professor at the Department of Statistics and O. R. of the University of Granada, Spain.

References

A. Cadet, Polar coordinates in $\mathbf{R}^{np}$; Application to the computation of the Wishart and beta laws, Sankhyā A 58 (1996) 101-113.

A. C. Constantine, Noncentral distribution problems in multivariate analysis, Ann. Math. Statist. 34 (1963) 1270–1285.

Y. Chikuse, Invariant polynomials with matrix arguments and their applications, in: R. P. Gupta, (ed.) Multivariate Statistical Analysis, North-Holland Publishing Company, pp. 53-68, 1980.

A. W. Davis, Invariant polynomials with two matrix arguments, extending the zonal polynomials, in: P. R. Krishnaiah (ed.) Multivariate Analysis V, North-Holland Publishing Company, pp. 287-299, 1980.

J. A. Díaz-García, R. Gutiérrez-Jáimez, and K. V. Mardia, Wishart and Pseudo-Wishart distributions and some applications to shape theory, J. Multivariate Anal. 63 (1997) 73-87.

J. A. Díaz-García, and J. R. Gutiérrez, Proof of the conjectures of H. Uhlig on the singular multivariate beta and the jacobian of a certain matrix transformation, Ann. Statist. 25 (1997) 2018-2023.

J. A. Díaz-García, and G. González-Farías, QR, SV and Polar Decomposition and the Elliptically Contoured Distributions, Technical Report No. I-99-22 (PE/CIMAT), Guanajuato, México, (1999), http://www.cimat.mx/biblioteca/RepTec.

J. A. Díaz-García, and G. González-Farías, Singular Random Matrix decompositions: Jacobians, J. Multivariate Anal. 93(2) (2005a) 196-212.

J. A. Díaz-García, and G. González-Farías, Singular Random Matrix decompositions: Distributions, J. Multivariate Anal. 94(1)(2005b) 109-122.

J. A. Díaz-García, and R. Gutiérrez-Jáimez, Distribution of the generalised inverse of a random matrix and its applications, J. Stat. Plann. Infer. 136(1) (2006) 183-192..

J. A. Díaz-García, R. Gutiérrez-Jáimez, The expected value of zonal polynomials, Test 10 (2001) 133-145.

J. A. Díaz-García, R. Gutiérrez-Jáimez, Noncentral matrix variate beta distribution, Comunicacin Técnica, No. I-06-06 (PE/CIMAT), Guanajuato, México, 2006, http://www.cimat.mx/biblioteca/RepTec/index.html?m=2.

J. A. Díaz-García, R. Gutiérrez-Jáimez, Doubly noncentral matrix variate beta distribution. Comunicacin Técnica, No. I-06-08 (PE/CIMAT), Guanajuato, México, (2006), Guanajuato, Mxico, http://www.cimat.mx/biblioteca/RepTec/index.html?m=2.

I. Dimitriu, A. Edelman, G. Shuman, MOPSJ: Multivariate orthogonal polynomials (simbolically). Mathworld, URL (2005), http://mathworld.wolfram.com/.

J. Demmel, P. Koev, Accuarate and efficient evaluation of Schur and Jack functions. Math. Comp. (2004). To appear.

R. H. Farrell, Multivariate Calculation: Use of the Continuous Groups, Springer Series in Statistics, Springer-Verlag, New York, 1985.

M. J. Greenacre, Symmetrized multivariate distributions, S. Afr. Statist. J. 7 (1973) 95-101.

C. R. Goodall, K. V. Mardia, Multivariate aspects of Shape Theory, Ann. Statist. 21 (1993) 848-866.

A. K. Gupta, D. K. Nagar, Matrix variate distributions, Chapman & Hall/CR, New York, 2000.

R. Gutiérrez, J. Rodríguez, A. J. Sáez, Approximation of hypergeometric functions with matricial argument throught development in series of zonal polynomials, Electron. Trans. Numer. Anal. 11 (2000) 121-130 (electronic).

MR2002b:33004.
A. T. James, Distributions of matrix variates and latent roots derived from normal samples Ann. Math. Statist. 35 (1964) 475-501.
C. G. Khatri, Some results for the singular normal multivariate regression nodels, Sankhyā A, 30 (1968) 267-280.
C. G. Khatri, A note on Mitra's paper "A density free approach to the matrix variate beta distribution", Sankhyā A 32 (1970) 311-318.
P. Koev, (2004), http://www.math.mit.edu/˜plamen.
P. Koev, J. Demmel, The efficient evaluation of the hypergeometric function of matrix argument, Math. Comp. (2004). Submitted.
R. J. Muirhead, Aspects of Multivariate Statistical Theory, John Wiley & Sons, New York, 1982.
I. Olkin, H. Rubin, Multivariate beta distributios and independence properties of Wishart distribution. Ann. Math. Statist. 35, 261-269. Correction 1966, 37(1) (1964) 297.
A. J. Sáez, Softwere for calculus of zonal polynomials (2004), http://estio.ujaen.es/Profesores/ajsaez/sofwere.html.
S. M. Srivastava, C. G. Khatri, An Introduction to Multivariate Statistics, North Holland, New York, 1979.
S. M. Srivastava, On the distribution of a multiple correlacion matrix: Non-central multivariate beta distributions, Ann. Math. Statist. 39 (1968) 227-232.
T. Ratnarajah, R. Vaillancourt, Complex singular Wishart matrices and applications Computers & Mathematics with Appliocations 50(3-4) (2005) 399-411.
T. Ratnarajah, R. Vaillancourt, Quadratic forms on complex random matrices and multiple-antenna systems. IEEE Trans. on Information Theory 15(8) (2005) 2979-2984.
J. J. J. Roux, New families of multivariate distributions, in: G. P. Patil, S. Kotz, J. K. Ord, (eds.) A Modern course on Statistical distributions in scienctific work, Volume I, Model and structures, D. Reidel, Dordrecht-Holland, 281-297, 1975.
H. Uhlig, On singular Wishart and singular multivartiate beta distributions, Ann. Statistic. 22 (1994) 395-405.
S. S. Wilks, Certain generalizations in the analysis of variance, Biometrika 24 (1932) 471-494.