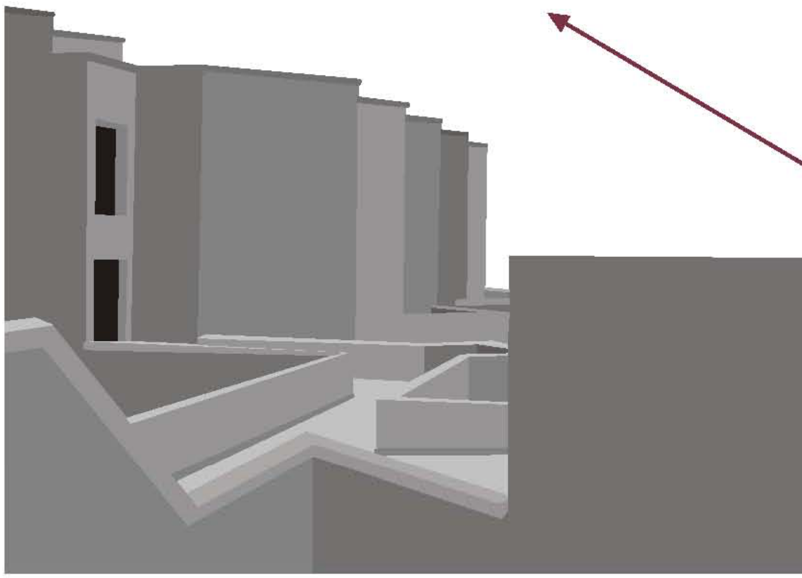

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GLOBAL EXISTENCE AND FINITE-TIME BLOW UP OF A SEMI-LINEAR NONAUTONOMOUS SYSTEM WITH DIRICHLET BOUNDARY CONDITION

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Global existence and finite-time blow up of a semi-linear nonautonomous system with Dirichlet boundary condition

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Abstract

We give conditions for existence of global solutions and for blow up in finite time of semi-linear systems of the form $\partial u_i(t,x)/\partial t = k(t)\left(-(-\Delta)^{\alpha/2}\right) u_i(t,x) + u_j^{\beta_i}(t,x)$, with Dirichlet boundary conditions on a bounded domain $D \subset \mathbb{R}^d$, where $k : [0,\infty) \to [0,\infty)$ is continuous, $0 < \alpha \le 2$, $\beta_i > 1$, $i \in \{1,2\}$ and $j \in \{1,2\} \backslash \{i\}$. Our approach uses in an essential way certain properties of a related time-inhomogeneous Markov process.

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1 Introduction

In this paper we investigate semi-linear equations of the form

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} &= k(t)\Delta_\alpha u(t,x) + u^\beta(t,x), \\ u(0,x) &= f(x), \quad x \in D, \end{aligned} \tag{1}$$

where $k : [0,\infty) \to [0,\infty)$ is continuous, $D \subseteq \mathbb{R}^d$ is an open connected set, Δ_α is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \le 2$, $\beta > 1$ is a constant and $f \ge 0$ is a bounded measurable function. Equations of this kind are relevant in applications as they allow for nonlocal integro-differential diffusion terms in partial differential equations that arise in many mathematical models; see [2, 11, 20] and the references therein. Moreover, their positive solutions can exhibit finite-time blow up, in the sense that $\|u(t,\cdot)\|_\infty$ becomes ∞ in a bounded time interval. For these reasons, determining for which equation parameters a solution either is global, or undergoes blow up in finite time, constitutes a very interesting and challenging topic in contemporary mathematical research; see [1, 6, 10, 17] for surveys.

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It is know that the factor $k(t)$ has a strong effect on the asymptotic behavior of positive solutions of (1): when $D = \mathbb{R}^d$, we proved in [12] that integrability of k already excludes existence of global solutions, and if $\int_0^t k(s)\, ds \sim t^\rho$ as $t \to \infty$ for some $\rho \geq 1$, then the blow up behavior of (1) (i.e. finite-time blow up vs. existence of global solutions) parallels that of the equation $\partial u(t)/\partial t = \Delta_{\alpha/\rho} u + u^\beta$. If D is a bounded smooth domain, H. Fujita [7] proved, in the case of $k \equiv 1$ and $\alpha = 2$, that for any nontrivial, nonnegative initial condition $f \in L^2(D)$ such that

$$\int_D f(x)\varphi_0(x)\, dx > \lambda_0^{1/\beta}, \tag{2}$$

the solution of equation (1) blows up in finite time. Here $\lambda_0 > 0$ is the first eigenvalue of the Laplacian on D, and φ_0 the corresponding eigenfunction normalized so that $\|\varphi_0\|_{L^1} = 1$. A similar condition was obtained by López-Mimbela and Torres [13] for the case of $0 < \alpha \leq 2$, under the additional assumption that D is a bounded domain of class $C^{1,1}$.

In this paper we investigate the dichotomy finite-time blow up versus existence, globally in time, of positive solutions of the nonautonomous semi-linear system with Dirichlet boundary condition

$$\begin{aligned}
\frac{\partial u_1(t,x)}{\partial t} &= k(t)\,\Delta_\alpha u_1(t,x) + u_2^{\beta_1}(t,x) \\
\frac{\partial u_2(t,x)}{\partial t} &= k(t)\,\Delta_\alpha u_2(t,x) + u_1^{\beta_2}(t,x), \quad t > 0,\ x \in D \\
u_i(0,x) &= f_i(x), \quad x \in D, \quad u_i|_{\partial D} \equiv 0,\ i = 1,2.
\end{aligned} \tag{3}$$

Here $\beta_i > 1$ are constants, $D \subset \mathbb{R}^d$ is a bounded $C^{1,1}$ domain with $d \geq 2$, and the initial values f_i, $i = 1,2$, are nonnegative functions belonging to the space $C_0(D)$ of continuous functions on D that vanish at ∂D. As before, the function $k : [0,\infty) \to [0,\infty)$ is assumed to be continuous and not identically zero. We define

$$K(t,s) = \int_s^t k(r)\, dr,\ 0 \leq s \leq t. \tag{4}$$

Our main result, Theorem 4 below, gives information in terms of the system parameters, on how large the initial values should be in order to get finite-time explosion of system (3). In particular, system (3) exhibits finite-time blow up provided that

$$\min_{i\in\{1,2\}} \int_D f_i(x)\varphi_0(x)\, dx > \text{Const.} \left[\int_0^\infty \min_{i\in\{1,2\}} \left(e^{-\lambda_0 K(r,0)}\right)^{\beta_i - 1} dr\right]^{-(\beta_1+1)/(\beta_1\beta_2-1)}.$$

This shows that, for an integrable k, positive solutions corresponding to initial values satisfying the above inequality cannot be global. The approach we use to prove this result consists of an adaptation of the eigenfunction method (see e.g. [17], §17), and relies on the property that the semigroup with generator Δ_α is intrinsically ultracontractive for $d \geq 2$ [4].

In Section 3 we prove existence of local solutions of (3) using an adaptation of the classical fixed-point argument. Conditions ensuring existence of a global positive solution of (3) are given in Section 4. Theorem 4 is proved in Section 5. In the next section, we recall several basic properties of a Markov evolution system and its associated (time-inhomogeneous) Markov process, that we will need to develop our arguments.

2 Killed aditive process

Recall [19] that the linear operator Δ_α is the generator of the d–dimensional symmetric α-stable process $Z \equiv \{Z(t)\}_{t\geq 0}$. Let $\{W(t)\}_{t\geq 0}$ be the time-inhomogeneous Markov process in $\mathbb{R}^d$, corresponding to the family of generators $\{k(t)\Delta_\alpha\}_{t\geq 0}$. The family of random variables $\{W(t)\}_{t\geq 0}$ constitutes an additive process such that

$$P[W(t) \in B] = P[Z(K(t,0)) \in B], \quad B \in \mathcal{B}(\mathbb{R}^d), \quad t \geq 0, \tag{5}$$

where $\mathcal{B}(\mathbb{R}^d)$ stands for the system of Borel sets in $\mathbb{R}^d$. For any $t > 0$ let us denote by $p(t,x,y) \equiv p(t,x-y)$, $x,y \in \mathbb{R}^d$, the positive and continuous density function of $Z(t)$. Letting

$$p(s,x,t,y) \equiv p(K(t,s),x,y), \quad 0 \leq s \leq t, \quad x,y \in \mathbb{R}^d,$$

we see that $p(s,x,t,y)$ is a positive and continuous transition density function for the process $\{W(t)\}_{t\geq 0}$.

Let us define

$$\tau_D = \inf\{t > 0 : W(t) \notin D\} \quad \text{and} \quad \widehat{\tau}_D = \inf\{t > 0 : Z(t) \notin D\}.$$

Using (5) we obtain that

$$\widehat{\tau}_D = K(\tau_D, 0). \tag{6}$$

Let $\{S_D(t)\}_{t\geq 0}$ be the semigroup associated to the process $\{Z(t)\}_{t\geq 0}$ killed on exiting D, and let $p_D(t,x,y)$ be the transition density function of $\{S_D(t)\}_{t\geq 0}$, i.e.

$$\begin{aligned} S_D(t) f(x) &= E^x\left[f(Z(t)); t < \widehat{\tau}_D\right] \\ &= \int_D f(y)\, p_D(t,x,y)\, dy, \quad x \in D, \ t > 0, \ f \in \mathbb{B}^+\left(\mathbb{R}^d\right), \end{aligned}$$

where P^x denotes the distribution of $\{Z(t)\}_{t\geq 0}$ such that $P^x[Z(0) = x] = 1$, E^x represents the expectation with respect to P^x and $\mathbb{B}^+\left(\mathbb{R}^d\right)$ is the space of nonnegative bounded measurable functions on $\mathbb{R}^d$. It is known that

$$p_D(t,x,y) = p_D(t,y,x) \quad \text{and} \quad 0 < p_D(t,x,y) \leq p(t,x,y), \ t > 0, \ x,y \in D,$$

and that $\{S_D(t)\}_{t\geq 0}$ is a strongly continuous Feller semigroup of contractions on the space $L^2(D)$; see e.g. [9], p. 326. Moreover, the linear operators $S_D(t)$, $t \geq 0$, are compact, and there exists an orthonormal basis of eigenfunctions $\{\varphi_n\}_{n=0}^\infty$ with corresponding eigenvalues $\left\{e^{-\lambda_n t}\right\}_{n=0}^\infty$ satisfying

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \quad \text{and} \quad \lim_{n\to\infty} \lambda_n = \infty.$$

All eigenfunctions φ_n are continuous and real-valued. The eigenfunction φ_0 is strictly positive on D.

Let us consider the aditive process $\{W(t)\}_{t\geq 0}$ killed on exiting D; that is, let

$$W_D(t) = \begin{cases} W(t), & \text{on } \{t < \tau_D\} \\ \partial, & \text{on } \{t \geq \tau_D\}, \end{cases}$$

where ∂ is a cemetery point. The state space of $\{W_D(t)\}_{t\geq 0}$ is the set $D_\partial = D \cup \{\partial\}$, and its transition function is given by

$$P_D(s,x,t,\Gamma) = P^{s,x}[W(t) \in \Gamma; t < \tau_D],\ 0 \leq s < t,\ x \in D,\ \Gamma \in \mathcal{B}(D),$$

where $\mathcal{B}(D)$ denotes the Borel σ-field on D, and $P^{s,x}$ is the distribution of $\{W(t)\}_{t\geq 0}$ such that $P^{s,x}(W(s) = x) = 1$. We define, for $f \in L^2(D)$,

$$U_D(t,s) f(x) = \int_D f(y) P^{s,x}[W(t) \in dy; t < \tau_D], \quad 0 \leq s < t, \quad x \in D.$$

Using (6) it is easy to see that, for $t > s \geq 0$, $x \in D$ and $\Gamma \in \mathcal{B}(D)$,

$$\begin{aligned} P_D(s,x,t,\Gamma) &= P^{s,x}[W(t) \in \Gamma; t < \tau_D] \\ &= P^x[Z(K(t,s)) \in \Gamma; K(t,s) < \widehat{\tau}_D] \\ &= P_D(K(t,s), x, \Gamma). \end{aligned}$$

Hence for all $f \in L^2(D)$,

$$U_D(t,s) f(x) = S_D(K(t,s)) f(x),\ x \in D. \tag{7}$$

For $t > 0$ and $x, y \in \mathbb{R}^d$, let $r_D(t,x,y) = E^x[p(t - \widehat{\tau}_D, Z(\widehat{\tau}_D), y); \widehat{\tau}_D < t]$. It is known ([4], Theorem 2.4) that

$$p_D(t,x,y) = p(t,x,y) - r_D(t,x,y).$$

We define

$$r_D(s,x,t,y) = E^{s,x}[p(\tau_D, W(\tau_D), t, y); \tau_D < t]$$

and

$$p_D(s,x,t,y) = p(s,x,t,y) - r_D(s,x,t,y).$$

Since

$$E^{s,x}[p(\tau_D, W(\tau_D), t, y); \tau_D < t] = E^x[p(K(t,s) - \widehat{\tau}_D, Z(\widehat{\tau}_D), y); \widehat{\tau}_D < K(t,s)],$$

we have that

$$p_D(s,x,t,y) = p(K(t,s), x, y) - r_D(K(t,s), x, y) = p_D(K(t,s), x, y).$$

Proposition 1 *The function $p_D(s,x,t,y)$ is a density of $P_D(s,x,t,\Gamma)$, wich is strictly positive, symmetric and continuous on $D \times D$.*

Proof. This follows easily from the fact that $p_D(t,x,y)$ is a density of $P_D(t,x,\Gamma)$, wich is strictly positive, symmetric and continuous on $D \times D$. ∎

Using (7) and the fact that $\{S_D(t)\}_{t\geq 0}$ is a strongly continuous Feller semigroup of contractions on $L^2(D)$, we obtain that $\{U_D(t,s)\}_{t\geq s\geq 0}$ is a Feller evolution family of contractions on $L^2(D)$. In [4] and [9] it is proved that $\{S_D(t)\}_{t\geq 0}$ is an intrinsically ultracontractive semigroup, meaning that for every $t>0$ there are constants $c=c(t,D,\alpha)$ and $C=C(t,D,\alpha)$ such that

$$c\varphi_0(x)\varphi_0(y) \leq p_D(t,x,y) \leq C\varphi_0(x)\varphi_0(y), \quad t>0,\ x,y \in D.$$

The above property is equivalent to the fact that, for all $t>0$, there exists a positive constant $c=c(t,D,\alpha)$ such that

$$|S_D(t) f(x)| < c\varphi_0(x)\|f\|_2, \quad x \in D; \tag{8}$$

see [5], Theorem 3.2.

3 Local existence of a mild solution

The integral representation of system (3) is given, for $f_i \in L^2(D)$, by

$$u_i(t,x) = U_D(t,0) f_i(x) + \int_0^t U_D(t,r) u_{i'}^{\beta_i}(r,x)\,dr, \quad t\geq 0,\ x \in D, \tag{9}$$

where, here and in the sequel, $i \in \{1,2\}$ and $i' = 3-i$. A solution of the integral system (9) is known as a mild solution of (3).

In this section we are going to asume that $f_i \in C_b(D)$, where $C_b(D)$ is the space of real-valued continuous and bounded functions defined on D.

Our proof of existence of local solutions is an adaptation, to our case, of the proof given in [21]. Let $\tau > 0$ and

$$E_\tau \equiv \{(u_1,u_2) : [0,\tau] \to C_b(D) \times C_b(D), |||(u_1,u_2)||| < \infty\},$$

where

$$|||(u_1,u_2)||| \equiv \sup_{0\leq t\leq \tau} \{||u_1(t,\cdot)||_\infty + ||u_2(t,\cdot)||_\infty\}.$$

The couple $(E_\tau, |||\ |||)$ is a Banach space and $P_\tau \equiv \{(u_1,u_2) \in E_\tau : u_1 \geq 0, u_2 \geq 0\}$ and $C_R \equiv \{(u_1,u_2) \in E_\tau : |||(u_1,u_2)||| \leq R\}$, $R>0$, are closed subsets of E_τ.

Theorem 2 *There exists a constant $\tau = \tau(f_1,f_2) > 0$ such that the integral system (9) has a local solution in $C_b([0,\tau]\times D) \times C_b([0,\tau]\times D)$.*

Proof. Define the operator Ψ on $C_b([0,\tau]\times D) \times C_b([0,\tau]\times D)$ by

$$\begin{aligned}\Psi(u_1,u_2) &= (U_D(t,0) f_1(x), U_D(t,0) f_2(x)) \\ &\quad + \left(\int_0^t U_D(t,r) u_2^{\beta_1}(r,x)\,dr, \int_0^t U_D(t,r) u_1^{\beta_2}(r,x)\,dr\right).\end{aligned}$$

Choosing $\tau > 0$ small enough and $R > 0$ sufficiently large, it is easy to verify that Ψ is a contraction mapping on $C_R \cap P_\tau$. In fact, if $(u_1, u_2), (\tilde{u}_1, \tilde{u}_2) \in C_R \cap P_\tau$, then

$$\begin{aligned} |||\Psi(u_1,u_2) - \Psi(\tilde{u}_1,\tilde{u}_2)||| &\leq \sup_{0\leq t\leq\tau} \int_0^t \|u_2^{\beta_1}(r,\cdot) - \tilde{u}_2^{\beta_1}(r,\cdot))\|_\infty dr \\ &+ \sup_{0\leq t\leq\tau} \int_0^t \|u_1^{\beta_2}(r,\cdot) - \tilde{u}_1^{\beta_2}(r,\cdot))\|_\infty dr \end{aligned}$$

and, using the elementary inequality

$$|a^p - b^p| \leq p(a \vee b)^{p-1}|a-b|,$$

which holds for all $a, b > 0$ and $p \geq 1$, we get

$$\begin{aligned} |||\Psi(u_1,u_2) - \Psi(\tilde{u}_1,\tilde{u}_2)||| &\leq \beta_1 R^{\beta_1-1} \int_0^\tau \|u_2(r,\cdot) - \tilde{u}_2(r,\cdot)\|_\infty dr \\ &\quad +\beta_2 R^{\beta_2-1} \int_0^\tau \|u_1(r,\cdot) - \tilde{u}_1(r,\cdot)\|_\infty dr \qquad (10) \\ &\leq (\beta_1 R^{\beta_1-1} \vee \beta_2 R^{\beta_2-1})|||(u_1,u_2) - (\tilde{u}_1,\tilde{u}_2)|||\tau. \end{aligned}$$

From (10), we observe that for $\tau > 0$ small enough and $R > 0$ sufficiently large, Ψ is a contraction mapping on $C_R \cap P_\tau$, hence the result follows by the fixed point theorem. ■

4 Global existence of the mild solution

Here we suppose again that $f_i \in C_b(D)$. Our proof of the next theorem follows closely the proof of Theorem 2.2 in [12].

Theorem 3 *Let f_i be nonnegative, and let $g \equiv f_1 \vee f_2$. If*

$$(\beta_i - 1)\int_0^\infty \|U_D(t,0)g\|_\infty^{\beta_i-1} dt < 1, \quad i = 1, 2,$$

then the solution of the integral system (9) is global.

Proof. Putting

$$B_i(t) = \left[1 - (\beta_i - 1)\int_0^t \|U_D(r,0)g\|_\infty^{\beta_i-1} dr\right]^{-\frac{1}{\beta_i-1}}$$

we get $B_i(0) = 1$ and

$$\begin{aligned} \frac{d}{dt}B_i(t) &= -\frac{1}{\beta_i-1}\left[1-(\beta_i-1)\int_0^t \|U_D(r,0)g\|_\infty^{\beta_i-1}dr\right]^{-\frac{1}{\beta_i-1}-1}[-(\beta_i-1)\|U_D(t,0)g\|_\infty^{\beta_i-1}] \\ &= \|U_D(t,0)g\|_\infty^{\beta_i-1} B_i^{\beta_i}(t), \end{aligned}$$

which gives

$$B_i(t) = 1 + \int_0^t \|U_D(r,0)g\|_\infty^{\beta_i - 1} B_i^{\beta_i}(r)dr, \quad i = 1, 2. \tag{11}$$

Since the evolution system $\{U_D(t,s)\}_{t\geq s\geq 0}$ is positive-preserving, we can consider a continuous function $v_i : [0,\infty) \times D \to [0,\infty)$ such that $v_i(t,\cdot) \in C_b(D)$ for all $t \geq 0$ and

$$0 \leq v_i(t,x) \leq (B_1(t) \wedge B_2(t)) U_D(t,0)g(x), \quad t \geq 0.$$

In this way

$$\mathcal{F}_i v_i(t,x) := U_D(t,0)f_i(x) + \int_0^t U_D(t,r) v_{i'}^{\beta_i}(r,x)dr$$

satisfies

$$\begin{aligned}
0 \leq \mathcal{F}_i v_i(t,x) &\leq U_D(t,0)g(x) + \int_0^t B_i^{\beta_i}(r) U_D(t,r) \left(U_D(r,0)g(x)\right)^{\beta_i} dr \\
&\leq U_D(t,0)g(x) + \int_0^t B_i^{\beta_i}(r) U_D(t,r) U_D(r,0)g(x) \|U_D(r,0)g\|_\infty^{\beta_i - 1} dr \\
&= U_D(t,0)g(x) \left[1 + \int_0^t \|U_D(r,0)g\|_\infty^{\beta_i - 1} B_i^{\beta_i}(r)dr \right] \\
&= B_i(t) U_D(t,0)g(x),
\end{aligned}$$

where we used (11) in the last equality. Therefore,

$$0 \leq \mathcal{F}_i v_i(t,x) \leq (B_1(t) \vee B_2(t))\, U_D(t,0)g(x), \quad t \geq 0, \ x \in D.$$

We now define

$$u_{i,0}(t,x) = U_D(t,0)f_i(x) \quad \text{and} \quad u_{i,n+1}(t,x) = \mathcal{F}_i u_{i,n}(t,x), \quad n = 0, 1, \ldots.$$

Using that $u_{i,0}(t,x) \leq u_{i,1}(t,x)$ for all $t \geq 0$, $x \in D$, and the fact that $U_D(t,s)$ preserves positivity, it follows by induction that $u_{i,n}(t,x) \leq u_{i,n+1}(t,x)$, $n \geq 0$. Hence

$$u_i(t,x) \equiv \limsup_{n\to\infty} u_{i,n}(t,x) \leq (B_1(t) \vee B_2(t))\, U_D(t,0)g(x) < \infty$$

for all $t \geq 0$ and $x \in D$. From the monotone convergence theorem we conclude that $u_i(t,x)$ satisfies

$$u_i(t,x) = U_D(t,0)f_i(x) + \int_0^t U_D(t,r) u_{i'}^{\beta_i}(r,x)dr, \quad t \geq 0, \ x \in D.$$

Therefore, (u_1, u_2) is a global mild solution of (3). ■

5 Blow up in finite time of the positive mild solution

Without loss of generality, we assume in (3) that $\beta_2 \geq \beta_1 > 1$.

Theorem 4 *Let $f_1, f_2 \in C_0(D)$ be two nonnegative functions. If*

$$\min_{i\in\{1,2\}} \langle f_i, \varphi_0\rangle > \left[\frac{1}{\left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^\infty \min\limits_{i\in\{1,2\}} \left(\frac{e^{-\lambda_0 K(r,0)}}{\|\varphi_0\|_1}\right)^{\beta_i-1} dr}\right]^{\frac{\beta_1+1}{\beta_1\beta_2-1}}, \tag{12}$$

then the mild solution of (3) blows up in finite time.

Remark Notice that the above theorem is consistent with the corresponding result for the case of a single equation obtained in [13].

Before starting the proof of Theorem 4, notice the following.

Let $t_0 > 0$. If for some $0 < t \le t_0$,

$$\sup_{x\in D} u_1(t,x) + \sup_{x\in D} u_2(t,x) = \infty,$$

then the mild solution of (3) blows up in finite time and there is nothing to prove. Therefore, assume that

$$\sup_{x\in D,\ 0<t\le t_0} u_1(t,x) + \sup_{x\in D,\ 0<t\le t_0} u_2(t,x) < \infty.$$

Hence (see [14], pages 4 and 5),

$$\begin{aligned} u_i(t+t_0,x) &= U_D(t+t_0,t_0)\, u_i(t_0,x) \\ &+ \int_0^t U_D(t+t_0, r+t_0)\, u_{i'}^{\beta_i}(r+t_0,x)\, dr, \quad t\ge 0, \quad x\in D. \end{aligned} \tag{13}$$

Moreover, due to (9) and the fact that the integral in the right of (9) is positive,

$$u_i(t_0,x) \ge U_D(t_0,0)\, f_i(x), \quad x\in D. \tag{14}$$

Let (v_1, v_2) be the mild solution of (3) with initial values $v_i(0,\cdot) = U_D(2t_0,t_0)\, f_i(\cdot)$. Then

$$\begin{aligned} v_i(t+t_0,x) &= U_D(t+t_0,t_0)\, v_i(t_0,x) \\ &+ \int_0^t U_D(t+t_0, r+t_0)\, v_{i'}^{\beta_i}(r+t_0,x)\, dr, \quad t\ge 0, \quad x\in D. \end{aligned} \tag{15}$$

Since $\{U_D(t,s)\}_{t\ge s\ge 0}$ is an evolution system of contractions on $L^2(D)$, we have that for all $f\in L^2(D)$,

$$\|U_D(t,r) f\|_2 = \|U_D(t,s)\, U_D(s,r) f\|_2 \le \|U_D(s,r) f\|_2, \quad t\ge s\ge r\ge 0.$$

Now, for each $x\in D$, letting $g(y) = \frac{1}{\mathrm{Vol}(D)} f(x)$, $y\in D$, we get that $g\in L^2(D)$ and

$$U_D(t,r) f(x) = \|U_D(t,r)\, g\|_2 \le \|U_D(s,r)\, g\|_2 = U_D(s,r) f(x) \tag{16}$$

for all $t \geq s \geq r \geq 0$. From (16) we obtain

$$\begin{aligned}
v_i(t_0,x) &= \int_D p_D(K(t_0,0), x-y) U_D(2t_0,t_0) f_i(y)\,dy \\
&\quad + \int_0^{t_0} \left(\int_D p_D(K(t_0,r), x-y) v_{i'}^{\beta_i}(r,y)\,dy \right) dr \\
&= U_D(2t_0,0) f_i(x) + \int_0^{t_0} U_D(t_0,r) v_{i'}^{\beta_i}(r,x)\,dr \\
&\leq U_D(t_0,0) f_i(x) + \int_0^{t_0} U_D(t_0,0) v_{i'}^{\beta_i}(r,x)\,dr,
\end{aligned}$$

which gives $v_i(t_0,x) \leq u_i(t_0,x)$, $x \in D$. Plugging this into (15) yields

$$\begin{aligned}
v_i(t+t_0,x) &\leq U_D(t+t_0,t_0) u_i(t_0,x) \\
&\quad + \int_0^t U_D(t+t_0, r+t_0) v_{i'}^{\beta_i}(r+t_0,x)\,dr.
\end{aligned}$$

It follows that

$$v_i(t+t_0,x) \leq u_i(t+t_0,x), \quad t \geq 0, \quad x \in D.$$

Therefore, it suffices to proof that, under the hypothesis of Theorem 4, the mild solution of (3) blows up in finite time for all initial conditions of the form $u_i(0,x) = U_D(2t_0,t_0) f_i(x)$, $x \in D$, where f_1 and f_2 are not identically zero. Moreover, it is enough to show that (3) blows up for all initial values of the form $U_D(2t_0,t_0) h_i$, where h_1, h_2 are any two continuous functions with support contained in D, and such that $0 \leq h_i \leq f_i$, $i = 1,2$.

On the other hand, intrinsic ultracontractivity (8) implies that

$$\frac{U_D(2t_0,t_0) h_i}{\varphi_0} = \frac{S_D(K(2t_0,t_0)) h_i}{\varphi_0} \in C_b(D).$$

Thus, we can assume that the initial conditions in (3) are of the form $f_i = g_i \varphi_0$, with $0 \leq g_i \in C_b(D)$. In particular, we can assume that $0 \leq g_i \leq 1$.

We define

$$T(t,s) g(x) = \frac{e^{\lambda_0 K(t,s)}}{\varphi_0(x)} S_D(K(t,s)) (g\varphi_0)(x), \ x \in D, \ g \in C_b(D), \ t \geq 0$$

and

$$E[h] := \int h(x) \varphi_0^2(x)\,dx, \quad h \in C_b(D).$$

It is known ([13], p. 287) that $\varphi_0^2(x)\,dx$ is the unique invariant measure of the semigroup $\{Q(t)\}_{t \geq 0}$ given by

$$Q(t) g(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} S_D(t) (g\varphi_0)(x), \ x \in D, \ g \in C_b(D), \ t \geq 0,$$

and that $\{Q(t)\}_{t\geq 0}$ is a strongly Feller's semigroup with generator

$$\widehat{\Delta}_\alpha(\cdot) = \frac{\left(\overline{\Delta}_\alpha + \lambda_0\right)(\cdot\varphi_0)}{\varphi_0},$$

where $\overline{\Delta}_\alpha$ is the infinitesimal generator of the semigroup $\{S_D(t)\}_{t\geq 0}$.

Lemma 5 *The family of operators $\{T(t,s)\}_{t\geq s\geq 0}$ is an evolution family on $C_b(D)$ that solves the homogeneous nonautonomous Cauchy system*

$$\begin{aligned} \frac{\partial w(t,x)}{\partial t} &= k(t)\left(\overline{\Delta}_\alpha + \lambda_0\right)(\cdot\varphi_0)\, w(t,x), \quad t > s \geq 0, \quad x \in D, \\ w(s,x) &= g(x), \quad g \in C_b(D), \quad x \in D. \end{aligned}$$

Proof. The fact that $\{T(t,s)\}_{t\geq s\geq 0}$ is an evolution family on $C_b(D)$ follows directly from the semigroup property and strong continuity of $\{S_D(t)\}_{t\geq 0}$. For the last statement we have

$$\begin{aligned} \frac{\partial T(t,s)\,g}{\partial t} &= \frac{\lambda_0 k(t)\exp(\lambda_0 K(t,s))}{\varphi_0} S_D(K(t,s))(g\varphi_0) \\ &\quad + \frac{k(t)\exp(\lambda_0 K(t,s))}{\varphi_0}\overline{\Delta}_\alpha S_D(K(t,s))(g\varphi_0) \\ &= T(t,s)\left[k(t)\left(\overline{\Delta}_\alpha + \lambda_0\right)(\cdot\varphi_0)\right](g), \quad t \geq s \geq 0,\ g \in C_b(D). \end{aligned}$$

■

Lemma 6 *For any $t \geq s \geq 0$ and $g \in D\left(\widehat{\Delta}_\alpha\right)$,*

$$E[T(t,s)\,g] = E[g].$$

Proof. This is a consequence of the fact that $\varphi_0^2(x)\,dx$ is the unique invariant measure of $\{Q(t)\}_{t\geq 0}$, and of the relation $T(t,s)\,g = Q(K(t,s))\,g$. ■

In the sequel we are going to assume that the initial values in (3) are of the form $f_i = g_i\varphi_0$ with $g_i \in D\left(\widehat{\Delta}_\alpha\right)$, $i = 1,2$. Notice that $\langle f_i, \varphi_0\rangle = E[g_i]$, $i = 1,2$.

We define

$$w_i(t,x) = \frac{e^{\lambda_0 K(t,0)} u_i(t,x)}{\varphi_0(x)} \quad \text{and} \quad z(t,x) = e^{-\lambda_0 K(t,0)}\varphi_0(x), \ x \in D,\ t \geq 0,$$

where (u_1, u_2) is the mild solution of (3), i.e., (u_1, u_2) solves the integral system (9). Multiplying both sides of (9) by $\varphi_0(x)^{-1}\exp(\lambda_0 K(t,0))$, we get

$$
\begin{aligned}
& w_i(t,x) \\
= \;& T(t,0)\,g_i(x) + \int_0^t \frac{\exp(\lambda_0 K(t,0))}{\varphi_0(x)} U_D(t,r)\, u_{i'}^{\beta_i}(r,x)\,dr \\
= \;& T(t,0)\,g_i(x) + \int_0^t \frac{\exp(\lambda_0 K(t,0))}{\varphi_0(x)} U_D(t,r) \left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i-1}(x)} \varphi_0^{\beta_i-1}(x) \right) dr \\
= \;& T(t,0)\,g_i(x) + \int_0^t \exp(\lambda_0 K(t,0)) \exp(-\lambda_0 K(r,0)) \\
& \cdot \frac{\exp(\lambda_0 K(r,0))}{\varphi_0(x)} U_D(t,r) \left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i-1}(x)} \varphi_0^{\beta_i-1}(x) \right) dr \\
= \;& T(t,0)\,g_i(x) + \int_0^t \exp(\lambda_0 K(r,0)) \frac{\exp(\lambda_0 K(t,r))}{\varphi_0(x)} U_D(t,r) \left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i-1}(x)} \varphi_0^{\beta_i-1}(x) \right) dr \\
= \;& T(t,0)\,g_i(x) + \int_0^t \exp(\lambda_0 K(r,0))\, T(t,r) \left(\frac{u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i}(x)} \varphi_0^{\beta_i-1}(x) \right) dr \\
= \;& T(t,0)\,g_i(x) + \int_0^t T(t,r) \left(\frac{\exp(\lambda_0 K(r,0)\,\beta_i)\, u_{i'}^{\beta_i}(r,x)}{\varphi_0^{\beta_i}(x)} \right) \\
& \cdot \exp(-\lambda_0 K(r,0)(\beta_i - 1))\, \varphi_0^{\beta_i-1}(x)\,dr \\
= \;& T(t,0)\,g_i(x) + \int_0^t T(t,r)\, w_{i'}^{\beta_i}(r,x)\, z^{\beta_i-1}(r,x)\,dr.
\end{aligned}
$$

Proof of Theorem 4. From the last equality we get

$$
E[w_i(t,\cdot)] = E[T(t,0)\,g_i] + \int_0^t E\left[T(t,r)\, w_{i'}^{\beta_i}(r,\cdot)\, z^{\beta_i-1}(r,\cdot) \right] dr. \tag{17}
$$

Due to Lemma 6,

$$
\begin{aligned}
E\left[T(t,r)\, w_{i'}^{\beta_i}(r,\cdot)\, z^{\beta_i-1}(r,\cdot) \right] &= E\left[w_{i'}^{\beta_{i_i}}(r,\cdot)\, z^{\beta_i-1}(r,\cdot) \right] \\
&= e^{-\lambda_0 K(r,0)(\beta_i-1)} \int [w_{i'}(r,x)\,\varphi_0(x)]^{\beta_i} \varphi_0(x)\,dx \\
&\geq e^{-\lambda_0 K(r,0)(\beta_i-1)} \|\varphi_0\|_1 \left(\int w_{i'}(r,x) \frac{\varphi_0^2(x)}{\|\varphi_0\|_1} dx \right)^{\beta_i} \\
&= \left(\frac{\exp(-\lambda_0 K(r,0))}{\|\varphi_0\|_1} \right)^{\beta_i-1} E[w_{i'}(r,\cdot)]^{\beta_i},
\end{aligned}
$$

where we have used Jensen's inequality (with respect to the probability measure $\frac{\varphi_0(x)}{\|\varphi_0\|_1}dx$). Let $h_i(t) := E[w_i(t,\cdot)]$. Differentiating (17) with respect to t, we get

$$\begin{aligned} h_i'(t) &\geq \left(\frac{\exp(-\lambda_0 K(t,0))}{\|\varphi_0\|_1}\right)^{\beta_i-1} h_{i'}^{\beta_i}(t) \\ h_i(0) &= \langle f_i, \varphi_0\rangle . \end{aligned} \tag{18}$$

Let $c(t) = \min_{i\in\{1,2\}}\left\{\left(\frac{e^{-\lambda_0 K(t,0)}}{\|\varphi_0\|_1}\right)^{\beta_i-1}\right\}$, $N = \min_{i\in\{1,2\}}\{\langle f_i,\varphi_0\rangle\}$ and consider the ordinary differential system

$$p_1'(t) = c(t)p_2^{\beta_1}(t), \quad p_2'(t) = c(t)p_1^{\beta_2}(t), \quad p_i(0) = N, \ i = 1,2. \tag{19}$$

It follows that

$$\int_0^t p_1^{\beta_2}(r)p_1'(r)dr = \int_0^t p_2^{\beta_1}(r)p_2'(r)dr,$$

that is

$$\frac{1}{\beta_2+1}\left[p_1^{\beta_2+1}(t) - N^{\beta_2+1}\right] = \frac{1}{\beta_1+1}\left[p_2^{\beta_1+1}(t) - N^{\beta_1+1}\right].$$

Since $0 \leq N \leq 1$ and by assumption $\beta_2 \geq \beta_1$, we get

$$\frac{1}{\beta_2+1}p_1^{\beta_2+1}(t) \leq \frac{1}{\beta_1+1}p_2^{\beta_1+1}(t)$$

or

$$p_2(t) \geq \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{1}{\beta_1+1}} p_1^{\frac{\beta_2+1}{\beta_1+1}}(t).$$

Substituting this into the first equation of (19), we get

$$p_1'(t) \geq c(t)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} p_1^{\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t),$$

which is the same as

$$p_1^{-\frac{\beta_1(\beta_2+1)}{\beta_1+1}}(t)p_1'(t) \geq c(t)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}}.$$

Integrating the above inequality from 0 to t yields

$$\frac{\beta_1+1}{1-\beta_1\beta_2}\left[p_1^{\frac{1-\beta_1\beta_2}{\beta_1+1}}(t) - N^{\frac{1-\beta_1\beta_2}{\beta_1+1}}\right] \geq \left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)dr.$$

Thus, in view of $\beta_2 \geq \beta_1 > 1$,

$$p_1(t) \geq \frac{1}{\left[N^{\frac{1-\beta_1\beta_2}{\beta_1+1}} - \left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)dr\right]^{\frac{\beta_1}{\beta_1\beta_2-1}}}. \tag{20}$$

Since the function $\int_0^t c(r)\,dr$ is continuous and increases to $\int_0^\infty c(r)\,dr$, (20) implies finite-time blow up of (3) provided that

$$N^{\frac{1-\beta_1\beta_2}{\beta_1+1}} < \left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^t c(r)\,dr$$

or, equivalently,

$$\min_{i\in\{1,2\}} \langle f_i, \varphi_0\rangle > \left[\frac{1}{\left(\frac{\beta_1\beta_2-1}{\beta_1+1}\right)\left(\frac{\beta_1+1}{\beta_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_0^\infty \min_{i\in\{1,2\}} \left(\frac{e^{-\lambda_0 K(r,0)}}{\|\varphi_0\|_1}\right)^{\beta_i-1} dr}\right]^{\frac{\beta_1+1}{\beta_1\beta_2-1}}.$$

■

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