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ABSTRACT. We compute the Chow motive of certain subvarieties of the Flags manifold and show that it is an Artin motiv.

INTRODUCTION

It is well known (see [6]) that if $\mathbf{G}$ is a connected algebraic semisimple group defined over an algebraiclly closed field $\mathbb{K}$, with universal separable covering, U the variety of unipotent elements of $\mathbf{G}$, B a Borel subgroup of $\mathbf{G}$ and

$$Y := \{(x, gB) \in U \times \mathbf{B} | g^{-1}xg \in B\}$$

then

$$\pi : Y \longrightarrow U$$

is a desingularization, where π is the natural projection and $\mathbf{B} = \mathbf{G}/B$.

If $\mathbf{G} = SL_n$, then $\mathbf{B} = \mathbb{F}$, the variety of complete flags, and the fiber $\pi^{-1}(x)$ is isomorphic to the variety of fixed flags under the unipotent element x. Moreover, in that case J.A. Vargas (see [7]) has given a description of a dense open set for every irreducible component of the fiber and N. Spaltenstein (see [4]) has constructed a stratification of the fiber (which unfortunately is not completely compatible with the decomposition on irreducible components).

The purpose of this work is to describe the motive of the irreducible components of the fibers $\pi^{-1}(x)$ when x is of type (p, q). Since several algebraic singularities already appear within the Unipotent variety, it is interesting to know the geometry and K-theory of the fibers of its desingularization. It is also important for applications such as computing zeta functions and counting points over finite fields.

The paper is divided as follows: In section 1 we introduce some notation as well as Spaltenstein stratification mentioned above. In section 2 we give a description of the irreducible components of the fiber of x when x is a unipotent element of type (p, q). In section 3 we use the above description to compute the Chow motive and show that the image of the Chow motive of the irreducible components in Voevodski's category is an Artin motive. We also compute the Motive of some irreducible components of a slightly more general type, showing that these motives are extension by Artin motives of the motive of a product of Flag varieties.

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1. Preliminaries

Let $\mathbf{G}$ be the group SL_n with coefficients in an algebraically closed field $\mathbb{K}$, consider a Borel subgroup B of $\mathbf{G}$ and T a maximal torus on B. If V is a $\mathbb{K}$-vector space of dimension n, then the variety $\mathbf{B}$ is isomorphic to the variety of complete flags $\mathbb{F} = \mathbb{F}(V)$. For any unipotent element $x \in U \subset \mathbf{G}$, $\mathbb{F}_x$ will denote the fiber of π. One says that x is of type $(\lambda_1, \cdots, \lambda_s)$ if the Jordan canonical form of x consist exactly of s blocks of sizes $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s$.
If we write $x = 1 + n$ where n is the nilpotent part of x, then there is a basis

$$\{e_{i,j} | 1 \leq j \leq s \, , 1 \leq i \leq \lambda_j\}$$

of V adapted to x in the following sense: $n(e_{i,j}) = e_{i-1,j}$, where $e_{0,j} = 0$ for every j. Therefore we can write $V = V_1 \oplus \cdots \oplus V_{\lambda_1}$ and $n : V_{\lambda_1} \longrightarrow V_{\lambda_1 - 1} \longrightarrow \cdots V_1 \longrightarrow 0$, where V_i is the space generated by $\{e_{i,j}\}$ with i fix.
It is well known that if $x \in U \subset SL_n$ is unipotent of type $(\lambda_1, \cdots, \lambda_s)$, then the fiber $\mathbb{F}_x$ has as many irreducible components as there are standard tableaux of type $(\lambda_1, \cdots, \lambda_s)$. The shape of a standard tableau is as follows:

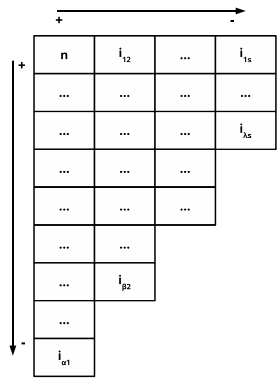


Figure 1. Standard tableau of type $(\alpha, \beta, \cdots, \lambda)$.

where the numeration strictly decrease from the top to the bottom and from left to right.
Spaltenstein constructed a stratification of $\mathbb{F}_x$, which we will explain in the particular case of x of type (p, q).
Given $x \in U \subset SL_n$ of type (p, q) and a $\mathbb{K}$ vector space V of dimension n, consider a basis $\{e_{i,j} | 1 \leq j \leq 2 \, , 1 \leq i \leq \lambda_j\}$ of V adapted to x as above. Given a number $t = 1$ or 2 we construct a subset of $\mathbb{F}_x$ as follows:
For $t = 1$ let $X_1 = X_1(V)$ be the set of flags $F_1 \subset \cdots \subset F_n \in \mathbb{F}$ such that $F_1 :=< e_{1,1} >$.
For $t = 2$ let $X_2 = X_2(V)$ be the set of flags $F_1 \subset \cdots \subset F_n \in \mathbb{F}$ such that $F_1 :=< ae_{1,1} + e_{1,2} >$ for some number $a \in \mathbb{K}$ not necessarily different from zero.
One can define inductively the sets

$$X_{i,j}(V) := \{F_1 \subset \cdots \subset F_n \in X_j(V) \mid F_2/F_1 \subset \cdots \subset F_n/F_1 \in X_i(V/F_1)\},$$

etcetera. It is not difficult to see that if you continue this process n times, you will end up with a locally closed subset of flags which actually belongs to $\mathbb{F}_x$, since a flag $F := F_1 \subset \cdots \subset F_n$ is in $\mathbb{F}_x$ if and only if $n(F_i) \subset F_{i-1}$ for every i. Moreover the sets you get form a stratification of $\mathbb{F}_x$.

It is clear from the construction that all spaces of the stratification are affine spaces and that the affine strata of maximal dimension (which is precisely q if x is of type (p, q)) are open disjoint sets. You can also count the number of irreducible components of $\mathbb{F}_x$ (which coincides with the number of standard tableaux of the given type).

Remark 1.1. *The natural projection from $X_1(V)$ to $\mathbb{F}(V/F_1)$ is an isomorphism.*

Remark 1.2. *Given a nilpotent element $x \in U$ of type $(\lambda_1, \cdots, \lambda_s)$, a standard tableau σ as above, a $\mathbb{K}$-vector space V of dimension $n = \lambda_1 + \cdots \lambda_s$ and a basis $\{e_{i_1,\cdots,i_s}\}$ of V adapted to x, there is a maximal affine space among those obtained from Spaltenstein's stratification of $\mathbb{F}_x$, which can be associated to σ: Let $\psi : \{1, \cdots, n\} \longrightarrow \{1, \cdots, s\}$ be the function given by $\psi(k) = i$ if k appears in the $i-$th column of σ, counting from left to right, and consider the stratum $X_\sigma := X_{\psi(n)\cdots\psi(1)}$. X_σ is of maximal dimension and the corresponding irreducible component will be denoted by Y_σ.*

2. Decomposable irreducible components

Let $A := \{a_1, \cdots, a_r\}$ be a totally ordered set and we assume that $a_1 < \cdots < a_r$. Let $\phi : A \longrightarrow [1, \cdots, s]$ be a surjective map. Then the pair (A, ϕ) induces a tableau $\sigma := \sigma(A, \phi)$ as follows:

The tableau σ has s columns, the i-th column of the tableau has $b_i := |\phi^{-1}(i)|$ boxes and one fill them up according to ϕ, i.e. starting with r down to 1 the number k should appear in the column $\phi(k)$. The numbers inside a column will decrease to the bottom and the tableau will be of type $(b_1, \cdots, b_s)$.
Not every tableau obtained this way is a Young tableau, unless $b_1 \geq \cdots \geq b_s$, and even then it need not be standard. For instance the tableau associated to $\phi(1) = \phi(2) = 1$, $\phi(3) = \phi(4) = 2$ and $\phi(5) = 3$ will be a Young tableau but will not be standard, whereas the tableau associated to $\phi(1) = \phi(4) = 2$, $\phi(2) = 3$ and $\phi(5) = \phi(3) = 1$ will be a standard young tableau.
If ϕ is a decreasing bijection and $|A| > 1$ we will say that the pair (A, ϕ) is of flag type.

Remark 2.1. *Any standard Young tableau of type $(\lambda_1 \cdots \lambda_s)$ is obtained this way from the pair $([1, \cdots, n], \psi)$, where $\psi : [1, \ldots, n] \longrightarrow [1, \cdots, s]$ is the function described in 1.2.*

Definition 2.2. *A standard Young tableau σ of type $(\lambda_1 \cdots \lambda_s)$, with $n = \lambda_1 + \cdots + \lambda_s$, is called* decomposable *if there is a partition $[1, \cdots, n] = (\sqcup_k A_k) \bigsqcup (\sqcup_t B_t)$ by totally ordered sets such that the pair $(A_k, \psi|_{A_k})$ is of flag type for all k (i.e. if $|A_k| = m_k$ then $\psi|_{A_k}$ is a decreasing bijection between A_k and $[1, \cdots, m_k]$), and the pair $(B_t, \psi|_{B_t})$ induces a standard Young tableau of type (p_t, q_t) for every t, where ψ is as above. In particular $Im\ \psi|_{B_t} = \{1, 2\}$.*

Remark 2.3. *A pair (A, ϕ) is of flag type if and only if the associated standard tableau σ is of type $(1, \cdots, 1)$, which corresponds to the identity in SL_{n+1}, in which case there is only one irreducible component, namely the whole Flag variety.*

3. Irreducible components of $\mathbb{F}_x$ for x of type (p,q)

The description of irreducible components of type (p,q) given here appears already in [1]. We decided to included it for the shake of completeness, thought the present proof is a bit better that the one in [1].
Let $x = 1 + n$ be a unipotent element of type (p,q) in SL_n with nilpotent part n, V be a $\mathbb{K}$-vector space of dimension $n = p + q$ and $\mathbb{F} = \mathbb{F}(V)$ be the variety of complete flags on V.

Lemma 3.1. *The set*

$$A = \{F := F_1 \subset \cdots \subset F_n \in \mathbb{F} \mid n^i(F_k) \subset F_m\}$$

is closed in $\mathbb{F}$ for all $i, k, m \in \mathbb{N}$ fixed. Here $F_0 = 0$.

Lemma 3.2. *The set*

$$L = \{F := F_1 \subset \cdots \subset F_n \in \mathbb{F} \mid n^i(F_k) \subset S\}$$

is closed in $\mathbb{F}$ for all $i, k \in \mathbb{N}$ fixed. Here S is some fixed subspace of V.

Lemma 3.3. *The set*

$$H = \{F := F_1 \subset \cdots \subset F_n \in \mathbb{F} \mid dim(F_r + n^k(F_r)) \leq d\}$$

is closed in $\mathbb{F}$ for all $r, k, d \in \mathbb{N}$ fixed.

For the three lemmas above one shows that the corresponding set (either A, L or H) is a determinantal set and therefore algebraic.

Let $\pi_r : \mathbb{F} \longrightarrow \mathbb{G}r(1,n) \times \cdots \mathbb{G}r(r,n)$ be the composition of the natural embedding of $\mathbb{F}$ in $\mathbb{G}r(1,n) \times \cdots \mathbb{G}r(n-1,n)$ followed by the projection to the first r factors.

Theorem 3.4. *Let σ be a standard tableau of type (p,q), Y_σ the corresponding irreducible component of $\mathbb{F}_x$, $1 \leq r \leq n$ a natural number and $Y_\sigma(r) := \pi_r(Y_\sigma)$. Let $Y_\sigma(0)$ be a point. If we denote by f_r the natural projection $Y_\sigma(r) \longrightarrow Y_\sigma(r-1)$, then for all $p \in Y_\sigma(r-1)$ one has*

$$f_r^{-1}(p) = \begin{cases} 1pt & \text{if } r \text{ appears in the left column of } \sigma, \\ \mathbb{P}^1 & \text{if } r \text{ appears in the right column of } \sigma. \end{cases}$$

In particular $Y_\sigma(r) \longrightarrow Y_\sigma(r-1)$ is either a $\mathbb{P}^1$-bundle or an isomorphism for every $1 \leq r \leq n$.

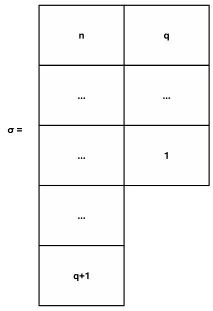
Proof. We will only prove the theorem in the case where $\mathbb{K} = \mathbb{C}$, to avoid some technical difficulties which do not really bring more light into the geometric problem.

First of all we need a description of the corresponding irreducible component, since so far we only know an open subset of it, namely X_σ. Observe that if you restrict yourself to that open set and consider its image $X_\sigma(r) := \pi_r(X_\sigma)$ under π_r, then the fibers of f_r restricted to $X_\sigma(r)$ are isomorphic to a point or to the affine line, depending on whether r appears in the left or in the right column of σ.

I) Fix a basis $\{e_{i,j}\}$ of V adapted to x as before and let σ be a standard Young tableau of type (p,q). The map $\psi : [1, \cdots, n] \to \{1,2\}$ defined in 1.2. induces a partition $[1, \cdots, n] = (\sqcup_{k=1}^t A_k) \bigsqcup (\sqcup_{k=1}^t B_k)$ such that $A_k \subset \psi^{-1}(1)$ and $B_k \subset \psi^{-1}(2)$ are made up by consecutive integers for all k. Being σ a standard tableau, it is clear that either

$A_1 < B_1 < A_2 < B_2 < \cdots < A_t$ and $B_t = \emptyset$ or $B_1 < A_1 < B_2 < A_t < \cdots < B_t < A_t$, where $C < D$ means $c < d$ for all $c \in C$ and all $d \in D$.
Without lose of generality we can assume $B_1 < A_1 < \cdots < B_t < A_t$.
Indeed, if $A_1 < B_1$ then $A_1 = \{1, \cdots, a_1\}$. Observe that in this case $p - a_1 > q$ since σ is standard and the numeration inside it decreases to the right, therefore for every flag $F_1 \subset \cdots \subset F_n \in X_\sigma$ one has $F_i = \text{im } n^{p-i}$ for $1 \leq i \leq a_1$. But these are closed conditions and so they should be fullfilled by all flags in Y_σ as well. In this situation 1.1 implies that $Y_\sigma \cong Y_{\sigma'}(V/(\text{im } n^{p-a_1}))$; where σ' is the standard young tableau induced by $([a_1+1, \cdots, n], \psi|_{[a_1+1,\cdots,n]})$.

Let us define $a_0 = b_0 = 0$ and for $1 \leq i \leq t$ let $a_i := |A_i|$, $b_i := |B_i|$, $s_j := \sum_{i<j} a_i$ and $S_j := \sum_{i<j} b_i$. In order to simplify notation we will also define $F_0 := 0$.
II) Consider first the case



Let W be a hermitian vector space of dimension 2 with basis $\{w_1, w_2\}$. For every point $(P_1 \cdots P_q) \in \mathbb{P}(W)^q$ write $P_i = (a_i : b_i)$ and let $R_i = (c_i : d_i)$ be the point of $\mathbb{P}(W)$ that represents the orthogonal complement of the subspace $< a_i w_1 + b_i w_2 > \subset W$.
For all $1 \leq j \leq q$ consider the vectors (in V)

$$\begin{aligned} P_{1,j} &= a_1 e_{j,1} + b_1 e_{j,2} \\ R_{1,j} &= c_1 e_{j,1} + d_1 e_{j,2} \end{aligned}$$

and for all i, j with $i > 1$ and $i + j \leq q + 1$ construct the vectors

$$\begin{aligned} P_{i,j} &= a_i R_{i-1,j} + b_i P_{i-1,j+1} \\ R_{i,j} &= c_i R_{i-1,j} + d_i P_{i-1,j+1} \end{aligned}$$

With the above notation define the sets Y_k fot $1 \leq k \leq q$ by the conditions:
$F_1 \subset \cdots \subset F_n \in Y_k$ if and only if there exist $(P_1, \cdots, P_q) \in \mathbb{P}(W)^k \times (\mathbb{P}(W) - \{(1:0)\})^{q-k}$ such that
a) $F_m =< P_{1,1}, \cdots, P_{m,1} >$ for all $1 \leq m \leq q$,
b) $F_{q+t} =< P_{1,1}, \cdots, P_{q,1}, R_{q,1}, \cdots, R_{q-t+1,t} >$ for all $1 \leq t \leq q$
c) $F_s = \text{Ker } n^s$ for all $2q \leq t \leq n$

Observe that if the P_i's are different from $(1:0)$ for all i, then

$$< P_{1,1}, \cdots, P_{q,1}, R_{q,1}, \cdots, R_{q-t+1,t} >=< P_{1,1}, \cdots, P_{q,1}, e_{1,1}, \cdots, e_{1,t} >$$

for all t, therefore $Y_0 = X_\sigma$. It can also be described as follows:
$F_1 \subset \cdots \subset F_n \in X_\sigma$ if and only if

a) $F_1 \subset \text{Ker } n - < e_{1,1} >$,
b) $n(F_i) \subset F_{i-1}$ but $n(F_i) \not\subset F_{i-2}$ for every $2 \leq i \leq q$,
c) ker $n^t \subset F_{q+t}$ for every $1 \leq t \leq p$.

Similarly the sets Y_k for $1 \leq k < q$ satisfy $F_1 \subset \cdots \subset F_n \in Y_k$ if and only if:
a) $F_m \subset \text{Ker } n^m$ for all $1 \leq m \leq k$,
b) $n(F_i) \subset F_{i-1}$ but $n(F_i) \not\subset F_{i-2}$ for every $k+1 \leq i \leq q$,
c) ker $n^t \subset F_{q+t}$ for every $1 \leq t \leq p$.

Finally the set Y_q satisfies $F_1 \subset \cdots \subset F_n \in Y_k$ if and only if:
a) $F_m \subset \text{Ker } n^m$ for all $1 \leq m \leq k$,
b) $n(F_i) \subset F_{i-1}$,
c) ker $n^t \subset F_{q+t}$ for every $1 \leq t \leq p$.

Now Y_q contains $X_\sigma = Y_0$ and is irreducible by construction, moreover it is actually a closed set because of 3.1, 3.2 and 3.3; therefore $Y_q = Y_\sigma$.
The natural projection $f_k : Y_\sigma(k) \longrightarrow Y_\sigma(k-1)$ is nothing more than the map

$$F_0 \subset \cdots \subset F_{k-1} \subset F_k \mapsto F_0 \subset \cdots \subset F_{k-1}$$

and, because of the construction of Y_q, the fiber of this map is $\mathbb{P}^1$ for all $1 \leq k \leq q$. On the other hand

$$(3.1) < P_{1,1}, \cdots, P_{q,1}, R_{q,1}, \cdots, R_{q-k+1,k} > = < P_{1,k+1}, \cdots, P_{q-k,k+1} > \oplus \text{ ker } n^k$$

for all $1 \leq k \leq q$, therefore f_{q+k} is an isomorphism for $1 \leq k \leq q$ since the space F_{q+k} is already determined by the space F_{q+k-1}. Finally $F_{2q+t} = \text{ker } n^{q+t}$ for all $t \geq 0$ and the theorem follows in this case.

III) Assume that $s_j < S_j$ for all $1 \leq j < t$.

With the notation as in (I), we define Y_k as follows: $F_1 \subset \cdots \subset F_n = V \in Y_k$ if and only if there exist $(P_1, \cdots, P_q) \in \mathbb{P}(W)^k \times (\mathbb{P}(W) - \{(1:0)\})^{q-k}$ such that
a) $F_{S_j+m} = \text{ker } n^{s_j} \oplus < P_{1,s_j+1}, \cdots, P_{S_j-s_j+m,s_j+1} >$ for all $1 \leq m \leq b_j$, for all $1 \leq j \leq t$.
b) $F_{S_{j+1}+s_j+m} = \text{ker } n^{s_j} \oplus < P_{1,s_j+1}, \cdots, P_{S_{j+1}-s_j,s_j+1}, R_{S_{j+1}-s_j,s_j+1}, \cdots, R_{S_{j+1}-s_j-m+1,s_j+m} >$ for all $1 \leq m \leq a_j$, for all $1 \leq j \leq t$.
c) $F_s = \text{Ker } n^s$ for all $2q \leq t \leq n$

One sees inmediatly that $X_\sigma = Y_0 \subset \cdots \subset Y_q$, Y_q is irreducible by construction and similarly as in (I) it can be described by the conditions:

$F_1 \subset \cdots \subset F_n \in Y_k$ if and only if:
a) ker $n^{s_j} \subset F_{S_j+m} \subset \text{Ker } n^{S_j+m}$ for all $1 \leq m \leq b_j$, for all $1 \leq j \leq t$,
b) ker $n^{s_j+m} \subset F_{S_{j+1}+s_j+m} \subset \text{ker } n^{S_{j+1}+m}$,
c) ker $n^m = F_m$ for all $2q \leq m \leq p$.
where condition b) is a consequence of the following equality:

$$(3.2) \quad \begin{aligned} \text{ker } n^s \quad &\oplus < P_{1,s+1}, \cdots, P_{m,s+1}, R_{m,s+1}, \cdots, R_{m-t+1,s+t} > \\ &= \text{ ker } n^{s+t} \oplus < P_{1,s+t+1}, \cdots, P_{m-t,s+t+1} > . \end{aligned}$$

where $1 \leq t \leq m$ and $s+t+1 \leq q$.
As in (II), it follows from the construction of Y_q that the fiber of f_k is isomorphic to $\mathbb{P}^1$ if $k \in B_s$ for some s and either from condition c) or from equation 3.2 above, one gets that f_k will be an isomorphism if $k \in A_s$ for some s.

IV) With the notation as in (III), consider a standard tableau σ such that the condition $s_j < S_j$ for all $1 \leq j \leq t$ is not satisfied. Then there exist a $j < t$ such that $s_j \geq S_j$. Let j_0 be the smallest index such that $s_j \geq S_j$. Since the A_i's and the B_i's are made up by consecutive numbers and for every index $k > m$ the number appearing in B_k and A_k are bigger than those appearing in A_m or B_m then by induction one shows that $max\ \{A_m\} = s_{m+1} + S_{m+1}$ and $max\ \{B_m\} = s_m + S_{m+1}$ for every m, in particular $max\ \{A_{j_0}\} \geq 2S_{j_0}$. Moreover since j_0 was minimal then $max\ \{B_{j_0}\} < 2S_{j_0}$, i.e. $2S_{j_0} \in A_{j_0}$. Then for every flag $0 = F_0 \subset \cdots \subset F_n \in X_\sigma$ one has $F_{2S_{j_0}} = \ker\ n^{S_{j_0}}$. Since this is a closed condition it should also be true for all flags in the irreducible component Y_σ and therefore $Y_\sigma \cong Y_{\sigma'}(\ker\ n^{S_{j_0}}) \times Y_{\sigma''}(V/\ker\ n^{S_{j_0}})$, where σ' is the standard tableau induced by the pair $([1, \cdots, 2S_{j_0}]; \psi|_{[1,\cdots,2S_{j_0}]})$ and σ'' is the standard tableau induced by the pair $([2S_{j_0}+1, \cdots, n]; \psi|_{[2S_{j_0}+1,\cdots,n]})$, both of them of type (a,b) for some $a \geq b$. The proposition follows by induction on the dimension.

Q.E.D.

Remark 3.5. *The hipotesis $\mathbb{K} = \mathbb{C}$ was only used in order to have a natural choise of the points R_i's.*

4. The motive of the irreducible components

If X is a scheme and G a group, the G-torsors on X for the étale cohomology are parametrized by $H^1_{et}(X, G)$. The exact sequence

$$0 \longrightarrow \mathbb{G}_m \longrightarrow GL_2 \longrightarrow PGL_2 \longrightarrow 0$$

gives us a connection map

$$\delta :\ H^1_{et}(X, PGL_2) \longrightarrow H^2_{et}(X, \mathbb{G}_m) = Br(X).$$

Moreover, $z \in\ \ker \delta \Leftrightarrow z$ can be extended to a GL_2 torsor on X, i.e. if z can be extended to a vector bundle on X for the étale topology. Therefore, in order to show that a torsor corresponding to an element $z \in H^1_{et}(X, PGL_2)$ is the projective bundle associated to a rank 2 vector bundle on X, it is enough to show that its image in $Br(X)$ is zero. In this chapter we will deal with varieties over an algebraically closed field $\mathbb{K}$ of characteristic zero and therefore the Brauer group of X coincide with the "geometric" Brauer group in this case.

Theorem 4.1. *Let σ be a standard tableau of type (p,q) and Y_σ be the corresponding irreducible component of $\mathbb{F}_x$. Then the motive $h(Y_\sigma)$ is isomorphic to $(1+L)^q$. In particular it is an Artin motive.*

Proof. Since the open cell $X_{\psi(1),\cdots,\psi(n)} \subset Y_\sigma$ is an affine space, it follows from the proof of 3.4 that all the varieties $Y_\sigma(r)$ are rational and so $Br(Y_\sigma(r)) = 0$ for all r, therefore $Y_\sigma(r+1) \longrightarrow Y_\sigma(r)$ is either an isomorphism or the projective bundle associated to a rank 2 vector bundle on $Y_\sigma(r)$, in which case $h(Y_\sigma(r+1)) \cong (1+L) \otimes h(Y_\sigma(r))$, see [2]. Since dim $Y_\sigma = q$ the conclusion follows by induction.

Q.E.D.

Remark 4.2. *Keeping the notation of 1.2, if Y_σ is an irreducible component for which* $\psi(t) = \begin{cases} 2 & \text{fot } t \text{ odd,} \\ \\ 1 & \text{for } t \text{ even,} \end{cases}$ *then $Y_\sigma \cong (\mathbb{P}^1)^q$ and the multiplicative structure of the corresponding motive is clear. In general one needs to find sections of the maps $Y_\sigma(r+1) \to Y_\sigma(r)$ that correspond to the bundle $\mathcal{O}_{Y_\sigma(r+1)}(1)$ and compute their autointersection numbers to explicitly find a normalized rank 2 vector bunlde which induces the $\mathbb{P}^1$- bundle over $Y_\sigma(r)$and therefore being able to actually compute the multiplicative structure of the motive (see* [2]*).*

Corollary 4.3. *If σ is a decomposable standard tableau then its motive is a Tate motive.*

PROOF. If σ is decomposable then the irreducible component Y_σ is isomorphic to a product of towers of $\mathbb{P}^1$-bundles over flag varieties. Since flag varieties are rational then all the Brauer groups involved are zero. Moreover since the flag varieties are themselves towers of projective bundles associated to vector bundles, the motive we get is a Tate motiv. Q.E.D.

REFERENCES

[1] del Angel R, P.L.: *The Geometry of* $\mathbb{F}_U$. International Seminar on Algebra and its Applications, Aportaciones Mat. Notas Investigación, 6, Soc. Mat. Mexicana, Mxico (1992), 91–107.

[2] Manin, Ju. I.: *Correspondences, motifs and monoidal transformations.*(Russian) Mat. Sb. (N.S.) **77** 119 (1968) 475–507.

[3] Milne, J.S.: *Étale cohomology.* Princeton Mathematical Series, 33. Princeton University Press, Princeton, N.J., 1980. xiii+323 pp.

[4] Spaltenstein, N.: *The Fixed Point Set of a Unipotent Transformation on the Flag Manifold*, Indag. Math. **38** (1976), 452-456.

[5] Steinberg, R.: *Conjugacy Classes in Algebraic Groups*, LNM **366**, Springer-Verlag, Berlin, Heidelberg, New York (1974).

[6] Steinberg, R.: *On the Desingularization of the Unipotent Variety*, Inv. Math. **36** (1976), 209-224.

[7] Vargas, J.A.: *Fixed Points Under the Action of Unipotent Elements of* SL_n, Bol. Soc. Mat. Mex. (1) **24** (1979), 1-14.

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