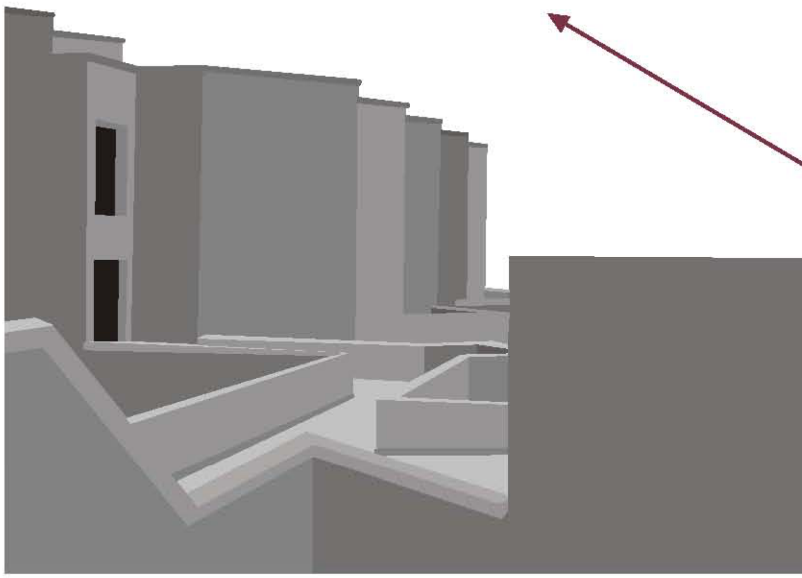

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DOUBLE SINGULAR RANDOM MATRIX VARIATE DISTRIBUTION: T DISTRIBUTION

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Abstract

A sort of singularities for the matric and matrix variate t distributions are studied and their corresponding densities are derived. Finally, an application is given in the context of sensitivity analysis.

Key words: Matrix-variate t distribution, singular distribution, matricvariate t distribution, externally studentised residual.
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1 Introduction

The multivariate t distribution has been studied by several authors. An excellent compendium of that topic can be found in Kotz and Nadarajah (2004).

Recall that the r-dimensional t distribution with n degrees of freedom , mean vector $\mu \in \Re^r$, and positive definite correlation matrix $\mathbf{R} \in \Re^{r\times r}$ has the following two representations, see Kotz and Nadarajah (2004, pp. 2-7),

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- $$\mathbf{t} = S^{-1}\mathbf{Y} + \mu; \tag{1}$$
  where $\mathbf{Y} \sim \mathcal{N}_r(\mathbf{0}, \mathbf{\Sigma})$, i.e. $\mathbf{Y}$ is a *non-singular* r-dimensional normal random vector with mean $\mathbf{0}$ and positive definite covariance matrix $\mathbf{\Sigma}$, $\mathbf{\Sigma} > \mathbf{0}$; and $nS/\sigma^2 \sim \chi^2(n)$, independent of $\mathbf{Y}$, it is the chi-squared random variable with n degrees of freedom.
- $$\mathbf{t} = \left(\mathbf{V}^{1/2}\right)^{-1}\mathbf{Y} + \mu; \tag{2}$$
  where $\mathbf{Y} \sim \mathcal{N}_r(\mathbf{0}, n\mathbf{I}_r)$ is independent of $\mathbf{V}$, $\mathbf{I}_r$ is the r-dimensional identity matrix, $\mathbf{V}^{1/2}$ is the symmetric square root of $\mathbf{V}$, i.e.
  $$\mathbf{V}^{1/2}\mathbf{V}^{1/2} = \mathbf{V} \sim \mathcal{W}_m\left(n + r - 1, \mathbf{R}^{-1}\right)$$
  and $\mathcal{W}_r(n, \mathbf{\Sigma})$ is the *non-singular* Wishart distribution with n degrees of freedom n and positive definite matrix parameter $\mathbf{\Sigma}$.

Now, the matricial versions of these representations are given next:

- For the first generalisation assume that $\mathbf{Y} \sim \mathcal{N}_{m\times r}(\mathbf{0}, \mathbf{I}_m \otimes \mathbf{\Sigma})$ (Muirhead, 1982, pp. 89-80), then $\mathbf{T} \in \Re^{m\times r}$ is defined similar to (1), but $\mu \in \Re^{m\times r}$ and the corresponding density is called the matrix-variate non-singular t distribution, see Fang and Anderson (1990, p. 208), Sutradhar and Ali (1989) and Gupta and Varga (1993, p. 76), among others.
- For the generalisation of (2), assume that $\mathbf{Y} \sim \mathcal{N}_{m\times r}(\mathbf{0}, \mathbf{I}_m \otimes n\mathbf{I}_r)$, then the density of $\mathbf{T} \in \Re^{m\times r}$ is called the non singular matricvariate t distribution, see Dickey (1967), Press (1982, pp. 138-141), Box and Tiao (1972, pp. 441-448) and Kotz and Nadarajah (2004, Section 5.11).

Note that the rows of the matrices $\mathbf{T}'^{s}$ have the same marginal distribution in both representations, but the densities of the matrix $\mathbf{T}$ do not satisfy that equality, see Dickey (1967) and Sutradhar and Ali (1989) among others.

A common problem in the matric and matrix variate t distributions arrives when the matrix $\mathbf{Y}$ is singular. More explicitly, if the matrix $\mathbf{Y} \in \Re^{m\times r}$ represents the information of m objects or observations where r variables (characteristics) are evaluated per object, then, the singularity is due to the following reasons: the rows (observations) and/or the columns (characteristics) of the matrix $\mathbf{Y}$ are linearly dependent, see Díaz-García et al. (1997) and Díaz-García and González-Farías (2005). In addition, for the matricvariate distribution we need to consider another class of singularity; it is when $\mathbf{V}$ is positive semidefinite ($\mathbf{V} \geq \mathbf{0}$). From a practical point of view, this singularity ($\mathbf{V} \geq \mathbf{0}$), can be explained by many reasons; for example, when the number of observations is less that the number of characteristics (dimension) of the problem, $m < r$, see Díaz-García et al. (1997).

In any case, if the matrix $\mathbf{Y}$ and/or the matrix $\mathbf{V}$ are singular, then the matrix $\mathbf{T}$ is singular, and the corresponding densities of the matric and matrix variate distributions exist respect to the Hausdorff measure, see Billingsley (1986), Díaz-García et al. (1997) and Díaz-García and González-Farías (2005). In the case of the matrix-variate t distribution there is only a type of singularity as a consequence of the singularity of $\mathbf{Y}$. However, in the case of the matricvariate t distribution we have the following cases:

$$\mathbf{T} = \begin{cases} \text{Non singular,} & \mathbf{V} > \mathbf{0},\ \mathbf{Y} \text{ is non singular;} \\ \text{Singular 1,} & \mathbf{V} > \mathbf{0},\ \mathbf{Y} \text{ is singular;} \\ \text{Singular 2,} & \mathbf{V} \geq \mathbf{0},\ \mathbf{Y} \text{ is non singular;} \\ \text{Double singular ,} & \mathbf{V} \geq \mathbf{0},\ \mathbf{Y} \text{ is singular.} \end{cases}$$

In this work we propose an expression for the double singular matricvariate t distribution and as particular cases we find expressions for the non singular, singular 1 and singular 2 matricvariate t distributions. Similarly, we propose an expression for the singular matrix-variate t distribution. Finally, an application of the singular matricvariate t distribution is proposed by studying the generalised multivariate version of the externally studentised residual is studied when the number of observations is less than the dimension of the problem.

2 Double singular matricvariate t distribution

Given any symmetric matrix $\mathbf{A}$, let $\mathbf{A}^+$ and $\mathbf{A}^-$ be the Moore-Penrose inverse and any symmetric generalised inverse $\mathbf{A}$, respectively, see Rao (1973). If $\mathrm{ch}_i(\mathbf{A})$ denotes the i-th eigenvalue of the matrix $\mathbf{A}$ and we use the notation of Díaz-García et al. (1997) for the singular matrix-variate normal and Wishart and Pseudo-Wishart distributions, then we have:

Theorem 1 (Double singular matricvariate t distribution) *Let* $\mathbf{T}$ *be the random* $m \times r$ *matrix,*

$$\mathbf{T} = \left(\mathbf{V}^{1/2}\right)^+ \mathbf{Y} + \mu$$

where $\mathbf{V}^{1/2}\mathbf{V}^{1/2} = \mathbf{V} \sim \mathcal{W}_m^q(n, \boldsymbol{\Xi})$, $\boldsymbol{\Xi}(: m \times m) \geq \mathbf{0}$ *with rank*$(\boldsymbol{\Xi}) = r_\chi \leq m$ *and* $q = \min(m, n)$; *independent of* $\mathbf{Y} \sim \mathcal{N}_{m \times r}^{m, r_\Sigma}(\mathbf{0}, \mathbf{I}_m \otimes \boldsymbol{\Sigma})$, $\boldsymbol{\Sigma}(: r \times r) \geq \mathbf{0}$ *with rank*$(\boldsymbol{\Sigma}) = r_\Sigma \leq r$. *Then the singular random matrix* $\mathbf{T}$ *has the density,*

$$\frac{c(m, n, q, q_1, r_\chi, r_\Sigma, r_\alpha)}{\displaystyle\prod_{i=1}^{r_\Sigma} \mathrm{ch}_i(\boldsymbol{\Sigma})^{m/2} \prod_{j=1}^{r_\chi} \mathrm{ch}_j(\boldsymbol{\Xi})^{n/2}} \prod_{l=1}^{r_\alpha} \mathrm{ch}_l \left[\boldsymbol{\Xi}^- + (\mathbf{T} - \mu)\boldsymbol{\Sigma}^-(\mathbf{T} - \mu)'\right]^{-(n+r_\Sigma)/2} (d\mathbf{T}), \quad (3)$$

where

$$c(m,n,q,q_1,r_{\chi},r_{\Sigma},r_{\alpha}) = \frac{\pi^{n(q-r_{\chi})/2-(n+r_{\Sigma})(q_1-r_{\alpha})/2-mr_{\Sigma}/2}\Gamma_{q_1}[(n+r_{\Sigma})/2]}{2^{(mr_{\Sigma}+nr_{\chi})/2-(n-r_{\Sigma})r_{\alpha}/2}\Gamma_q[n/2]}, \quad (4)$$

where $q_1 = \min(m, n+r_{\Sigma})$, $r_{\alpha} = \mathit{rank}[\mathbf{\Xi}^- + (\mathbf{T}-\mu)\mathbf{\Sigma}^-(\mathbf{T}-\mu)'] \leq m$, *and* $(d\mathbf{T})$ *denotes the Hausdorff measure.*

Proof: From Díaz-García et al. (1997), the joint density of $\mathbf{V}$ and $\mathbf{Y}$ is

$$dF_{\mathbf{Y},\mathbf{V}}(\mathbf{Y},\mathbf{V}) \propto |\mathbf{L}|^{(n-m-1)/2}\operatorname{etr}\left(-\frac{1}{2}\left(\mathbf{\Xi}^-\mathbf{V}+\mathbf{Y}\mathbf{\Sigma}^-\mathbf{Y}'\right)\right)(d\mathbf{Y})(d\mathbf{V}),$$

where $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$ and the corresponding constant of proportionality is

$$c = \frac{2^{-nr_{\chi}/2}\ \pi^{n(q-r_{\chi})/2}}{(2\pi)^{mr_{\Sigma}/2}\Gamma_q[n/2]\prod_{i=1}^{r_{\Sigma}}\operatorname{ch}_i(\mathbf{\Sigma})^{m/2}\ \prod_{j=1}^{r_{\chi}}\operatorname{ch}_j(\mathbf{\Xi})^{n/2}},$$

where $\mathbf{V} = \mathbf{H}_1\mathbf{L}\mathbf{H}_1'$, is the nonsingular part of the spectral decomposition of $\mathbf{V}$, $\mathbf{H}_1$ is a $q \times m$ semi-orthogonal matrix, and $\mathbf{L} = \operatorname{diag}(l_1,\dots,l_q)$, $l_1 > \cdots > l_q$. First consider the transformation $\mathbf{T} = \left(\mathbf{V}^{1/2}\right)^+\mathbf{Y} = \left(\mathbf{V}^{1/2}\right)^+(\mathbf{Y}_1,\dots,\mathbf{Y}_r)$, where $\mathbf{Y}_j$, $j=1,\dots,r$ are the columns of $\mathbf{Y}$ such that the $\mathbf{Y}_j$'s are in the image of $\mathbf{V}^{1/2}$, then $\mathbf{Y} = \left(\mathbf{V}^{1/2}\right)\mathbf{T}$. Now make the change of variables $\mathbf{Y} = \left(\mathbf{V}^{1/2}\right)(\mathbf{T}-\mu)$ then by Díaz-García (2007),

$$(d\mathbf{Y}) = |\mathbf{L}|^{r_{\mathbf{\Sigma}}/2}(d\mathbf{T}).$$

The joint density of $\mathbf{T}$ and $\mathbf{V}$ is given by

$$dF_{\mathbf{T},\mathbf{V}}(\mathbf{T},\mathbf{V}) \propto \frac{|\mathbf{L}|^{(n+r_{\Sigma}-m-1)/2}}{\operatorname{etr}\left(\frac{1}{2}\left(\mathbf{\Xi}^-+(\mathbf{T}-\mu)\mathbf{\Sigma}^-(\mathbf{T}-\mu)'\right)\mathbf{V}\right)}(d\mathbf{T})(d\mathbf{V}).$$

Integrating with respect to $\mathbf{V} \geq \mathbf{0}$ we have (see Díaz-García et al. , 1997),

$$dF_{\mathbf{T}}(\mathbf{T}) \propto \frac{2^{(n+r_{\Sigma})r_{\alpha}/2}\Gamma_{q_1}[(n+r_{\Sigma})/2]}{\pi^{(n+r_{\Sigma})(q_1-r_{\alpha})/2}\prod_{l=1}^{r_{\alpha}}\operatorname{ch}_l\left(\mathbf{\Xi}^-+(\mathbf{T}-\mu)\mathbf{\Sigma}^-(\mathbf{T}-\mu)'\right)^{(n+r_{\Sigma})/2}}(d\mathbf{T}),$$

and the result follows by noting that $q_1 = \min(m, n+r_{\Sigma})$ and $r_{\alpha} \leq m$ is the rank of the matrix $(\mathbf{\Xi}^-+(\mathbf{T}-\mu)\mathbf{\Sigma}^-(\mathbf{T}-\mu)')$. □

Note that inside the singular 2 and double singular types there is a number of particular cases of the matricvariate t distribution according to the singularity sources of the density of $\mathbf{V}$, see Díaz-García et al. (1997). Some of these particular cases are obtained by the following substitutions:

(1) if $r_\alpha = m$, then

$$\frac{c(m, n, q, q_1, r_\chi, r_\Sigma, r_\alpha)}{\displaystyle\prod_{i=1}^{r_\Sigma} \mathrm{ch}_i(\mathbf{\Sigma})^{m/2} \prod_{j=1}^{r_\chi} \mathrm{ch}_j(\mathbf{\Xi})^{n/2}} \left|\mathbf{\Xi}^- + (\mathbf{T} - \mu)\mathbf{\Sigma}^-(\mathbf{T} - \mu)'\right|^{-(n+r_\Sigma)/2} (d\mathbf{T}).$$

(2) Case: Singular 1, then

$$\frac{c(m, n, m, m, m, r_\Sigma, m)}{\displaystyle\prod_{i=1}^{r_\Sigma} \mathrm{ch}_i(\mathbf{\Sigma})^{m/2} \ |\mathbf{\Xi}|^{n/2}} \left|\mathbf{\Xi}^{-1} + (\mathbf{T} - \mu)\mathbf{\Sigma}^-(\mathbf{T} - \mu)'\right|^{-(n+r_\Sigma)/2} (d\mathbf{T}).$$

This distribution was obtained previously in an alternative way by Díaz-García and Gutiérrez-Jáimez (2006b).

(3) Case: Singular 2, in general we have

$$\frac{c(m, n, q, n+r, r_\chi, r, m)}{\displaystyle|\mathbf{\Sigma}|^{m/2} \prod_{j=1}^{r_\chi} \mathrm{ch}_j(\mathbf{\Xi})^{n/2}} \left|\mathbf{\Xi}^- + (\mathbf{T} - \mu)\mathbf{\Sigma}^{-1}(\mathbf{T} - \mu)'\right|^{-(n+r)/2} (d\mathbf{T}),$$

or in a particular case, when $\mathbf{V}$ has a nonsingular Pseudo-Wishart distribution,

$$\frac{c(m, n, n, n+r, m, r, m)}{|\mathbf{\Sigma}|^{m/2} \ |\mathbf{\Xi}|^{n/2}} \left|\mathbf{\Xi}^{-1} + (\mathbf{T} - \mu)\mathbf{\Sigma}^{-1}(\mathbf{T} - \mu)'\right|^{-(n+r)/2} (d\mathbf{T}).$$

Observe that, the non singular case studied by Dickey (1967) and Box and Tiao (1972, p. 439-447) is obtained as a particular case of Theorem 1, by taking $q = q_1 = r_\chi = r_\alpha = m$ and $r_\Sigma = r$,

$$\frac{\Gamma_m[(n+r)/2]}{\pi^{mr/2}|\Gamma_m[n/2]\mathbf{\Sigma}|^{m/2} \ |\mathbf{\Xi}|^{n/2}} \left|\mathbf{\Xi}^{-1} + (\mathbf{T} - \mu)\mathbf{\Sigma}^{-1}(\mathbf{T} - \mu)'\right|^{-(n+r)/2} (d\mathbf{T}),$$

where $(d\mathbf{T})$ is the Lebesgue measure.

3 Singular matrix-variate t distribution

In this section we study the density function of the singular matrix-variate t distribution.

Theorem 2 (Singular matrix-variate t distribution) *Let $\mathbf{T}$ be the random $m \times r$ matrix,*

$$\mathbf{T} = S^{-1}\mathbf{Y} + \mu$$

where $nS/\sigma^2 \sim \chi^2(n)$ is independent of $\mathbf{Y} \sim \mathcal{N}_{m\times r}^{r_\Theta, r_\Sigma}(\mathbf{0}, \mathbf{\Theta}\otimes\mathbf{\Sigma})$, $\mathbf{\Sigma}(: r \times r) \geq \mathbf{0}$ with rank$(\mathbf{\Sigma}) = r_{\Sigma} \leq r$ and rank$(\mathbf{\Theta}) = r_{\Theta} \leq m$. Then the $\mathbf{T}$ has the density,

$$\frac{\pi^{-g/2}\Gamma[(n+g)/2]}{\Gamma[n/2]\displaystyle\prod_{i=1}^{r_\Sigma}\mathrm{ch}_i(\mathbf{\Sigma})^{r_\Theta/2}\prod_{j=1}^{r_\chi}\mathrm{ch}_j(\mathbf{\Theta})^{r_\Sigma/2}}\left(1+\mathrm{tr}\,\mathbf{\Theta}^{-}(\mathbf{T}-\mu)\mathbf{\Sigma}^{-}(\mathbf{T}-\mu)'\right)^{-(n+g)/2}(d\mathbf{T}),\quad(5)$$

where $g = r_{\Sigma}r_{\Theta}$ and $(d\mathbf{T})$ denotes the Hausdorff measure.

Proof: The joint density of $\mathbf{Y}$ and S is

$$dF_{\mathbf{Y},\mathbf{S}}(\mathbf{Y},\mathbf{s}) \propto s^{(n-2)/2}\exp\left(-\frac{1}{2}\left(s+\mathrm{tr}\,\mathbf{\Theta}^{-}\mathbf{Y}\mathbf{\Sigma}^{-}\mathbf{Y}'\right)\right)(d\mathbf{Y})(ds),$$

where the corresponding constant of proportionality is

$$c_1 = \frac{1}{2^{n/2}(2\pi)^{r_\Theta r_\Sigma/2}\Gamma[n/2]\displaystyle\prod_{i=1}^{r_\Sigma}\mathrm{ch}_i(\mathbf{\Sigma})^{r_\Theta/2}\prod_{j=1}^{r_\Theta}\mathrm{ch}_j(\mathbf{\Theta})^{r_\Sigma/2}}.$$

Let $\mathbf{Y} = S^{1/2}(\mathbf{T}-\mu)$, then by Díaz-García (2007), $(dY) = S^{r_\Theta r_\Sigma/2}(dT)$. The joint density of $\mathbf{Y}$ and S is given by

$$dF_{\mathbf{T},\mathbf{S}}(\mathbf{Y},\mathbf{s}) \propto \frac{s^{(n+r_\Theta r_\Sigma-2)/2}}{\exp\left(\frac{1}{2}\left(1+\mathrm{tr}\,\mathbf{\Theta}^{-}(\mathbf{T}-\mu)\mathbf{\Sigma}^{-}(\mathbf{T}-\mu)'\right)s\right)}(d\mathbf{T})(ds).$$

Integrating with respect to S and by using

$$\int_{s>0} s^{(n+r_\Theta r_\Sigma-2)/2}\exp\left(-\frac{1}{2}\left(1+\mathrm{tr}\,\mathbf{\Theta}^{-}(\mathbf{T}-\mu)\mathbf{\Sigma}^{-}(\mathbf{T}-\mu)'\right)s\right)(ds)$$

$$= 2^{(n+r_\Theta r_\Sigma)/2}\Gamma[(n+r_\Theta r_\Sigma)/2]\left(1+\mathrm{tr}\,\mathbf{\Theta}^{-}(\mathbf{T}-\mu)\mathbf{\Sigma}^{-}(\mathbf{T}-\mu)'\right)^{-(n+r_\Theta r_\Sigma)/2},$$

give the required marginal density function for $\mathbf{T}$. □

Theorem 2 allows generalisations of the results in Sutradhar and Ali (1989) and Fang and Anderson (1990, p. 208) to the singular matrix-variate case. Also observe that this distribution can be obtained in an alterative way by Theorem 1 in Díaz-García and Gutiérrez-Jáimez (2006c), see also Díaz-García and González-Farías (2005).

4 Application

The distributions of the certain classes of residuals associated to a multivariate general linear model play a fundamental role in the context of sensitivity analysis, see Caroni (1987) and Díaz-García and Gutiérrez-Jáimez (2006a), among many others.

Consider the multivariate general linear model

$$\mathbf{Y} = \mathbf{XB} + \mathbf{E}, \tag{6}$$

where $\mathbf{X}$ is the $m \times p$ (known) design matrix of rank $r_{\scriptscriptstyle X} \leq \min(m, p)$ and $\mathbf{B}$ is the $p \times r$ matrix of unknown parameters. We shall assume that $\mathbf{E} \sim \mathcal{N}_{m\times r}(\mathbf{0}, \mathbf{I}_m \otimes \mathbf{\Sigma})$ with $r \times r$ unknown matrix of covariance $\mathbf{\Sigma} > \mathbf{0}$. Also we shall assume that

$$m \leq r,$$

i.e. the number of observations is less than the dimension r.

The maximum likelihood or the least square estimator of $\mathbf{XB}$ is given by

$$\widehat{\mathbf{XB}} \equiv \mathbf{X}\widehat{\mathbf{B}} = \mathbf{X}(\mathbf{X}'\mathbf{X})^{-}\mathbf{X}'\mathbf{Y} = \mathbf{XX}^{+}\mathbf{Y},$$

which is invariant for any generalised inverse of $(\mathbf{X}'\mathbf{X})$.

The covariance matrix $\mathbf{\Sigma}$ can be unbiasedly estimated by

$$\widehat{\mathbf{\Sigma}} = \frac{1}{(m - r_{\scriptscriptstyle X})}(\mathbf{Y} - \mathbf{X}\widehat{\mathbf{B}})'(\mathbf{Y} - \mathbf{X}\widehat{\mathbf{B}}).$$

The matrix $\widehat{\mathbf{\Sigma}}$ is, however, a singular random matrix of rank $m - r_{\scriptscriptstyle X} < r$; moreover, $(m - r_{\scriptscriptstyle X})\widehat{\mathbf{\Sigma}} \sim \mathcal{PW}_r^{m-r_{\scriptscriptstyle X}}((m - r_{\scriptscriptstyle X}), \mathbf{\Sigma})$.

The residual matrix is defined as $\widehat{\mathbf{E}} = \mathbf{Y} - \widehat{\mathbf{Y}} = \mathbf{Y} - \mathbf{X}\widehat{\mathbf{B}} = (\mathbf{I}_m - \mathbf{XX}^{+})\mathbf{Y} = (\mathbf{I}_m - \mathbf{H})\mathbf{Y}$, where $\mathbf{H} = \mathbf{XX}^{+}$ is the orthogonal projector on the image of $\mathbf{X}$. Then $\widehat{\mathbf{E}}$ has a singular matrix-variate normal distribution of rank $r(m - r_{\scriptscriptstyle X})$, i.e. $\widehat{\mathbf{E}} \sim \mathcal{N}_{m\times r}^{(m-r_{\scriptscriptstyle X}),r}(\mathbf{0}, (\mathbf{I}_m - \mathbf{H}) \otimes \mathbf{\Sigma})$. Also, observe that the i-th row of $\widehat{\mathbf{E}}$, denoted as $\widehat{\mathbf{E}}_i$, has a nonsingular r-variate normal distribution, i.e., $\widehat{\mathbf{E}}_i \sim \mathcal{N}_r(0, (1 - h_{ii})\mathbf{\Sigma})$, $\mathbf{H} = (h_{ij})$, for all $i = 1, \cdots, n$. Given that the $\widehat{\mathbf{E}}_i$'s are linearly dependent, we need to define the index $I = \{i_1, \cdots, i_k\}$, with $i_s = 1, \cdots, n$; $s = 1, \cdots, k$ and $k \leq (n - r_{\scriptscriptstyle X})$, in such way that the vectors $\widehat{\mathbf{E}}_{i_1}, \cdots, \widehat{\mathbf{E}}_{i_k}$'s are linearly independent. Thus we write

$$\widehat{\mathbf{E}}_I = \begin{pmatrix} \widehat{\mathbf{E}}_{i_1}{}^T \\ \vdots \\ \widehat{\mathbf{E}}_{i_k}{}^T \end{pmatrix}. \tag{7}$$

Observe that $\widehat{\mathbf{E}}_I$ has a matrix-variate normal nonsingular distribution, moreover $\widehat{\mathbf{E}}_I \sim \mathcal{N}_{k\times p}(\mathbf{0}, (\mathbf{I}_k - \mathbf{H}_I) \otimes \mathbf{\Sigma})$. Here $\mathbf{H}_I$ is obtained by deleting the row and column of $\mathbf{H}$ non indexed by I.

A generalised multivariate version of the externally studentised residual when the number of observations is less than the dimension ($\widehat{\mathbf{\Sigma}} \geq \mathbf{0}$) is given by (see Caroni (1987) for classical definition),

$$\mathbf{u}_i = \frac{1}{\sqrt{1-h_{ii}}} \left(\widehat{\mathbf{\Sigma}}_{(i)}^{1/2}\right)^+ \widehat{\mathbf{E}}_i,$$

where $\widehat{\mathbf{\Sigma}}_{(i)}$ is obtained by removing the i-th observation from the sample, which is also $r \times r$ singular matrix of rank $(m - r_{_X} - 1)$. Given the index I, the following definition is established

$$\mathbf{u}_I = (\mathbf{I}_k - \mathbf{H}_I)^{-1/2} \widehat{\mathbf{E}}_I \left(\widehat{\mathbf{\Sigma}}_I^{1/2}\right)^+,$$

where $\widehat{\mathbf{\Sigma}}_I$ is obtained by removing the observations of the sample non indexed by I.

Theorem 3 (Externally studentised residual) *Under the general multivariate linear model (6),* $\mathbf{W} = \mathbf{u}_I/\sqrt{s}$, *has a singular matricvariate t distribution (singular 2), moreover*

$$dF_{\mathbf{W}}(\mathbf{W}) = \frac{\Gamma_{(m-r_{_X})}[(m-r_{_X})/2]}{\pi^{(m-r_{_X})(m-r_{_X}-r)/2+kr/2}\Gamma_s[s/2]} \left|\mathbf{I}_r + \mathbf{W}'\mathbf{W}\right|^{-(m-r_{_X})/2} (d\mathbf{W}).$$

where $s = m - r_{_X} - k$.

Proof: The demonstration follows from Theorem 1, by observing that $\mathbf{W} = \mathbf{T}'$, $(\mathbf{I}_k - \mathbf{H}_I)^{-1/2}\widehat{\mathbf{E}}_I \sim \mathcal{N}_{k\times r}(0, I \otimes \mathbf{\Sigma})$ and that $(m - r_{_X} - k)\widehat{\mathbf{\Sigma}}_I \sim \mathcal{PW}_r^{(m-r_{_X}-k)}((m - r_{_X} - k), \mathbf{\Sigma})$. $\square$

Conclusions

The singular matrix and matric variate t distribution are studied under a number of singularities. Several particular interesting cases in the statistical literature are proposed. In the same context, previously published results are found as particular cases of the general densities derived in this paper. Finally, some of these results are applied in the context of sensitivity analysis, specifically, we study the joint distribution of the externally studentised residual.

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