

Comunicaciones del CIMAT

Matrix Kummer-general relation

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Abstract

An extension of the known matrix Kummer relation of Hertz (1955) is proposed in this paper by assuming a general model which admits a Taylor expansion in zonal polynomials.

Key words: Matrix Kummer relations, hypergeometric function, zonal polynomials. *PACS:* 33C70, 33D70

1 Introduction

The work of Hertz (1955), which generalised the classical special functions of hypergeometric type to matrix variables, opened a perspective of important applications in many fields of knowledge. Then, Constantine (1963) placed the work of Herz in the context of the well known zonal polynomial theory and the applications in multivariate analysis and other areas were possible. For example, the information theory became one of the novel fields where these

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relations were applied successfully, see the works of Ratnarajah and Vaillancourt (2005a,b) and the references there in. In fact, some studies of Goodall and Mardia (1993); Caro-Lopera *et al.* (2008a) considered new relations in order to perform exact inference, which avoids the classical approximations. Some advances, including new relations are studied in the context of shape theory based on affine transformations, see Caro-Lopera *et al.* (2008b); explicitly, the general relations transform the elliptical configuration densities, which are infinite series of zonal polynomials, see Caro-Lopera *et al.* (2008a), into polynomials of lower degree easily computable.

Now, the classical Kummer relation can be generalised if we replace its exponential model by a function which admits a Taylor expansion in zonal polynomials; then a sort of new expressions can be derived easily.

However some applications demand the extension of the parameter definition domain, and this implies new integral representations of the series, new integral transforms, and new induction constructions. So, the domain extension problem places the generalised Kummer relations in an interesting mathematical task with promissory applications.

This work develops the above discussion by generalising the classical Kummer relation in section 2, then a number of known results are derived as corollaries, and a source for new relations is established by using some partitional formulae. Finally, a domain extension study is proposed at the end of the paper in section 3.

2 Generalized Kummer relation

Recall that the Kummer relation (due to Hertz (1955, equation (2.8), p. 488)) states that

$$_{1}F_{1}(a;c;\mathbf{X}) = \operatorname{etr}(\mathbf{X})_{1}F_{1}(c-a;c;-\mathbf{X}),$$
 (1)

see also Muirhead (1982, equation (6), p. 265).

Now, consider the following definition, see Caro-Lopera *et al.* (2008a).

Definition 1 Let $\mathbf{X} > 0$ be an $m \times m$ positive definite matrix. The hypergeometric generalised function $_1P_1$ of matrix argument is defined by

$${}_{1}P_{1}(f(t, \operatorname{tr}(\mathbf{X})) : a; c; \mathbf{X}) = \sum_{t=0}^{\infty} \frac{f(t, \operatorname{tr}(\mathbf{X}))}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} C_{\tau}(\mathbf{X}),$$
(2)

where \sum_{τ} denotes the summation over all partitions τ , $\tau = (t_1, \dots, t_m), t_1 \ge t_2 \dots \ge t_m > 0$, of t, $C_{\tau}(X)$ is the zonal polynomial of X corresponding to τ ,

the function $f(t, tr(\mathbf{X}))$ is independent of τ and the generalised hypergeometric coefficient $(b)_{\tau}$ is given by

$$(\beta)_{\tau} = \prod_{i=1}^{m} \left(\beta - \frac{1}{2} (i-1) \right)_{t_i},$$

where

$$(b)_t = b(b+1)\cdots(b+t-1), \quad (b)_0 = 1.$$

Here **X**, the argument of the function, is a complex symmetric $m \times m$ matrix and the parameters a, c are arbitrary complex numbers. The parameter c can not be zero or an integer or a half-integer $\leq (m-1)/2$. If the parameter a is a negative integer, say, a = -l, then the function (2) is a polynomial of degree ml, because for $t \geq ml + 1$, $(a)_{\tau} = (-l)_{\tau} = 0$, see Muirhead (1982, p. 258). In particular note that, $_1P_1(1:a;c;\mathbf{X}) = _1F_1(a;c;\mathbf{X})$.

So, using this notation we see that the Kummer relation (1) is a particular case of a general type of expressions with the following form

$${}_{1}P_{1}\left(f^{(t)}(0):a;c;\mathbf{X}\right) = {}_{1}P_{1}\left(f^{(t)}(\operatorname{tr}(\mathbf{X})):c-a;c;-\mathbf{X})\right),$$
(3)

where $f^{(t)}(y)$ denotes the t-th derivative of the function f(y).

With this objective, we consider a general integral representation of the left hand side of (3), assuming that f(y) has a convergent power series expansion.

Theorem 2 Let be $\mathbf{X} < \mathbf{I}$, $\operatorname{Re}(a) > (m-1)/2$, $\operatorname{Re}(c) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$. If the function f(y) admits a Taylor expansion in zonal polynomials, then the following integral representation holds

$${}_{1}P_{1}\left(f^{(t)}(0):a;c;\mathbf{X}\right) = \frac{\Gamma_{m}(c)}{\Gamma_{m}(a)\Gamma_{m}(c-a)}$$
$$\times \int_{0<\mathbf{Y}<\mathbf{I}_{m}} f(\operatorname{tr}(\mathbf{X}\mathbf{Y}))|\mathbf{Y}|^{a-(m+1)/2}|\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2}(d\mathbf{Y}).$$
(4)

Proof. First, we use an expansion power series in zonal polynomials

$$f(\operatorname{tr}(\mathbf{X}\mathbf{Y})) = \sum_{t=0}^{\infty} \frac{f^{(t)}(0)}{t!} [\operatorname{tr}(\mathbf{X}\mathbf{Y})]^t$$
$$= \sum_{t=0}^{\infty} \frac{f^{(t)}(0)}{t!} \sum_{\tau} C_{\tau}(\mathbf{X}\mathbf{Y})$$

Then integrating term by term using Muirhead (1982, theorem 7.2.10, p. 254), we have that

$$\begin{split} &\int_{0<\mathbf{Y}<\mathbf{I}_{m}} f(\mathrm{tr}(\mathbf{X}\mathbf{Y}))|\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2} (d\mathbf{Y}) \\ &= \sum_{t=0}^{\infty} \frac{f^{(t)}(0)}{t!} \sum_{\tau} \int_{0<\mathbf{Y}<\mathbf{I}_{m}} |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2} C_{\tau}(\mathbf{X}\mathbf{Y}) (d\mathbf{Y}) \\ &= \sum_{t=0}^{\infty} \frac{f^{(t)}(0)}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} \frac{\Gamma_{m}(a)\Gamma_{m}(c-a)}{\Gamma_{m}(c)} C_{\tau}(\mathbf{X}) \\ &= \frac{\Gamma_{m}(a)\Gamma_{m}(c-a)}{\Gamma_{m}(c)} \sum_{t=0}^{\infty} \frac{f^{(t)}(0)}{t!} \sum_{\tau} \frac{(a)_{\tau}}{(c)_{\tau}} C_{\tau}(\mathbf{X}) \\ &= \frac{\Gamma_{m}(a)\Gamma_{m}(c-a)}{\Gamma_{m}(c)} {}_{1}P_{1}\left(f^{(t)}(0):a;c;\mathbf{X}\right), \end{split}$$

and the required result follows.

Now we propose an expression for the Kummer relation based on the generalised hypergeometric function (2), which shall be termed generalised Kummer relation.

Theorem 3 Let be $\mathbf{X} > 0$, $\operatorname{Re}(a) > (m-1)/2$, $\operatorname{Re}(c) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$. If the function f(y) admits a Taylor expansion in zonal polynomials, then the generalised Kummer relation is given by

$${}_{1}P_{1}\left(f^{(t)}(0):a;c;\mathbf{X}\right) = {}_{1}P_{1}\left(f^{(t)}(\operatorname{tr}(\mathbf{X})):c-a;c;-\mathbf{X}\right).$$
(5)

Proof. Consider $\mathbf{W} = \mathbf{I} - \mathbf{Y}$ in (4), then we obtain

$${}_{1}P_{1}\left(f(t, \operatorname{tr}(\mathbf{X})) : a; c; \mathbf{X}\right) = \frac{\Gamma_{m}(c)}{\Gamma_{m}(a)\Gamma_{m}(c-a)}$$

$$\times \int_{0 < \mathbf{W} < \mathbf{I}_{m}} f(\operatorname{tr}[\mathbf{X}(\mathbf{I} - \mathbf{W})]) |\mathbf{W}|^{c-a-(m+1)/2} |\mathbf{I} - \mathbf{W}|^{a-(m+1)/2} (d\mathbf{W})$$

$$= \frac{\Gamma_{m}(c)}{\Gamma_{m}(a)\Gamma_{m}(c-a)}$$

$$\times \int_{0 < \mathbf{W} < \mathbf{I}_{m}} \sum_{t=0}^{\infty} \frac{f^{(t)}(\operatorname{tr} \mathbf{X})}{t!} \sum_{\tau} C_{\tau}(-\mathbf{X}\mathbf{W}) \frac{|\mathbf{W}|^{c-a-(m+1)/2}}{|\mathbf{I} - \mathbf{W}|^{-(a-(m+1))/2}} (d\mathbf{W})$$

$$= \frac{\Gamma_{m}(c)}{\Gamma_{m}(a)\Gamma_{m}(c-a)} \left[\frac{\Gamma_{m}(a)\Gamma_{m}(c-a)}{\Gamma_{m}(c)} {}_{1}P_{1}\left(f^{(t)}(\operatorname{tr} \mathbf{X}) : c-a; c; -\mathbf{X}\right) \right],$$

and the proof is complete.

Theorem 3 gives a number of new relations, including the classical Kummer

Hertz (see 1955, equation (2.8), p. 488). We end this section by deriving them as corollaries and proposing other particular expressions.

First, we start with the classical Kummer relation:

Corollary 4

 $_{1}P_{1}(1:a;c;\mathbf{X}) = _{1}P_{1}(\operatorname{etr}(\mathbf{X}):c-a;c;-\mathbf{X}) = \operatorname{etr}(\mathbf{X}) _{1}P_{1}(1:c-a;c;-\mathbf{X})$ (6)

Proof. The result follows by taking $f(y) = \exp(y)$ in (5), which implies that $f^{(t)}(0) = 1$ and $f^{(t)}(\operatorname{tr}(\mathbf{X})) = \operatorname{etr}(\mathbf{X})$.

This is the known Kummer relation derived by Hertz (1955) and placed in the zonal polynomial theory by Constantine (1963) (widely used by Muirhead (1982)).

Corollary 5

$${}_{1}P_{1}\left((b)_{t}:a;c;\mathbf{X}\right) = \left(1 - \operatorname{tr}\mathbf{X}\right)^{-b}{}_{1}P_{1}\left((b)_{t}\left(1 - \operatorname{tr}\mathbf{X}\right)^{-t}:c - a;c;-\mathbf{X}\right).$$
(7)

Proof. In this case the result follows by taking the Pearson VII model $f(y) = (1-y)^{-b}$, where $f^{(t)}(0) = (b)_t$ and $f^{(t)}(\operatorname{tr}(\mathbf{X})) = (b)_t (1 - \operatorname{tr}(\mathbf{X}))^{-b-t}$. \Box

The above expression is referred as the Kummer-Pearson VII relation because it is related with a Pearson VII distribution.

Now, the following result of Caro-Lopera et al. (2008a) can be used in the derivation of a sort of new Kummer relations.

Lemma 6 Let $f(y) = y^{T-1} \exp(-Ry^s)$, $R, s, T \in \mathbb{R}$; if $\sum_{\kappa \in P_r}$ denotes the summation over all the partitions $\kappa = (k^{\nu_k}, (k-1)^{\nu_{k-1}}, \cdots, 3^{\nu_3}, 2^{\nu_2}, 1^{\nu_1})$ of r, then

$$f^{(k)}(y) = y^{T-1} \exp(-Ry^{s}) \left\{ \sum_{\kappa \in P_{k}} \frac{k!(-R)^{\sum_{i=1}^{k} \nu_{i}} \prod_{j=0}^{k-1} (s-j)^{\sum_{i=j+1}^{k} \nu_{i}}}{\prod_{i=1}^{k} \nu_{i}!(i!)^{\nu_{i}}} y^{\sum_{i=1}^{k} (s-i)\nu_{i}} + \sum_{m=1}^{k} \binom{k}{m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] \times \sum_{\kappa \in P_{k-m}} \frac{(k-m)!(-R)^{\sum_{i=1}^{k-m} \nu_{i}} \prod_{j=0}^{k-m-1} (s-j)^{\sum_{i=j+1}^{k-m} \nu_{i}}}{\prod_{i=1}^{k-m} \nu_{i}!(i!)^{\nu_{i}}} y^{\sum_{i=1}^{k-m} (s-i)\nu_{i}-m} \right\},$$
(8)

thus the corresponding expressions for $f^{(t)}(0)$ and $f^{(t)}(tr(\mathbf{X}))$ give the required Kotz relations.

For example, if s = 1 and T is an integer such that $1 \le T - 1 \le t$, so we obtain that

Corollary 7

$${}_{1}P_{1}\left(\frac{t!(-R)^{t-T+1}}{(t-T+1)!}:a;c;X\right) = {}_{1}P_{1}\left(f^{(t)}(\operatorname{tr}(\mathbf{X})):c-a;c;-X\right),\qquad(9)$$

where $f^{(t)}(tr(\mathbf{X}))$ is given by

$$(-R)^{t} \operatorname{tr}^{T-1} \mathbf{X} \exp(-R \operatorname{tr}(\mathbf{X})) \left\{ 1 + \sum_{m=1}^{t} {t \choose m} \left[\prod_{i=0}^{m-1} (T-1-i) \right] (-R \operatorname{tr}(\mathbf{X}))^{-m} \right\}.$$

And if T = 1, we have that

Corollary 8

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$${}_{1}P_{1}\left(\sum_{\kappa^{*}}\frac{t!(-R)^{\sum_{i=1}^{t}\nu_{i}}\prod_{j=0}^{t}(s-j)^{\sum_{i=j+1}^{t}\nu_{i}}}{\prod_{i=1}^{t}\nu_{i}!(i!)^{\nu_{i}}}:a;c;\mathbf{X}\right)$$
$$={}_{1}P_{1}\left(f^{(t)}(\operatorname{tr}(\mathbf{X})):c-a;c;-\mathbf{X}\right),(10)$$

where \sum_{κ^*} represents the summation over all the partitions κ^* of t such that $t = s \sum_{i=1}^k \nu_i$ and $f^{(t)}(\operatorname{tr}(\mathbf{X}))$ is given by

$$\exp(-R\operatorname{tr}^{s} \mathbf{X}) \sum_{\kappa \in P_{t}} \frac{t!(-R)^{\sum_{i=1}^{t}\nu_{i}} \prod_{j=0}^{t-1} (s-j)^{\sum_{i=j+1}^{t}\nu_{i}}}{\prod_{i=1}^{t}\nu_{i}! (i!)^{\nu_{i}}} (\operatorname{tr}(\mathbf{X}))^{\sum_{i=1}^{t} (s-i)\nu_{i}},$$

in this case the summation $\sum_{\kappa \in P_t}$ holds for all the partitions of t.

The general Kotz relation involved in (8) certainly generalises (6), the classical Kummer relation, which in this sense is based on a simpler exponential model.

Finally, consider the following generalization of (8), see Caro-Lopera *et al.* (2008a).

Lemma 9 Let f(t) = s(t)r(g(t)), where $s(\cdot)$, $r(\cdot)$ and $g(\cdot)$ have derivatives of all orders, if $w^{(k)}$ denotes $\frac{d^k w}{dt^k}$ then

$$f^{(k)} = \sum_{m=0}^{k} \binom{k}{m} s^{(m)} [r(g(t))]^{(k-m)},$$
(11)

where

$$[r(g(t))]^{(k)} = \sum_{\kappa = (k^{\nu_k}, (k-1)^{\nu_{k-1}}, \dots, 3^{\nu_3}, 2^{\nu_2}, 1^{\nu_1})} \frac{k!}{\prod_{i=1}^k \nu_i! (i!)^{\nu_i}} r^{(\sum_{i=1}^k \nu_i)} \prod_{i=1}^k (g^{(i)})^{\nu_i}.$$
(12)

Note that the function f admits a Taylor expansion then the above expressions always exist for all k.

Then a so termed Kummer logistic relation can be obtained by setting

$$h(y) = \exp(-y) (1 + \exp(-y))^{-2},$$

and computing the *t*-th derivative by using lemma 9, i.e. $f^{(t)}(tr(\mathbf{X}))$ is given by

$$\sum_{m=0}^{t} {t \choose m} \sum_{\kappa \in P_{t-m}} \frac{(t-m)! \left(\sum_{i=1}^{t-m} \nu_i + 1\right)! \exp(-(1+\sum_{i=1}^{t-m} \nu_i) \operatorname{tr}(\mathbf{X}))}{(-1)^{m+\sum_{i=1}^{t-m} (1+i)\nu_i} \prod_{i=1}^{t-m} \nu_i! (i!)^{\nu_i} \left[1+\operatorname{etr}(-\mathbf{X})\right]^{2+\sum_{i=1}^{t-m} \nu_i}}$$

Thus

$${}_{1}P_{1}\left(\sum_{m=0}^{t} {\binom{t}{m}} \sum_{\kappa \in P_{t-m}} \frac{(t-m)! \left(\sum_{i=1}^{t-m} \nu_{i}+1\right)!}{(-1)^{m+\sum_{i=1}^{t-m}(1+i)\nu_{i}} \prod_{i=1}^{t-m} \nu_{i}! (i!)^{\nu_{i}} 2^{2+\sum_{i=1}^{t-m} \nu_{i}}} : a; c; \mathbf{X}\right)$$
$$= {}_{1}P_{1}\left(f^{(t)}(\operatorname{tr} \mathbf{X}) : c-a; c; -\mathbf{X}\right).$$

3 Domain Extensions

Unfortunately, many references which cite and use certain important properties and relations involving hypergeometric functions, do not provide the domain of the corresponding parameters; the classical Kummer relation is an important example, see Muirhead (1982, Theorem 7.4.3, p. 265) and James (1964, eq. (51)), among many others. However, Hertz (1955) studies deeply this relations according to the respective domains; for example, He proposes two ways for deriving the confluent hypergeometric function $_1F_1$. The first one, via the Cauchy inversion formula, which results

$${}_{1}F_{1}(a;c;\mathbf{X}) = \frac{\Gamma_{m}(c)}{(2\pi i)^{m(m+1)/2}} \int_{\operatorname{Re}(\mathbf{Z})=\mathbf{X}_{0}} \operatorname{etr}(\mathbf{Z}){}_{1}F_{0}(a;\mathbf{X}\mathbf{Z}^{-1})|\mathbf{Z}|^{-c}(d\mathbf{Z}), \quad (13)$$

where the integral is absolutely convergent for arbitrary complex matrix \mathbf{X} and a provided we take $\mathbf{X}_0 > \mathbf{0}$, $\mathbf{X}_0 > \operatorname{Re}(\mathbf{M})$ and $\operatorname{Re}(c) > (m-1)/2$. And the second one, via the most important integral representation of ${}_1F_1(a;c;\mathbf{X})$, see Hertz (1955, eq. (2.9)); which is given by

$$\frac{\Gamma_m(c)}{\Gamma_m(a)\Gamma_m(c-a)} \int_{0<\mathbf{Y}<\mathbf{I}_m} \operatorname{etr}(\mathbf{X}\mathbf{Y}) |\mathbf{Y}|^{a-(m+1)/2} |\mathbf{I}-\mathbf{Y}|^{c-a-(m+1)/2} (d\mathbf{Y}) \quad (14)$$

holding for all X, Re(a) > (m-1)/2, Re(c) > (m-1)/2 and Re(c-a) > (m-1)/2.

Some authors derived the classical Kummer relation (1) based on the integral (14), see Muirhead (1982, Theorem 7.4.3); so, any posterior results established with that relation inherits the domain of the parameters according to the absolute convergence of (14). In this case the Kummer relation holds in a weaker domain

$$\operatorname{Re}(a) > (m-1)/2$$
, $\operatorname{Re}(c) > (m-1)/2$ and $\operatorname{Re}(c-a) > (m-1)/2$. (15)

However, Muirhead (1982, Theorem 10.3.7) uses the same Kummer relation when the restrictions (15) are not satisfied. This can be explained by noting that the Kummer relation is still valid in wider domains, for example: it is valid for

arbitrary
$$a$$
 and $\operatorname{Re}(c) > (m-1)/2$, (16)

when the integral representation of the confluent hypergeometric is obtained by applying the Laplace transform to (13), see Hertz (1955, eq. (2.8)); or it is valid for

$$\operatorname{Re}(a) > (m-1)/2$$
, and $\operatorname{Re}(c) > (m-1)/2$, (17)

if the integral representation for $_1F_0$ is used, see Hertz (1955, p. 485) or Muirhead (1982, Corollary 7.3.5).

The Laplace procedure of Hertz (1955) (based on the domain (16)) can be applied to the generalised Kummer relation, then we have:

Theorem 10 If f(y) admits a Taylor expansion in zonal polynomials, then the generalised Kummer relation is given by

$${}_{1}P_{1}\left(f^{(t)}(0):a;c;\mathbf{X}\right) = {}_{1}P_{1}\left(f^{(t)}(\operatorname{tr}(\mathbf{X})):c-a;c;-\mathbf{X}\right),$$
(18)

where $\mathbf{X} > 0$, $\operatorname{Re}(c) > (m-1)/2$ and a is arbitrary (or at least $\operatorname{Re}(a) > (m-1)/2$, if the integral representation of $_1F_0$ is used, see Hertz (1955, p. 485) or Muirhead (1982, Corollary 7.3.5)).

The result follows by applying the Laplace transform to the left hand side of (18)

$$\int_{X>0} \operatorname{etr}(\mathbf{X}\mathbf{Z}) |\mathbf{X}|^{c-(m+1)/2} {}_1P_1(f^{(t)}(0):a;c;\mathbf{X})(d\mathbf{X}),$$

and noting that this integral converges absolutely for $\operatorname{Re}(c) > (m-1)/2$ and arbitrary a (or at least for $\operatorname{Re}(a) > (m-1)/2$), see Hertz (1955, p. 487) and Muirhead (1982, Theorem 7.2.7). Finally, by applying the same procedure to the right hand side and noting the same facts for the domain, the required result follows.

Conclusions

This work proposes an extension of the classical matrix Kummer relation based on a function f(y) which admits a Taylor expansion in zonal polynomials. This generalised Kummer relation involves applications in many fields of knowledge, see for example, Ratnarajah and Vaillancourt (2005a,b). Recently, this kind of expressions have played an important role in shape theory based on affine transformations, i.e. in exact inference of elliptical configuration models, see Caro-Lopera *et al.* (2008b); Goodall and Mardia (1993). The configuration applications require domain extensions of the relation type (18) in order to obtain expressions valid for c - a a negative integer and X a positive definite. But, indeed, this procedure can be applied in the remaining shape theory approaches based on shape, shape cone and shape disk distributions, see Goodall and Mardia (1993); Díaz-García *et al.* (1997); Díaz-García, and Gutiérrez (2006). Finally, the extensions of the parameter domains for the generalised Kummer relations are obtained by using the first definition of the confluent hypergeometric function proposed by Hertz (1955).

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