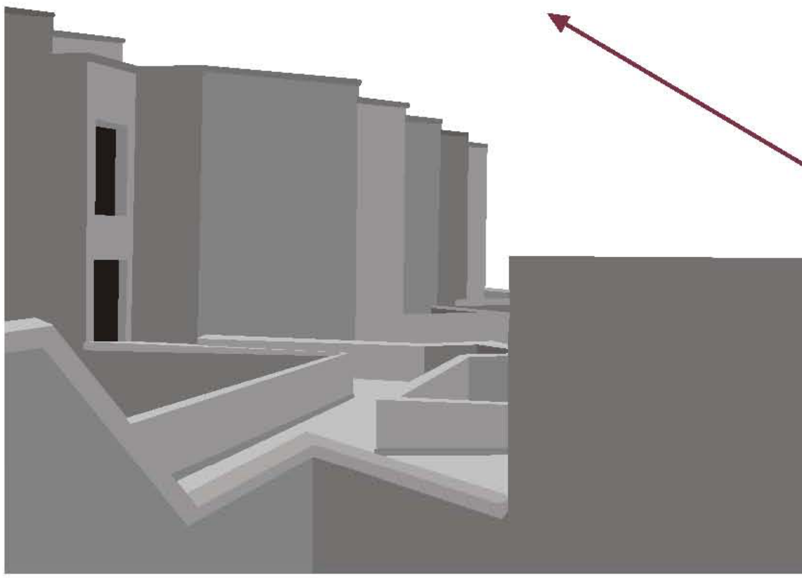

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DENDRIFORM ALGEBRAS AND ROTA-BAXTER OPERATORS REVISITED IN SEVERAL DIRECTIONS

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Dendriform algebras and Rota-baxter operators revisited in several directions

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Abstract

The main purpose of this article is to move the study of dendriform algebras and Rota-Baxter operators to a nonassociative setting beyond the Lie algebras. We show how to associate structures of dendriform type to alternative and flexible algebras and characterize the Rota-Baxter operators corresponding to them, in order to extend some results that have appeared in the literature for the associative case. These objects are studied in some detail. Also, we show that the usual version of Rota-Baxter operators acts on Leibniz algebras in the same form that they act on Lie algebras and in particular can be used into Leibniz-admissible algebras. As a consequence we arrive to the notion of admissible dendriform algebra. Additionally, we propose the concept of generalized dendriform algebra and describe a connection of it with the left-symmetric dialgebras recently introduced by the author.

Key words: Rota-Baxter operator, Dendriform algebras, Left-symmetric dialgebras. Leibniz-admissible algebras.

This paper is intended as a modest tribute to the work of J. L. Loday.

Introduction with some definitions and notations

In opinion of the author the Rota-Baxter operators and dendriform algebras have been studied to date, emphasizing their relation with associative algebras. Although there are some papers where these objects have been considered in their connections with Lie algebras, and in particular with left-symmetric algebras, statistically these publications represent a few exceptions. This work intends to fill this gap. Thus, our intention is to move the study of these things to the nonassociative algebras the best known of which are the flexible and alternative algebras. Thus, one of the objectives of this paper is to find the suitable dendriform analogs for flexible and alternative algebras and to propose their (flexible and alternative) Rota-Baxter operators.

Other main purpose of this work is to define a Rota-Baxter operator directly for the Leibniz bracket in such a way that one can construct other Leibniz bracket on the underlying vector space. Also, this allows us that if the Leibniz algebra is in particular a Leibniz-admissible algebra $(\mathfrak{L}, \dashv, \vdash)$ or a Lie-admissible algebra $(A, \cdot)$, the Rota-Baxter identity (8) written now in terms of the products of $\mathfrak{L}$ or the product of A can be induced easily. To our knowledge the classes of Rota-Baxter operators presented here have not been reported in the literature. Thus, in this sense we have proposed a version of the Rota-Baxter operator for a Leibniz algebra, which in particular can be used on an admissible-Leibniz algebra. As a consequence we arrived to the notion of admissible dendriform algebra. Also in this work, we have given the definition of generalized dendriform algebra which proves to be related with the concept of left-symmetric dialgebra, recently introduced by us in [9] (with relation to this comment the reader is referred to the paper [27]).

We wish to clarify that given the number of structures of dendriform type that are proposed in this work, we do not have space to deal with the representations of these structures on the specific spaces of trees, generalized power series rings or the set of words of some alphabet, etc.. Similarly, we do not have intention of to present examples of Rota-Baxter operators. This program could be developed in future works.

For an algebra $(A, \cdot)$ denote by A^- the algebra with multiplication $[x, y] = x \cdot y - y \cdot x$ defined on the vector space A. If A^- is a Lie algebra, then A is said to be a Lie-admissible algebra. Let $(\mathfrak{g}, [.,.])$ be a Lie algebra, then it is well know that if $R : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a linear operator such that $[Rx, Ry] = R([Rx, y] + [x, Ry])$, the product $[x, y]_R = [Rx, y] + [x, Ry]$ also constitutes a Lie bracket on $\mathfrak{g}$. This fact was used by Semenov-Tian-Shansky, Belavin and Drinfeld in the theory of integrable systems (see [8]). Next, we present other example of this type. A linear operator $N : \mathfrak{g} \longrightarrow \mathfrak{g}$ is called Nijenhuis if

$$[Nx, Ny] = N([Nx, y] + [x, Ny] + N[x, y]),$$

for all $x, y \in \mathfrak{g}$. Given a Nijenhuis operator, it can be showed that the operation

$$[.,.]_N = \mathfrak{g} \times \mathfrak{g}, [x, y]_N := [Nx, y] + [x, Ny] - N[x, y], x, y \in \mathfrak{g},$$

for all $x, y \in \mathfrak{g}$ is again a Lie algebra bracket (see [26]).

In the two cases presented above, there are analogous operators of R and N that satisfy well known identities for associative algebras and these linear operators are called Rota-Baxter operators. On other hand, Li, Hou and Bai [19] have considered operators of R type on the left-symmetric algebras, that is, for a special type of Lie-admissible algebra. We also report that P. A. Pozhidaev in [28] has constructed Rota-Baxter algebras from 0-dialgebras. A 0-dialgebra over a field K is a vector space D equipped with two binary operations $\dashv$ and $\vdash$ such that for all $x, y, z \in D$

$$x \dashv (y \dashv z) = x \dashv (y \vdash z), \tag{1}$$

$$(x \vdash y) \vdash z = (x \dashv y) \vdash z. \tag{2}$$

Pozhidaev gave some necessary and sufficient conditions for that the map $R(x) = \lambda_1 x + \lambda_2(x \vdash e) + \lambda_3(e \dashv x)$ can be a Rota-Baxter operator with respect to a new product $*$ on D defined by $x * y = \mu_1(x \dashv y) + \mu_2(x \vdash y) - \mu_3(y \dashv$

$x) + \mu_4(y \vdash x)$. On other hand, recently V. Yu. Gubarev and P.S. Kolesnikov address the question of embedding an arbitrary dendriform algebra (see [14]) into a Rota-Baxter algebra.

Leibniz algebras generalize Lie algebras, but with no symmetry requirements. The definition, given by Loday almost 20 years ago (see [20]), goes as follows: Let $\mathfrak{L}$ be a K-vector space and $[.,.] : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L}$ a bilinear map. We say that $(\mathfrak{L}, [.,.])$ is a Leibniz algebra if $[x,[y,z]] = [[x,y],z] - [[x,z],y]$ for all $x, y, z \in \mathfrak{L}$. Obviously, as it was indicated before all Lie algebra is a Leibniz algebra.

The following definition was given by the author in [9].

Definition 1 *Let $\mathfrak{L}$ be a K-vector space with two bilinear products $\vdash$ and $\dashv$. Denote by $\mathfrak{L}^-$ the algebra defined on the vector space $\mathfrak{L}$ with multiplication given by the bracket $[x,y] = x \dashv y - y \vdash x$. Then $\mathfrak{L}$ is said to be a* ***Leibniz-admissible algebra*** *if $\mathfrak{L}^-$ is a Leibniz algebra.*

It is clear that all Lie-admissible algebra is a Leibniz-admissible algebra in which both products coincide. Now, we recall the notion the Left-symmetric dialgebra (see also [9] for more details).

Definition 2 *Let S be a vector space over a field K. Let us assume that S is equipped with two bilinear products, not necessarily associative $\dashv: S \times S \to S$ $\vdash: S \times S \to S$ such that $(S, \vdash, \dashv)$ is a 0-dialgebra and*

$$x \dashv (y \dashv z) - (x \dashv y) \dashv z = y \vdash (x \dashv z) - (y \vdash x) \dashv z, \tag{3}$$

$$x \vdash (y \vdash z) - (x \vdash y) \vdash z = y \vdash (x \vdash z) - (y \vdash x) \vdash z, \tag{4}$$

for all $x, y, z \in S$, then we say that S is a ***left-symmetric dialgebra*** *(LSDA).*

Note that if $(S, \vdash, \dashv)$ is a left-symmetric dialgebra then $(S, \vdash)$ is a left-symmetric algebra and besides all left-symmetric algebra is a left-symmetric dialgebra.

The proof of the following result can be found in [9].

Theorem 3 *Let $(S, \vdash, \dashv)$ be a left-symmetric dialgebra. Then the Loday bracket*

$$[x,y]_S = x \dashv y - y \vdash x,$$

defines a structure of Leibniz algebra on S. In others words, $(S, [.,.]_S)$ is a Leibniz algebra denoted by S_{Leib}.

It is clear that any left-symmetric dialgebra is a Leibniz-admissible algebra.

An algebra $(A, \cdot)$ over a field K is **flexible** if it satisfies the identity $(x,y,x) = 0$ for all $x, y \in A$, where $(x,y,z) = x \cdot (y \cdot z) - (x \cdot y) \cdot z$ is the associator of three arbitrary elements x, y, z belong to A. If the characteristic of K is not two, then this identity is equivalent to $(x,y,z) = -(z,y,x)$. We say that $(A, \cdot)$ is **alternative** if $(x,y,z) = -(y,x,z)$ and $(x,y,z) = -(x,z,y)$. An alternative algebra is automatically flexible.

We remember the definition of dendriform algebra.

Definition 4 *A* ***dendriform algebra*** *is a vector space A together with maps: $\succ A \times A \longrightarrow A$ and $\succ A \times A \longrightarrow A$ such that for all $x, y, z \in A$*

$$x \succ (y \succ z) = (x \prec y + x \succ y) \succ z, \tag{5}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{6}$$

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z). \tag{7}$$

These algebras were introduced by Loday in [21] and in recent years have been the subject of intense study. They may be obtained of the following form: let $(A, \cdot)$ be an algebra, assume that the product can be split into two parts $x \cdot y = x \prec y + x \succ y$ satisfying the relation (5)-(7), then it is possible to establish associativity of the product $\cdot$. Thus, dendriform algebras are special associative algebras. From now on, a usual dendriform algebra will be called **Loday dendriform algebra**.

About 50 years ago, Baxter discovered the identity which is now known as the Rota-Baxter identity (formula (8)), it allowed him to deduce important results in the theory fluctuations of random variables. Subsequently Rota performed an algebraic and combinatorial analysis of this identity and defined the category of Rota-Baxter algebras. Since then, operators of type Rota-Baxter have appeared in several areas of pure and applied mathematics. For instance, lately in the Connes-Kreimer Hopf algebraic approach to renormalization.

Let $(A, \cdot)$ be an algebra. By an **ordinary Rota-Baxter operator** of weight θ we mean a linear map P from A into A satisfying the Baxter identity

$$P(x) \cdot P(y) = P(P(x) \cdot y + x \cdot P(y) + \theta x \cdot y), \tag{8}$$

for all $x, y \in A$. In this case, $(A, \cdot, P)$ is called a **ordinary Rota-Baxter algebra**. If A is associative then we say that P is an **associative Rota-Baxter operator** and $(A, \cdot, P)$ to be said an **associative Rota-Baxter algebra**.

In [2], the author indicated how one may associate an associative dendriform algebra to any associative algebra $(A, \cdot)$ equipped with a ordinary Rota-Baxter operator P of weight 0, i.e., it was shown that if $P : A \longrightarrow A$ is a Rota.Baxter operator then the pair of multiplications $x \succ y = P(x) \cdot y$ and $x \prec y = x \cdot P(y)$ is a dendriform structure on A.

Let A be a vector space with two bilinear operations $\prec$ and $\succ$. We define the **dendriform triple systems** for any $x, y, z \in A$ in the way

$$(x, y, z)_1 = x \succ (y \succ z) - (x \prec y + x \succ y) \succ z, \tag{9}$$

$$(x, y, z)_2 = x \succ (y \prec z) - (x \succ y) \prec z, \tag{10}$$

$$(x, y, z)_3 = (x \prec y) \prec z - x \prec (y \prec z + y \succ z). \tag{11}$$

These dendriform triple systems are called right, central and left dendriform triple system respectively. It is easy to show that

$$(x, y, z) = (x, y, z)_1 + (x, y, z)_2 - (x, y, z)_3, \tag{12}$$

where (x, y, z) is the associator for the product $x \cdot y = x \succ y + x \prec y$.

Then (12) tells us clearly that the study of associator (x, y, z) can be made through the dendriform triple systems. For instance, as we already know, from (12) follows that the conditions $(x, y, z)_1 = (x, y, z)_2 = (x, y, z)_3 = 0$ are enough in order to the algebra $(A, \cdot)$ becomes an associative algebra. However, in order that the associator go to vanish, it is not necessary that the dendriform triple systems vanish simultaneously. Thus, we have of following definition

Definition 5 *An* ***associative dendriform algebra*** $(A, \prec, \succ)$ *is a vector space with two bilinear maps* $\prec$ *and* $\succ$ *each of which maps* A *into* A *satisfying at least one of the following conditions:*

- $(x, y, z)_1 = (x, y, z)_2 = (x, y, z)_3 = 0,$ (13)
- $(x, y, z)_1 = 0, \quad (x, y, z)_2 = (x, y, z)_3,$ (14)
- $(x, y, z)_1 = (x, y, z)_3, \quad (x, y, z)_2 = 0,$ (15)
- $(x, y, z)_1 + (x, y, z)_2 = 0, \quad (x, y, z)_3 = 0,$ (16)
- $(x, y, z)_1 + (x, y, z)_2 - (x, y, z)_3 = 0.$ (17)

It is clear that any associative dendriform algebra $(A, \prec, \succ)$ is an associative algebra with respect to the new product defined by $x \cdot y = x \succ y + x \prec y$ for all $x, y \in A$. Observe that on any associative dendriform algebra

$$\begin{aligned} x \cdot y - y \cdot x &= (x \succ y + x \prec y) - (y \succ x + y \prec x) \\ &= (x \succ y - y \prec x) - (y \succ x - x \prec y) \\ &= x \circ y - y \circ x, \end{aligned}$$

where $x \circ y = x \succ y - y \prec x$. Thus, the products $\cdot$ and $\circ$ define the same Lie structure on A. If (13) holds it is well known that $(A, \circ)$ is a left-symmetric algebra, that is,

$$x \circ (y \circ z) - (x \circ y) \circ z = y \circ (x \circ z) - (y \circ x) \circ z.$$

Of course, (13)-(17) are not all cases for which the associator is canceled, but we can say that they are all cases related to the axioms initially proposed by Loday to define his remarkable Loday dendriform algebra. We recall that of similar way one can define right-symmetric algebras.

Again let $(A, \cdot)$ be an algebra. We can use three dendriform operations to express the initial multiplication $\cdot$, in other words, assume $x \cdot y = x \succ y + x \prec y + x \circ y$. Then

$$(x, y, z) = \sum_{k=1}^{7} ((x, y, z))_k, \tag{18}$$

for all $x, y, z \in A$, where

$$((x, y, z))_1 = x \prec (y \succ z + y \prec z + y \circ z) - (x \prec y) \prec z, \tag{19}$$
$$((x, y, z))_2 = x \succ (y \prec z) - (x \succ y) \prec z, \tag{20}$$
$$((x, y, z))_3 = x \succ (y \succ z) - (x \succ y + x \prec y + x \circ y) \succ z, \tag{21}$$
$$((x, y, z))_4 = x \succ (y \circ z) - (x \succ y) \circ z, \tag{22}$$
$$((x, y, z))_5 = x \circ (y \succ z) - (x \prec y) \circ z, \tag{23}$$
$$((x, y, z))_6 = x \circ (y \prec z) - (x \circ y) \prec z, \tag{24}$$
$$((x, y, z))_7 = x \circ (y \circ z) - (x \circ y) \circ z. \tag{25}$$

Loday and Ronco in [22] defined a dendriform trialgebra as those $(A, \prec, \succ, \circ)$ for which $((x, y, z))_k = 0$, for $k = 1, \ldots, 7$. In this case, $(A, \cdot)$ is necessarily an associative algebra. The triple systems $((x, y, z))_k$, for $k = 1, \ldots, 7$ could be called **dendriform triple systems**.

Let K be a field. Given two vector spaces V and U over K, we denote by $V \otimes U$ its tensor product. Denote by V^* the dual space of V, that is, the space of all linear functionals on V. Given $f \in V^*$ and $v \in V$, the symbol $< f, v >$ denotes the linear functional f evaluated at v.

Let A be a vector space over K and let $P : A \longrightarrow A$ be a linear operator. Then, it induces a linear operator $P_I : A^* \longrightarrow A^*$ defined of the following form $< P_I f, a > = < f, Pa >$ for all $f \in A^*$ and any $a \in A$.

Definition 6 *A pair $(A, \triangle)$ where A is a vector space and $A \longrightarrow A \otimes A$ is a linear mapping, will be called a* ***coalgebra****. The mapping $\triangle$ is called a* ***comultiplication****.*

In what follows, given an element a in a coalgebra $(A, \triangle)$, we will express to $\triangle(a)$ of two different manners, the first is $\triangle(a) = a^{'} \otimes a^{''}$ and the second is $\triangle(a) = \sum_a a_{(1)} \otimes a_{(2)}$. Let $(A, \triangle)$ be a coalgebra. Define on A^* a multiplication in the form

$$< fg, a > = \sum_a < f, a_{(1)} >< g, a_{(2)} >, \tag{26}$$

where $f, g \in A^*$, for any finite expansion $\triangle(a) = \sum_a a_{(1)} \otimes a_{(2)} \in A \otimes A$. The so-obtained algebra is called the **dual algebra** of the coalgebra $(A, \triangle)$. This dual algebra will be associative whenever the coproduct $\triangle$ is coassociative, that is, if $(Id \otimes \triangle) \circ \triangle(a) = (\triangle \otimes Id) \circ \triangle(a)$.

The dual algebra A^* of a coalgebra $(A, \triangle)$ determines two bimodule actions $(\circ)$ on A:

$$f \circ a = \sum_a a_{(1)} < f, a_{(2)} >, \quad a \circ f = \sum_a < f, a_{(1)} > a_{(2)}, \tag{27}$$

where $f \in A^*$ and $\triangle(a) = \sum_a a_{(1)} \otimes a_{(2)}$.

The following definition was given in [4]

Definition 7 *Let M be a variety of algebras over K. Then a pair $(A, \triangle)$ is called a M-coalgebra if the dual algebra A^* belongs to M.*

In particular, a study about alternative-coalgebras may be found in [11].

Let $(A, \cdot)$ be an algebra with comultiplication $\triangle$, and let A^* be the dual algebra of $(A, \triangle)$. The algebra A induces the bimodule action $(\bullet)$ on A^* by the formulas

$$< f \bullet a, b > = < f, a \cdot b >, \quad < b \bullet f, a > = < f, a \cdot b > . \tag{28}$$

Consider the space $D(A) = A \oplus A^*$ and equip it with multiplication by means of

$$(a + f) * (b + g) = (a \cdot b + f \circ b + a \circ g) + (fg + f \bullet b + a \bullet g), \tag{29}$$

then $D(A)$ is a usual algebra over K; A and A^* are subalgebras in $D(A)$. The algebra $D(A)$ is the **Drinfeld double** of the coalgebra $(A, \triangle)$.

We recall the definitions of left-symmetric coalgebra and Lie coalgebra [29], [30]. We define the associator of a coproduct $\triangle(a)$ as

$$A_{\triangle} = (Id \otimes \triangle) \circ \triangle - (\triangle \otimes Id) \circ \triangle. \tag{30}$$

For a K-vector spaces A, denote by σ the permutation $a \otimes b \longrightarrow b \otimes a$ in $A \otimes A = A^{\otimes 2}$. Furthermore, we denote by τ and ϕ the permutations $a \otimes b \otimes c \longrightarrow c \otimes a \otimes b$ and $a \otimes b \otimes c \longrightarrow b \otimes c \otimes a$ respectively in $A \otimes A \otimes A = A^{\otimes 3}$. Let $P_A^{i,j}$ be the endomorphism of $A \otimes A \otimes A$ permuting the i-th and j-th tensor factors. A **left-symmetric coalgebra** is a K-vector spaces A endowed with a coproduct $\triangle : A \longrightarrow A^{\otimes 2}$ such that

$$(Id_{A^{\otimes 3}} - P^{1,2})A_{\triangle} = 0 \in Hom_K(A, A^{\otimes 3}). \tag{31}$$

Clearly, any coassociative coalgebra is a left-symmetric coalgebra. Given a left-symmetric coalgebra $(A, \triangle)$, the coproduct $\triangle$ induces a structure of ordinary left-symmetric algebra over the dual algebra A^*.

A **Lie coalgebra** is a K-vector space A endowed with a coproduct $\triangle : A \longrightarrow A^{\otimes 2}$ such that $\sigma \circ \triangle = -\triangle$ and

$$(Id_{A^{\otimes 3}} + \tau + \tau^2) \circ (Id \otimes \triangle) \circ \triangle = 0 \in Hom_K(A, A^{\otimes 3}). \tag{32}$$

A lie coalgebra $(A, \triangle)$ gives rise to a structure of ordinary Lie algebra on the dual algebra A^*. On the other hand, for any left-symmetric coalgebra $(A, \triangle)$, the comultiplication $\triangle_I = \triangle - \sigma\triangle : A \longrightarrow A \otimes A$ is a Lie cobracket, that is, $(A, \triangle_I)$ is a Lie coalgebra.

The following definition is due to Foissy [10].

Definition 8 *A* ***dendriform coalgebra*** *is a triple* $(A, \triangle_{\succ}, \triangle_{\prec})$ *where* A *is a* K*-vector space and* $\triangle_{\succ}, \triangle_{\prec} : A \longrightarrow A \otimes A$, $a \longrightarrow \triangle_{\succ}(a) = a'_{\succ} \otimes a''_{\succ}$, $a \longrightarrow \triangle_{\prec}(a) = a'_{\prec} \otimes a''_{\prec}$ *such that*

$$(\triangle_{\prec} \otimes Id) \circ \triangle_{\prec}(a) = (Id \otimes \triangle_{\prec} + Id \otimes \triangle_{\succ}) \circ \triangle_{\prec}(a), \tag{33}$$

$$(\triangle_{\succ} \otimes Id) \circ \triangle_{\prec}(a) = (Id \otimes \triangle_{\prec}) \circ \triangle_{\succ}(a), \tag{34}$$

$$(\triangle_{\prec} \otimes Id + \triangle_{\succ} \otimes Id) \circ \triangle_{\succ}(a) = (Id \otimes \triangle_{\succ}) \circ \triangle_{\succ}(a). \tag{35}$$

It is easy to show that if $(A, \triangle_{\succ}, \triangle_{\prec})$ is a dendriform coalgebra, then from (33)-(35) follow that $a \longrightarrow \widetilde{\triangle}(a) = \triangle_{\succ}(a) + \triangle_{\prec}(a)$ is a coassociative coproduct. Thus, a dendriform coalgebra is a special coassociative coalgebra. On other hand, for (33)-(35) it follows that

$$A_{\widetilde{\triangle}} = A_1(\widetilde{\triangle}) + A_2(\widetilde{\triangle}) + A_3(\widetilde{\triangle}), \tag{36}$$

where

$$A_1(\widetilde{\triangle}) = (Id \otimes \triangle_{\prec} + Id \otimes \triangle_{\succ}) \circ \triangle_{\prec} - (\triangle_{\prec} \otimes Id) \circ \triangle_{\prec}, \tag{37}$$

$$A_2(\widetilde{\triangle}) = (Id \otimes \triangle_{\prec}) \circ \triangle_{\succ} - (\triangle_{\succ} \otimes Id) \circ \triangle_{\prec}, \tag{38}$$

$$A_3(\widetilde{\triangle}) = (Id \otimes \triangle_{\succ}) \circ \triangle_{\succ} - (\triangle_{\prec} \otimes Id + \triangle_{\succ} \otimes Id) \circ \triangle_{\succ}. \tag{39}$$

Eq. (36) is the reason for which the condition $A_1(\widetilde{\triangle}) = A_2(\widetilde{\triangle}) = A_3(\widetilde{\triangle}) = 0$ implies that $\widetilde{\triangle}$ is a coassociative coproduct.

Associative dendriform algebras and Rota-Baxter operators revisited as motivation

We begin by studying Rota-baxter operators on Drinfeld double.

Let $(A, \triangle)$ be an arbitrary coalgebra. Consider two linear operators $P : A \longrightarrow A$ and $P^* : A^* \longrightarrow A^*$. We define the action of the pair $\widetilde{P} = (P, P^*)$ over $D(A)$ as $\widetilde{P}(a+f) = (Pa + P^*f)$ for any $(a+f) \in D(A)$. It is clear that $\widetilde{P}$ is a linear operator mapping $D(A)$ into $D(A)$.

Theorem 9 *Let $(A, \triangle)$ be an arbitrary coalgebra. Then $(D(A), *, \widetilde{P})$ is an ordinary Rota-Baxter algebra of weight θ if and only if $(A, \cdot, P)$ and (A^*, P^*) are ordinary Rota-Baxter algebras of weight θ and*

$$Pa \circ P^*g = P(Pa \circ g + a \circ P^*g + \theta a \circ g), \tag{40}$$

$$P^*f \circ Pb = P(P^*f \circ b + f \circ Pb + \theta f \circ b), \tag{41}$$

$$Pa \bullet P^*g = P^*(Pa \bullet g + a \bullet P^*g + \theta a \bullet g), \tag{42}$$

$$P^*f \bullet Pb = P^*(P^*f \bullet b + f \bullet Pb + \theta f \bullet b), \tag{43}$$

for all $a, b \in A$ and any $f, g \in A^*$.

Proof. Necessity. Suppose that $\widetilde{P}$ is an ordinary Rota-baxter algebra then

$$\widetilde{P}(a+f)*\widetilde{P}(b+g) = \widetilde{P}(\widetilde{P}(a+f)*(b+g)+(a+f)*\widetilde{P}(b+g)+\theta(a+f)*(b+g)). \tag{44}$$

It is straightforward to check that

$$\begin{aligned}\widetilde{P}(a+f) * \widetilde{P}(b+g) = &(Pa \cdot Pb + P^*f \circ Pb + Pa \circ P^*g)\\ &+ ((P^*f)(P^*g) + P^*f \bullet Pb + Pa \bullet P^*g).\end{aligned} \tag{45}$$

$$\begin{aligned}\widetilde{P}(a+f) * (b+g) = &(Pa \cdot b + P^*f \circ b + Pa \circ g)\\ &+ ((P^*f)g + P^*f \bullet b + Pa \bullet g).\end{aligned} \tag{46}$$

$$\begin{aligned}(a+f) * \widetilde{P}(b+g) = &(a \cdot Pb + f \circ Pb + a \circ P^*g)\\ &+ (f(P^*g) + f \bullet Pb + a \bullet P^*g).\end{aligned} \tag{47}$$

Therefore, equations (29) and (45)-(47) allow us to project the Eq. (44) on A and A^* respectively. Thus, we have

$$\begin{aligned}(Pa \cdot Pb + P^*f \circ Pb + Pa \circ P^*g) = &P(Pa \cdot b + P^*f \circ b + Pa \circ g)\\ &+ P(a \cdot Pb + f \circ Pb + a \circ P^*g)\\ &+ \theta P(a \cdot b + f \circ b + a \circ g),\end{aligned} \tag{48}$$

and

$$\begin{aligned}((P^*f)(P^*g) + P^*f \bullet Pb + Pa \bullet P^*g) = &((P^*f)g + P^*f \bullet b + Pa \bullet g)\\ &+ (f(P^*g) + f \bullet Pb + a \bullet P^*g)\\ &+ \theta(fg + f \bullet b + a \bullet g).\end{aligned} \tag{49}$$

If $f = g = 0$, equation (49) is canceled, whereas computing both side of equation (48) we see that $(A, \cdot, P)$ must be an ordinary Rota-Baxter algebra. Taking $a = 0$ and $b = 0$ we arrive to the same requirement for (A, P^*). The remaining non trivial cases, that is, $a = 0$, $g = 0$ and $f = 0$, $b = 0$ lead us to equations (40)-(43).

It is immediate that the hypothesis of the theorem ensure that $(D(A), *, \widetilde{P})$ is an ordinary Rota-Baxter algebra of weight θ. ■

A basic problem is to define an analog of Rota-Baxter operator for dendriform coalgebras. Let $(A, \triangle)$ be an arbitrary coassociative coalgebra and let $P : A \longrightarrow A$ be a linear operator. For $a \in A$ we define $\triangle_{\succ}(a) = Pa' \otimes a''$ and $\triangle_{\prec}(a) = a' \otimes Pa''$ where $\triangle(a) = a' \otimes a''$. We come now to the description of those P for which $(A, \triangle_{\succ}, \triangle_{\prec})$ is a dendriform coalgebra. Being $\triangle(a) = a' \otimes a''$

$$(\triangle_{\prec} \otimes Id) \circ \triangle_{\prec}(a) = (\triangle_{\prec} \otimes Id)(a' \otimes Pa'') = (a')' \otimes P(a')'' \otimes Pa'', \tag{50}$$

$$\begin{aligned}(Id \otimes \triangle_{\prec} + Id \otimes \triangle_{\succ}) \circ \triangle_{\prec}(a) &= (Id \otimes \triangle_{\prec} + Id \otimes \triangle_{\succ})(a' \otimes Pa'') \\ &= a' \otimes (Pa'')' \otimes P(Pa'')'' \\ &\quad + a' \otimes P(Pa'')' \otimes (Pa'')'',\end{aligned} \tag{51}$$

so, of (50) and (51) we conclude that

$$(a')' \otimes P(a')'' \otimes Pa'' = a' \otimes (Pa'')' \otimes P(Pa'')'' + a' \otimes P(Pa'')' \otimes (Pa'')''.$$

It is a simple matter to check that (34) takes the following form

$$P(a')' \otimes (a')'' \otimes Pa'' = Pa' \otimes (a'')' \otimes P(a'')'',$$

Next, we will examine (35) for any $a \in A$ and $\triangle(a) = a' \otimes a''$. It is readily checked that (35) leads us to the equality

$$Pa' \otimes P(a'')' \otimes (a'')'' = P(Pa')' \otimes (Pa')'' \otimes a'' + (Pa')' \otimes P(Pa')'' \otimes a''.$$

We propose the following definition

Definition 10 *Let $(A, \triangle)$be an arbitrary coassociative coalgebra. A liner operator $P : A \longrightarrow A$ is called coassociative Rota-baxter operator if for all $a \in A$*

$$(a')' \otimes P(a')'' \otimes Pa'' = a' \otimes (Pa'')' \otimes P(Pa'')'' + a' \otimes P(Pa'')' \otimes (Pa'')'', \tag{52}$$

$$P(a')' \otimes (a')'' \otimes Pa'' = Pa' \otimes (a'')' \otimes P(a'')'', \tag{53}$$

$$Pa' \otimes P(a'')' \otimes (a'')'' = P(Pa')' \otimes (Pa')'' \otimes a'' + (Pa')' \otimes P(Pa')'' \otimes a'', \tag{54}$$

where $\triangle(a) = a' \otimes a''$. The triple $(A, \triangle, P)$ is called a Rota-Baxter coassociative coalgebra.

Let $(A, \triangle_{\succ}, \triangle_{\prec})$ be a dendriform coalgebra. Then, the coproducts $\triangle_{\succ}$ and $\triangle_{\prec}$ lead two multiplications on A^*

$$< f \succ g, a >=< f, a'_{\succ} >< g, a''_{\succ} >, \quad < f \prec g, a >=< f, a'_{\prec} >< g, a''_{\prec} >, \tag{55}$$

where $\triangle_{\succ}(a) = a^{'}_{\succ} \otimes a^{''}_{\succ}$ and $\triangle_{\prec}(a) = a^{'}_{\prec} \otimes a^{''}_{\prec}$. Clearly, the new coproduct $\widetilde{\triangle}(a) = \triangle_{\succ}(a) + \triangle_{\prec}(a) = a^{'}_{\succ} \otimes a^{''}_{\succ} + a^{'}_{\prec} \otimes a^{''}_{\prec}$ induces on A^* the following product

$$< fg >=< f, a^{'}_{\succ} >< g, a^{''}_{\succ} > + < f, a^{'}_{\prec} >< g, a^{''}_{\prec} >, \tag{56}$$

for all $f, g \in A^*$. Observe that $fg = f \succ g + f \prec g$. The product (56) coincides with (26) and since $\widetilde{\triangle}$ is coassociative it will be associative. But there is another reason whereby this product can be associative which will be explained below (see theorem 11). Let $R : a \longrightarrow a^{'} \otimes a^{''} \otimes a^{'''}$ be a map of A into $A \otimes A \otimes A$. It determines a triple system $\{f, g, h\}_R$ on A^* defined as

$$< \{f, g, h\}_R, a >=< f, a^{'} >< g, a^{''} >< h, a^{'''} > . \tag{57}$$

This can be extended by linearity to the most general case in which $R(a) = \sum_a a^{'} \otimes a^{''} \otimes a^{'''} \in A \otimes A$. Obviously, if $R(a) = R_1(a) + R_2(a)$ where $R_1 : a \longrightarrow a^{'}_1 \otimes a^{''}_1 \otimes a^{'''}_1$ and $R_2 : a \longrightarrow a^{'}_2 \otimes a^{''}_2 \otimes a^{'''}_2$ then $\{f, g, h\}_R = \{f, g, h\}_{R_1} + \{f, g, h\}_{R_2}$ as elements of A^*.

Observe now that within of the axioms of a dendriform coalgebra we can find eight different maps of A into $A \otimes A \otimes A$:

$$R_1(a) = (\triangle_{\prec} \otimes Id) \circ \triangle_{\prec}(a) = (a^{'}_{\prec})^{'}_{\prec} \otimes (a^{'}_{\prec})^{''}_{\prec} \otimes a^{''}_{\prec}, \tag{58}$$

$$R_2(a) = (Id \otimes \triangle_{\prec}) \circ \triangle_{\prec}(a) = a^{'}_{\prec} \otimes (a^{''}_{\prec})^{'}_{\prec} \otimes (a^{''}_{\prec})^{''}_{\prec}, \tag{59}$$

$$R_3(a) = (Id \otimes \triangle_{\succ}) \circ \triangle_{\prec}(a) = a^{'}_{\prec} \otimes (a^{''}_{\prec})^{'}_{\succ} \otimes (a^{''}_{\prec})^{''}_{\succ}, \tag{60}$$

$$R_4(a) = (\triangle_{\succ} \otimes Id) \circ \triangle_{\prec}(a) = (a^{'}_{\prec})^{'}_{\succ} \otimes (a^{'}_{\prec})^{''}_{\succ} \otimes a^{''}_{\prec}, \tag{61}$$

$$R_5(a) = (Id \otimes \triangle_{\prec}) \circ \triangle_{\succ}(a) = a^{'}_{\succ} \otimes (a^{''}_{\succ})^{'}_{\prec} \otimes (a^{''}_{\succ})^{''}_{\prec}, \tag{62}$$

$$R_6(a) = (Id \otimes \triangle_{\succ}) \circ \triangle_{\succ}(a) = a^{'}_{\succ} \otimes (a^{''}_{\succ})^{'}_{\succ} \otimes (a^{''}_{\succ})^{''}_{\succ}, \tag{63}$$

$$R_7(a) = (\triangle_{\prec} \otimes Id) \circ \triangle_{\succ}(a) = (a^{'}_{\succ})^{'}_{\prec} \otimes (a^{'}_{\succ})^{''}_{\prec} \otimes a^{''}_{\succ}, \tag{64}$$

$$R_8(a) = (\triangle_{\succ} \otimes Id) \circ \triangle_{\succ}(a) = (a^{'}_{\succ})^{'}_{\succ} \otimes (a^{'}_{\succ})^{''}_{\succ} \otimes a^{''}_{\succ}. \tag{65}$$

we are ready to state the following theorem.

Theorem 11 *Let $(A, \triangle_{\succ}, \triangle_{\prec})$ be a dendriform coalgebra. Then $(A^*, \prec, \succ)$ is a Loday dendriform algebra, where $\succ$ and $\prec$ are the products on A^* which were associated above with $\triangle_{\succ}$ and $\triangle_{\prec}$ respectively.*

Proof. On the one hand, for any $f, g, h \in A^*$ we have

$$\{f, g, h\}_{R_1} = (f \prec g) \prec h, \quad \{f, g, h\}_{R_2} = f \prec (g \prec h), \tag{66}$$

$$\{f, g, h\}_{R_3} = f \prec (g \succ h), \quad \{f, g, h\}_{R_4} = (f \succ g) \prec h, \tag{67}$$

$$\{f, g, h\}_{R_5} = f \succ (g \prec h), \quad \{f, g, h\}_{R_6} = f \succ (g \succ h), \tag{68}$$

$$\{f, g, h\}_{R_7} = (f \prec g) \succ h, \quad \{f, g, h\}_{R_8} = (f \succ g) \succ h. \tag{69}$$

And secondly, taking into account that $(A, \triangle_{\succ}, \triangle_{\prec})$ is a dendriform coalgebra, from (58)-(65) follow that $R_1 = R_2 + R_3$, $R_4 = R_5$ and $R_6 = R_7 + R_8$.

The theorem is proved. ■

Far as we know the following result has not been reported in the literature.

Theorem 12 *Let $(A, \triangle_{\succ}, \triangle_{\prec})$ be a dendriform coalgebra. Define $\widehat{\triangle} = \triangle_{\succ} - \sigma \circ \triangle_{\prec}$, then $(A, \widehat{\triangle})$ is a left-symmetric coalgebra.*

Proof. One can see that for any $a \in A$

$$
\begin{aligned}
A_{\widehat{\triangle}}(a) = {} & (Id \otimes \triangle_{\succ}) \circ \triangle_{\succ}(a) - (Id \otimes \triangle_{\succ}) \circ \sigma\triangle_{\prec}(a) \\
& - (Id \otimes \sigma\triangle_{\prec}) \circ \triangle_{\succ}(a) + (Id \otimes \sigma\triangle_{\prec}) \circ \sigma\triangle_{\prec}(a) \\
& - (\triangle_{\succ} \otimes Id) \circ \triangle_{\succ}(a) + (\triangle_{\succ} \otimes Id) \circ \sigma\triangle_{\prec}(a) \\
& + (\sigma\triangle_{\prec} \otimes Id) \circ \triangle_{\succ}(a) - (\sigma\triangle_{\prec} \otimes Id) \circ \sigma\triangle_{\prec}(a).
\end{aligned} \tag{70}
$$

Thus, if we calculate explicitly each term of (70) one obtains

$$
\begin{aligned}
A_{\widehat{\triangle}}(a) = {} & a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime\prime}_{\succ} - a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime}_{\succ} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\succ} \\
& - a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime\prime}_{\prec} \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\prec} + a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime}_{\prec} \\
& - (a^{\prime}_{\succ})^{\prime}_{\succ} \otimes (a^{\prime}_{\succ})^{\prime\prime}_{\succ} \otimes a^{\prime\prime}_{\succ} + (a^{\prime\prime}_{\prec})^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\prec})^{\prime\prime}_{\succ} \otimes a^{\prime}_{\prec} \\
& + (a^{\prime}_{\succ})^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\succ})^{\prime}_{\prec} \otimes a^{\prime\prime}_{\succ} - (a^{\prime\prime}_{\prec})^{\prime\prime}_{\prec} \otimes (a^{\prime\prime}_{\prec})^{\prime}_{\prec} \otimes a^{\prime}_{\prec}.
\end{aligned} \tag{71}
$$

It implies that

$$
\begin{aligned}
P^{1,2}A_{\widehat{\triangle}}(a) = {} & (a^{\prime\prime}_{\succ})^{\prime}_{\succ} \otimes a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime\prime}_{\succ} - (a^{\prime}_{\prec})^{\prime}_{\succ} \otimes a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\succ} \\
& - (a^{\prime\prime}_{\succ})^{\prime\prime}_{\prec} \otimes a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\prec} + (a^{\prime}_{\prec})^{\prime\prime}_{\prec} \otimes a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime}_{\prec} \\
& - (a^{\prime}_{\succ})^{\prime\prime}_{\succ} \otimes (a^{\prime}_{\succ})^{\prime}_{\succ} \otimes a^{\prime\prime}_{\succ} + (a^{\prime\prime}_{\prec})^{\prime\prime}_{\succ} \otimes (a^{\prime\prime}_{\prec})^{\prime}_{\succ} \otimes a^{\prime}_{\prec} \\
& + (a^{\prime}_{\succ})^{\prime}_{\prec} \otimes (a^{\prime}_{\succ})^{\prime\prime}_{\prec} \otimes a^{\prime\prime}_{\succ} - (a^{\prime\prime}_{\prec})^{\prime}_{\prec} \otimes (a^{\prime\prime}_{\prec})^{\prime\prime}_{\prec} \otimes a^{\prime}_{\prec}.
\end{aligned} \tag{72}
$$

We claim that $A_{\widehat{\triangle}}(a) - P^{1,2}A_{\widehat{\triangle}}(a) = 0$ for all $a \in A$. To prove the claim it will be useful to write the axioms (33)-(35) of an explicit manner through their tensorial factors, that is

$$
(a^{\prime}_{\prec})^{\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\prec} \otimes a^{\prime\prime}_{\prec} = a^{\prime}_{\prec} \otimes (a^{\prime\prime}_{\prec})^{\prime}_{\prec} \otimes (a^{\prime\prime}_{\prec})^{\prime\prime}_{\prec} + a^{\prime}_{\prec} \otimes (a^{\prime\prime}_{\prec})^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\prec})^{\prime\prime}_{\succ}, \tag{73}
$$

$$
(a^{\prime}_{\prec})^{\prime}_{\succ} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\succ} \otimes a^{\prime\prime}_{\prec} = a^{\prime} \succ \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\prec} \otimes (a^{\prime\prime}_{\succ})^{\prime\prime}_{\prec}, \tag{74}
$$

$$
a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime\prime}_{\succ} = (a^{\prime}_{\succ})^{\prime}_{\prec} \otimes (a^{\prime}_{\succ})^{\prime\prime}_{\prec} \otimes a^{\prime\prime}_{\succ} + (a^{\prime}_{\succ})^{\prime}_{\succ} \otimes (a^{\prime}_{\succ})^{\prime\prime}_{\succ} \otimes a^{\prime\prime}_{\succ}. \tag{75}
$$

Next, we will identify in the expression $A_{\widehat{\triangle}}(a) - P^{1,2}A_{\widehat{\triangle}}(a)$ eight groups of terms which vanish. First of all, note that one of the groups match with the axiom (35), therefore there is nothing to prove in this case (terms 1 and 5 of (71) and term 7 of (72) with their respective signs). Now, we claim that

$$
(a^{\prime}_{\prec})^{\prime}_{\succ} \otimes a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\succ} = a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime\prime}_{\prec} \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\prec}. \tag{76}
$$

In fact, the equality (76) can be obtained from (74) by applying to both sides of this identity the homomorphism $P^{2,3}$. In the same way

$$
a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime}_{\succ} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\succ} = (a^{\prime\prime}_{\succ})^{\prime\prime}_{\prec} \otimes a^{\prime}_{\succ} \otimes (a^{\prime\prime}_{\succ})^{\prime}_{\prec},
$$

since we arrive to this by applying τ to the equality (74). Notice that necessarily

$$
a^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime\prime}_{\prec} \otimes (a^{\prime}_{\prec})^{\prime}_{\prec} = (a^{\prime\prime}_{\prec})^{\prime\prime}_{\prec} \otimes (a^{\prime\prime}_{\prec})^{\prime}_{\prec} \otimes a^{\prime}_{\prec} + (a^{\prime\prime}_{\prec})^{\prime\prime}_{\succ} \otimes (a^{\prime\prime}_{\prec})^{\prime}_{\succ} \otimes a^{\prime}_{\prec}.
$$

In fact, this is the result of applying $P^{1,3}$ to (73). On other hand, if to (73) we apply ϕ we obtain

$$(a'_{\prec})''_{\prec} \otimes a''_{\prec} \otimes (a'_{\prec})'_{\prec} = (a''_{\prec})'_{\prec} \otimes (a''_{\prec})''_{\prec} \otimes a'_{\prec} + (a''_{\prec})'_{\succ} \otimes (a''_{\prec})''_{\succ} \otimes a'_{\prec}.$$

Finally, observe that the following formula is true

$$(a''_{\succ})'_{\succ} \otimes a'_{\succ} \otimes (a''_{\succ})''_{\succ} = (a'_{\succ})''_{\prec} \otimes (a'_{\succ})'_{\prec} \otimes a''_{\succ} + (a'_{\succ})''_{\succ} \otimes (a'_{\succ})'_{\succ} \otimes a''_{\succ},$$

because it is obtained of (75) after applying $P^{1,2}$. This completes the proof of theorem. ■

The following remark is interesting

Remark 13 *Let $(A, \triangle_{\succ}, \triangle_{\prec})$ be a dendriform coalgebra. We know that one can define two coproducts $\widetilde{\triangle} = \triangle_{\succ} + \triangle_{\prec}$ and $\widehat{\triangle} = \triangle_{\succ} - \sigma \circ \triangle_{\prec}$ such that $\widetilde{\triangle}$ is coassociative whereas that $\widehat{\triangle}$ is a left-symmetric coproduct. However, observe that*

$$\begin{aligned} \triangle_{lie} &= \widetilde{\triangle} - \sigma\widetilde{\triangle} \\ &= (\triangle_{\succ} + \triangle_{\prec}) - \sigma(\triangle_{\succ} + \triangle_{\prec}) \\ &= (\triangle_{\succ} - \sigma\triangle_{\prec}) + (\triangle_{\prec} - \sigma\triangle_{\succ}) \\ &= (I - \sigma)(\triangle_{\succ} - \sigma\triangle_{\prec}) \\ &= \widehat{\triangle} - \sigma\widehat{\triangle}. \end{aligned}$$

In other words, $\widetilde{\triangle}$ and $\widehat{\triangle}$ give rise to the same lie coalgebra structure on A.

Definition 14 *By definition a* ***dendriform tricoalgebra*** *is a K-vector space with three coproducts $\triangle_{\succ} : A \longrightarrow A \otimes A$, $\triangle_{\prec} : A \longrightarrow A \otimes A$ and $\triangle_{\circ} : A \longrightarrow A \otimes A$ such that*

$$(\triangle_{\prec} \otimes Id) \circ \triangle_{\prec}(a) = (Id \otimes \triangle_{\prec} + Id \otimes \triangle_{\succ} + Id \otimes \triangle_{\circ}) \circ \triangle_{\prec}(a), \quad (77)$$

$$(\triangle_{\succ} \otimes Id) \circ \triangle_{\prec}(a) = (Id \otimes \triangle_{\prec}) \circ \triangle_{\succ}(a), \quad (78)$$

$$(\triangle_{\prec} \otimes Id + \triangle_{\succ} \otimes Id + \triangle_{\circ} \otimes Id) \circ \triangle_{\succ}(a) = (Id \otimes \triangle_{\succ}) \circ \triangle_{\succ}(a), \quad (79)$$

$$(\triangle_{\succ} \otimes Id) \circ \triangle_{\circ}(a) = (Id \otimes \triangle_{\circ}) \circ \triangle_{\succ}(a), \quad (80)$$

$$(\triangle_{\prec} \otimes Id) \circ \triangle_{\circ}(a) = (Id \otimes \triangle_{\succ}) \circ \triangle_{\circ}(a), \quad (81)$$

$$(\triangle_{\circ} \otimes Id) \circ \triangle_{\prec}(a) = (Id \otimes \triangle_{\prec}) \circ \triangle_{\circ}(a), \quad (82)$$

$$(\triangle_{\circ} \otimes Id) \circ \triangle_{\circ}(a) = (Id \otimes \triangle_{\circ}) \circ \triangle_{\circ}(a). \quad (83)$$

Any dendriform tricoalgebra will be denoted by $(A, \triangle_{\succ}, \triangle_{\prec}, \triangle_{\circ})$.

Being $(A, \triangle_{\succ}, \triangle_{\prec}, \triangle_{\circ})$ an arbitrary dendriform tricoalgebra, then the coproduct $\widetilde{\triangle} = \triangle_{\succ} + \triangle_{\prec} + \triangle_{\circ}$ is coassociative and moreover $(A^{*}, \succ, \prec, \circ)$ is a dendriform trialgebra, where $f \circ g$ is the element of A^{*} defined of the following form $\langle f \circ g, a\rangle = \langle f, a'_0\rangle\langle g, a''_0\rangle$ if $\triangle_0(a) = a'_0 \otimes a''_0$, that is, $f \circ g$ is the linear functional associated to the coproduct $\triangle_{\circ}$.

Flexible and alternative dendriform algebras and Rota-Baxter operators

In this section dendriform analogs for alternative and flexible algebras and their respective Rota-Baxter operators are presented and studied. Next, we introduce the notions of flexible and alternative dendriform algebras.

Definition 15 *A* ***flexible dendriform algebra*** *is a vector space A together with bilinear operations: $\succ A \times A \longrightarrow A$ and $\prec A \times A \longrightarrow A$ such that for all $x, y, z \in A$*

$$(x, y, z)_1 = -(z, y, x)_1, \quad (x, y, z)_2 = -(z, y, x)_2, \quad (x, y, z)_3 = -(z, y, x)_3, \quad (84)$$

for all $x, y, z \in A$.

Let A be a flexible dendriform algebra. Define $x \cdot y = x \prec y + x \succ y$ for any $x, y \in A$, then it is easy to see that $(A, \cdot)$ is a usual flexible algebra.

Remark 16 *With the help of dendriform triple systems one can define also the concept of* ***flexible dendriform trialgebra****, but this structure will be not studied in this paper.*

We go to the second main definition quickly.

Definition 17 *Let A be a vector space together with bilinear operations: $\succ A \times A \longrightarrow A$ and $\prec A \times A \longrightarrow A$. We say that A is an* ***alternative dendriform algebra*** *if for any $x, y, z \in A$*

$$(x, y, z)_1 = -(y, x, z)_1, \quad (x, y, z)_2 = -(y, x, z)_2, \quad (x, y, z)_3 = -(y, x, z)_3, \quad (85)$$

and

$$(x, y, z)_1 = -(x, z, y)_1, \quad (x, y, z)_2 = -(x, z, y)_2, \quad (x, y, z)_3 = -(x, z, y)_3, \quad (86)$$

for all $x, y, z \in A$.

Assume that A is an alternative dendriform algebra, then A can be converted in an alternative algebra with the respective product defined by $x \cdot y = x \prec y + x \succ y$ for all $x, y, z \in A$. Note that any alternative dendriform algebra is a flexible dendriform algebra.

Remark 18 *In the same way, it is possible to define the notion of alternative dendriform trialgebra.*

On all flexible dendriform algebra the relations

$$(x, y, x)_1 = (x, y, x)_2 = (x, y, x)_3 = 0, \quad (87)$$

hold for all $x, y, \in A$. The analogous identities corresponding to an alternative dendriform algebra are

$$(x, x, z)_1 = (x, x, z)_2 = (x, x, z)_3 = 0, \quad (88)$$

and

$$(x, z, z)_1 = (x, z, z)_2 = (x, z, z)_3 = 0, \tag{89}$$

for all $x, z \in A$.

Conversely, one can see that (87) implies (84) and on other hand from (88) and (89) follow (85) and (86). Thus, $(A, \prec, \succ)$ is a flexible dendriform algebra if and only if

$$x \succ (y \succ x) = (x \prec y + x \succ y) \succ x, \tag{90}$$

$$x \succ (y \prec x) = (x \succ y) \prec x, \tag{91}$$

$$(x \prec y) \prec x = x \prec (y \prec x + y \succ x), \tag{92}$$

for all $x, y \in A$. It follows that all dendriform algebra is a flexible dendriform algebra.

Explicit expressions of (88) and (89) are

$$x \succ (x \succ z) = (x \prec x + x \succ x) \succ z, \tag{93}$$

$$x \succ (x \prec z) = (x \succ x) \prec z, \tag{94}$$

$$(x \prec x) \prec z = x \prec (x \prec z + x \succ z), \tag{95}$$

and

$$x \succ (z \succ z) = (x \prec z + x \succ z) \succ z, \tag{96}$$

$$x \succ (z \prec z) = (x \succ z) \prec z, \tag{97}$$

$$(x \prec z) \prec z = x \prec (z \prec z + z \succ z), \tag{98}$$

for $x, z \in A$. Of course any dendriform algebra is an alternative dendriform algebra.

Definition 19 *Let $(A, \cdot)$ be a flexible algebra. We say that a linear map $P : A \longrightarrow A$ is a **flexible Rota-Baxter operator of weight** θ if*

$$P(x) \cdot P(y) = P(P(x) \cdot y + x \cdot P(y) + \theta x \cdot y), \tag{99}$$

$$(x \cdot P(y)) \cdot P(x) = x \cdot (P(y) \cdot P(x)), \tag{100}$$

$$P(x) \cdot (P(y) \cdot x) = (P(x) \cdot P(y)) \cdot x, \tag{101}$$

*for all $x, y \in A$. In this case, the triple $(A, \cdot, P)$ is called **flexible Rota-Baxter algebra of weight** θ.*

Since all associative algebra is a flexible algebra, then any associative Rota-Baxter operator is a flexible Rota-Baxter operator.

Lemma 20 *If $(A, \cdot, P)$ is a flexible Rota-Baxter algebra of weight 0, then*

$$(x \cdot P(P(y))) \cdot P(P(x)) = x \cdot (P(P(y)) \cdot P(P(x))), \tag{102}$$

and

$$P(P(x)) \cdot (P(P(y)) \cdot x) = (P(P(x)) \cdot P(P(y))) \cdot x, \tag{103}$$

for all $x, y \in A$.

Proof. Actually, the validity of Eq. (102) is equivalent to the following

$$(z\cdot P(v))\cdot P(w)+(w\cdot P(v))\cdot P(z)=z\cdot(P(v)\cdot P(w))+w\cdot(P(v)\cdot P(z)),\quad (104)$$

for any $z, v, w \in A$ by letting in (100), $x \longrightarrow z+\lambda w$ and $y \longrightarrow v$. Substituting in (104), x, $P(y)$ and $P(x)$ in place of z, v and w respectively, and using the fact that $(A,\cdot)$ is flexible, we obtain (102). The proof of (103) is similar and basically it consists in to verify the relation

$$P(z)\cdot(P(v)\cdot w)+P(w)\cdot(P(v)\cdot z)=(P(z)\cdot P(v))\cdot w+(P(w)\cdot P(v))\cdot z\,.\quad (105)$$

■

We would like recall that (104) and (105) are equivalent to (100) and (101).

Theorem 21 *Let $(A,\cdot,P)$ be a flexible Rota-Baxter algebra of weight 0. Let us define*

$$x\circ y=P(x)\cdot y+x\cdot P(y),\quad (106)$$

for all $x, y \in A$. Then $(A,\circ,P)$ is also a flexible Rota-Baxter algebra of weight 0.

Proof. We have

$$\begin{aligned}(x\circ y)\circ x &= (P(x)\cdot y+x\cdot P(y))\circ x\\ &= P(P(x)\cdot y+x\cdot P(y))\cdot x+(P(x)\cdot y+x\cdot P(y))\cdot P(x)\\ &= (P(x)\cdot P(y))\cdot x+(P(x)\cdot y)\cdot P(x)+(x\cdot P(y))\cdot P(x)\\ &= P(x)\cdot(P(y)\cdot x+y\cdot P(x))+x\cdot(P(y)\cdot P(x))\\ &= P(x)\cdot(P(y)\cdot x+y\cdot P(x))+x\cdot P((P(y)\cdot x+y\cdot P(x)))\\ &= x\circ(P(y)\cdot x+y\cdot P(x))=x\circ(y\circ x).\end{aligned}$$

That $P(x)\circ P(y)=P(P(x)\circ y+x\circ P(y))$ it is well known. Now, from lemma 20 we have

$$\begin{aligned}(x\circ P(y))\circ P(x) &= (P(x)\cdot P(y)+x\cdot P(P(y)))\circ P(x)\\ &= P(P(x)\cdot P(y)+x\cdot P(P(y)))\cdot P(x)\\ &\quad+(P(x)\cdot P(y)+x\cdot P(P(y)))\cdot P(P(x))\\ &= (P(x)\cdot P(P(y)))\cdot P(x)+P(x)\cdot(P(y)\cdot P(P(x)))\\ &\quad+(x\cdot P(P(y)))\cdot P(P(x))\\ &= P(x)\cdot(P(P(y))\cdot P(x))+P(x)\cdot(P(y)\cdot P(P(x)))\\ &\quad+(x\cdot P(P(y)))\cdot P(P(x))\\ &= P(x)\cdot(P(y)\circ P(x))+(x\cdot P(P(y)))\cdot P(P(x))\\ &= P(x)\cdot(P(y)\circ P(x))+x\cdot(P(P(y))\cdot P(P(x)))\\ &= P(x)\cdot(P(y)\circ P(x))\\ &\quad+x\cdot P(P(P(y))\cdot P(x)+P(y)\cdot P(P(x)))\\ &= P(x)\cdot(P(y)\circ P(x))+x\cdot P(P(y)\circ P(x))\\ &= x\circ(P(y)\circ P(x)).\end{aligned}$$

On other hand, of (103) follows

$$
\begin{aligned}
P(x)\circ(P(y)\circ x) &= P(x)\circ(P(P(y))\cdot x + P(y)\cdot P(x)) \\
&= P(P(x))\cdot(P(P(y))\cdot x + P(y)\cdot P(x)) \\
&\quad + P(x)\cdot P(P(P(y))\cdot x + P(y)\cdot P(x)) \\
&= P(P(x))\cdot(P(P(y))\cdot x) + P(P(x))\cdot(P(y)\cdot P(x)) \\
&\quad + P(x)\cdot(P(P(y))\cdot P(x)) \\
&= P(P(x))\cdot(P(P(y))\cdot x) + (P(P(x))\cdot P(y))\cdot P(x) \\
&\quad + (P(x)\cdot P(P(y)))\cdot P(x) \\
&= (P(P(x))\cdot P(P(y)))\cdot x + (P(x)\circ P(y))\cdot P(x) \\
&= P(P(x)\circ P(y))\cdot x + (P(x)\circ P(y))\cdot P(x) \\
&= (P(x)\circ P(y))\circ x.
\end{aligned}
$$

■

Proposition 22 *Let $(A,\cdot,P)$ be a flexible Rota-Baxter algebra of weight 0. Define new operations on A by*

$$x \succ y = P(x)\cdot y, \quad x \prec y = x\cdot P(y), \quad \forall x,y\in A.$$

Then, $(A,\prec,\succ)$ is a flexible dendriform algebra.

Proof. The proof is simple and thus it will be omitted. ■

Definition 23 *Let $(A,\cdot)$ be an alternative algebra. We say that a linear map $P: A \longrightarrow A$ is an* ***alternative Rota-Baxter operator of weight θ*** *if it satisfies the following relations*

$$P(x)\cdot P(y) = P(P(x)\cdot y + x\cdot P(y) + \theta x\cdot y), \tag{107}$$

$$P(x)\cdot(P(x)\cdot y) = (P(x)\cdot P(x))\cdot y, \tag{108}$$

$$P(x)\cdot(x\cdot P(y)) = (P(x)\cdot x)\cdot P(y), \tag{109}$$

$$(x\cdot P(x))\cdot P(y) = x\cdot(P(x)\cdot P(y)), \tag{110}$$

$$P(x)\cdot(P(y)\cdot y) = (P(x)\cdot P(y))\cdot y, \tag{111}$$

$$P(x)\cdot(y\cdot P(y)) = (P(x)\cdot y)\cdot P(y), \tag{112}$$

$$(x\cdot P(y))\cdot P(y) = x\cdot(P(y)\cdot P(y)), \tag{113}$$

for all $x,y\in A$. The triple $(A,.,P)$ is called ***alternative Rota-Baxter algebra of weight θ****.*

Next, we show equivalent relations of (108)-(113). For all $z,w,v\in A$ we have

$$P(z)\cdot(P(w)\cdot v) + P(w)\cdot(P(z)\cdot v) = (P(z)\cdot P(w))\cdot v + (P(w)\cdot P(z))\cdot v, \tag{114}$$

$$P(z)\cdot(w\cdot P(v)) + P(w)\cdot(z\cdot P(v)) = (P(z)\cdot w)\cdot P(v) + (P(w)\cdot z)\cdot P(v), \tag{115}$$

$$(z\cdot P(w))\cdot P(v) + (w\cdot P(z))\cdot P(v) = z\cdot(P(w)\cdot P(v)) + w\cdot(P(z)\cdot P(v)), \tag{116}$$

$$P(v)\cdot(P(z)\cdot w) + P(v)\cdot(P(w)\cdot z) = (P(v)\cdot P(z))\cdot w + (P(v)\cdot P(w))\cdot z, \tag{117}$$

$$P(v)\cdot(z\cdot P(w)) + P(v)\cdot(w\cdot P(z)) = (P(v)\cdot z)\cdot P(w) + (P(v)\cdot w)\cdot P(z), \tag{118}$$

$$(v\cdot P(z))\cdot P(w) + (v\cdot P(w))\cdot P(z) = v\cdot(P(z)\cdot P(w)) + v\cdot(P(w)\cdot P(z)). \tag{119}$$

These latter relations will be useful to prove the following result

Theorem 24 *Let $(A, \cdot, P)$ be an alternative Rota-Baxter algebra of weight 0. The following operation*

$$x \circ y = P(x) \cdot y + x \cdot P(y), \tag{120}$$

equips A with a structure of alternative Rota-Baxter algebra of weight 0 with respect to P.

Proof. We must check that $(x \circ x) \circ y = x \circ (x \circ y)$ and $x \circ (y \circ y) = (x \circ y) \circ y$. From (107)-(110) follow that

$$\begin{aligned}
(x \circ x) \circ y &= (P(x) \cdot x + x \cdot P(x)) \circ y \\
&= P(P(x) \cdot x + x \cdot P(x)) \cdot y + (P(x) \cdot x + x \cdot P(x)) \cdot P(y) \\
&= (P(x) \cdot P(x)) \cdot y + (P(x) \cdot x + x \cdot P(x)) \cdot P(y) \\
&= (P(x) \cdot P(x)) \cdot y + (P(x) \cdot x) \cdot P(y) + (x \cdot P(x)) \cdot P(y) \\
&= P(x) \cdot (P(x) \cdot y) + P(x) \cdot (x \cdot P(y)) + x \cdot (P(x) \cdot P(y)) \\
&= P(x)(P(x) \cdot y + x \cdot P(y)) + x \cdot P(P(x) \cdot y + x \cdot P(y)) \\
&= x \circ (P(x) \cdot y + x \cdot P(y)) = x \circ (x \circ y).
\end{aligned}$$

On other hand, (107) and (111)-(113) imply

$$\begin{aligned}
x \circ (y \circ y) &= x \circ (P(y) \cdot y + y \cdot P(y)) \\
&= P(x) \cdot (P(y) \cdot y + y \cdot P(y)) + x \cdot P(P(y) \cdot y + y \cdot P(y)) \\
&= P(x) \cdot (P(y) \cdot y) + P(x) \cdot (y \cdot P(y)) + x \cdot (P(y) \cdot P(y)) \\
&= (P(x) \cdot P(y)) \cdot y + (P(x) \cdot y) \cdot P(y) + (x \cdot P(y)) \cdot P(y) \\
&= P(P(x) \cdot y + x \cdot P(y)) \cdot y + (P(x) \cdot y + x \cdot P(y)) \cdot P(y) \\
&= (P(x) \cdot y + x \cdot P(y)) \circ y = (x \circ y) \circ y.
\end{aligned}$$

That (107) is true is well known. We do not prove all relations (108)-(113), instead we will check only one of them, for instance

$$(x \circ P(x)) \circ P(y) = x \circ (P(x) \circ P(y)).$$

Observe that from (108) and (116) follow

$$(x \cdot P(P(x))) \cdot P(P(y)) = x \cdot (P(P(x)) \cdot P(P(y))).$$

Hence, using (108), (110) and the ordinary Rota-Baxter relation, we have

$$\begin{aligned}
(x \circ P(x)) \circ P(y) &= (P(x) \cdot P(x) + x \cdot P(P(x))) \circ P(y) \\
&= P(P(x) \cdot P(x) + x \cdot P(P(x))) \cdot P(y) \\
&\quad + (P(x) \cdot P(x) + x \cdot P(P(x))) \cdot P(P(y)) \\
&= (P(x) \cdot P(P(x))) \cdot P(y) + (P(x) \cdot P(x)) \cdot P(P(y)) \\
&\quad + (x \cdot P(P(x))) \cdot P(P(y)) \\
&= P(x) \cdot (P(x) \circ P(y)) + x \cdot (P(P(x)) \cdot P(P(y))) \\
&= P(x) \cdot (P(x) \circ P(y)) + x \cdot P(P(x) \circ P(y)) \\
&= x \circ (P(x) \circ P(y)).
\end{aligned}$$

■

Proposition 25 *Let $(A, \cdot, P)$ be an alternative Rota-Baxter algebra of weight 0. Introduce two new operations on A of the following form*

$$x \succ y = P(x) \cdot y, \quad x \prec y = x \cdot P(y), \quad \forall x, y \in A.$$

Then, $(A, \prec, \succ)$ is an alternative dendriform algebra.

Proof. The proof of the proposition involves very simple calculations thus it will be omitted. ∎

Definition 26 *Let $(A, \triangle)$ be a coalgebra over K. We say that A is a flexible coalgebra if*

$$(Id_{A^{\otimes 3}} + P^{1,3})A_{\triangle} = 0 \in Hom_K(A, A^{\otimes 3}).$$

Given a flexible coalgebra $(A, \triangle)$, the coproduct $\triangle$ indices a structure of flexible algebra in A^*, that is, $(A, \triangle)$ is a flexible-coalgebra.

Definition 27 *We call to a coalgebra $(A, \triangle)$ an alternative coalgebra if by definition*

$$(Id_{A^{\otimes 3}} + P^{1,2})A_{\triangle} = 0 \in Hom_K(A, A^{\otimes 3}),$$

and

$$(Id_{A^{\otimes 3}} + P^{2,3})A_{\triangle} = 0 \in Hom_K(A, A^{\otimes 3}).$$

The coproduct $\triangle$ of an alternative coalgebra $(A, \triangle)$ induces a structure of alternative algebra in A^*, in other words, $(A, \triangle)$ is an alternative-coalgebra.

Remark 28 *All alternative coalgebra is a flexible coalgebra. In fact, if $(A, \triangle)$ is an alternative coalgebra then $(Id_{A^{\otimes 3}} + P^{2,3})P^{1,2}A_{\triangle} = 0$ hence $(Id_{A^{\otimes 3}} + P^{1,2}P^{2,3}P^{1,2})A_{\triangle} = (Id_{A^{\otimes 3}} + P^{1,3})A_{\triangle} = 0$.*

To complete this section, we now consider Rota-Baxter operators in Cayley-Dickson algebras. Let $(A, \cdot)$ be an algebra with the unit 1, over a field K and an involution $a \longrightarrow \overline{a}$, where $a + \overline{a}, a \cdot \overline{a} \in K$, for every $a \in K$. Let us fix $\alpha \neq 0 \in K$. Define on the vector space $A \oplus A$ the following multiplication

$$(a_1, a_2) \circ (a_3, a_a) = (a_1 \cdot a_3 - \alpha a_4 \cdot \overline{a_2}, \overline{a_1} \cdot a_4 + a_3 \cdot a_2). \tag{121}$$

The resulting algebra is denoted by (A, α) and it is called the algebra derived from A by the Cayley-Dickson process. The element $(e, 0)$ is the unit of (A, α) and $(A, \alpha) = A \oplus vA$ where $v = (0, 1)$. Note that $v^2 = -\alpha 1$. If $x = a_1 + va_2 \in (A, \alpha)$, then the mapping $x \longrightarrow \overline{x} = \overline{a_1} - va_2$ is an involution on (A, α) and $x + \overline{x} = a_1 + \overline{a_1}, x \cdot \overline{x} = a_1 \cdot \overline{a_1} + \alpha a_2 \cdot \overline{a_2} \in K$.

Theorem 29 *Let $(A, \cdot)$ be an algebra with the unit 1, over a field K and an involution $a \longrightarrow \overline{a}$. Assume that P is a Rota-Baxter operator defined on A in the following sense*

$$Pa \cdot Pb = P(Pa \cdot b + a \cdot Pb), \quad \text{and} \quad \overline{Pa} = P\overline{a}, \tag{122}$$

for all $a, b \in A$. Let us define $\widehat{P}(a, b) = (Pa, Pb)$ then

$$\widehat{P}(a_1, a_2) \circ \widehat{P}(a_3, a_4) = \widehat{P}(\widehat{P}(a_1, a_2) \circ (a_3, a_4) + (a_1, a_2) \circ \widehat{P}(a_3, a_4)), \tag{123}$$

that is $\widehat{P}$ is a usual Rota-Baxter operator over $A \oplus A$. Moreover, $\overline{\widehat{P}(a, b)} = \widehat{P}\ \overline{(a, b)}$.

Proof. From (122) follows that

$$\begin{aligned}
\widehat{P}(a_1,a_2)\circ\widehat{P}(a_3,a_4) &= (Pa_1\cdot Pa_3-\alpha Pa_4\cdot P\overline{a_2}, P\overline{a_1}\cdot Pa_4+Pa_3\cdot Pa_2)\\
&= (P(Pa_1\cdot a_3+a_1\cdot Pa_3)-\alpha P(Pa_4\cdot\overline{a_2}+a_4\cdot P\overline{a_2}),\\
&\quad P(P\overline{a_1}\cdot a_4+\overline{a_1}\cdot Pa_4)+P(Pa_3\cdot a_2+a_3\cdot Pa_2))\\
&= P((Pa_1\cdot a_3+a_1\cdot Pa_3)-\alpha(Pa_4\cdot\overline{a_2}+a_4\cdot P\overline{a_2}),\\
&\quad (P\overline{a_1}\cdot a_4+\overline{a_1}\cdot Pa_4)+(Pa_3\cdot a_2+a_3\cdot Pa_2))\\
&= \widehat{P}((Pa_1\cdot a_3-\alpha a_4\cdot P\overline{a_2}, P\overline{a_1}\cdot a_4+a_3\cdot Pa_2),\\
&\quad (a_1\cdot Pa_3-\alpha Pa_4\cdot\overline{a_2},\overline{a_1}\cdot Pa_4+Pa_3\cdot a_2))\\
&= \widehat{P}((Pa_1,Pa_2)\circ(a_3,a_4)+(a_1,a_2)\circ(Pa_3,Pa_4))\\
&= \widehat{P}(\widehat{P}(a_1,a_2)\circ(a_3,a_4)+(a_1,a_2)\circ\widehat{P}(a_3,a_4)).
\end{aligned}$$

On other hand, observe that $\overline{\widehat{P}(a,b)} = (\overline{Pa},-Pb) = (P\overline{a},-Pb) = \widehat{P}(\overline{a},-b) = \widehat{P}\ \overline{(a,b)}$ ■

Suppose that $(A\oplus A,\circ)$ turns out to be an associative algebra, then under the hypothesis of previous theorem $A\oplus A$ becomes a dendriform algebra with the products $\succeq$ and $\preceq$ defined as

$$(a_1,a_2)\succeq(a_3,a_4)=\widehat{P}(a_1,a_2)\circ(a_3,a_4)=(Pa_1\cdot a_3-\alpha a_4\cdot P\overline{a_2}, P\overline{a_1}\cdot a_4+a_3\cdot Pa_2), \tag{124}$$

and

$$(a_1,a_2)\preceq(a_3,a_4)=\widehat{P}(a_1,a_2)\circ\widehat{P}(a_3,a_4)=(a_1\cdot Pa_3-\alpha Pa_4\cdot\overline{a_2},\overline{a_1}\cdot Pa_4+Pa_3\cdot a_2). \tag{125}$$

Observe that if initially the algebra $(A,\cdot)$ is associative the products $a\succ b = Pa\cdot b$ and $a\prec b = a\cdot Pb$ equip A of a structure of dendriform algebra, in this case (124) and (125) can be written in the following form

$$(a_1,a_2)\succeq(a_3,a_4)=(a_1\succ a_3-\alpha a_4\prec\overline{a_2},\overline{a_1}\succ a_4+a_3\prec a_2), \tag{126}$$

$$(a_1,a_2)\preceq(a_3,a_4)=(a_1\prec a_3-\alpha a_4\succ\overline{a_2},\overline{a_1}\prec a_4+a_3\succ a_2). \tag{127}$$

Note that $\overline{a\succ b}=\overline{Pa\cdot b}=\overline{b}\cdot\overline{Pa}=\overline{b}\cdot P\overline{a}=\overline{b}\prec\overline{a}$ for all $a,b\in A$. In the same way one can see that $\overline{a\prec b}=\overline{b}\succ\overline{a}$.

Lemma 30 *Under the hypotheses of the previous theorem*

$$\overline{(a_1,a_2)\succeq(a_3,a_4)}=\overline{(a_3,a_4)}\preceq\overline{(a_1,a_2)}, \tag{128}$$

and

$$\overline{(a_1,a_2)\preceq(a_3,a_4)}=\overline{(a_3,a_4)}\succeq\overline{(a_1,a_2)}. \tag{129}$$

Proof. The equations (128) and (129) follow of (126) and (127). ■

Rota-Baxter operator for Leibniz algebras

In 1993, J. L. Loday introduced the notion of Leibniz algebras (see [20]), which is a generalization of the Lie algebras where the skew-symmetric of the bracket is dropped and the Jacobi identity is changed by the Leibniz identity. In this section we introduce Rota-Baxter operator for Leibniz algebras.

Let $(\mathfrak{L}, [.,.])$ be a Leibniz algebra. A linear operator $F : \mathfrak{L} \longrightarrow \mathfrak{L}$ is called a Rota-Baxter operator of weight $\theta \in K$ for $\mathfrak{L}$ if

$$[Fx, Fy] = F([Fx, y] + [x, Fy] + \theta[x, y]), \tag{130}$$

for all $x, y \in \mathfrak{L}$.

Theorem 31 *Let $(\mathfrak{L}, [.,.])$ be a Leibniz algebra and let F be a Rota-Baxter operator for $\mathfrak{L}$. Then the bracket*

$$[.,.]_F = [F.,.] + [.,F.] + \theta[.,.], \tag{131}$$

converts $\mathfrak{L}$ in a Leibniz algebra.

We include the proof for completeness of this work.
Proof. We have for all $x, y, z \in \mathfrak{L}$

$$\begin{aligned}
[x, [y.z]_F]_F &= [x, [Fy, z] + [y, Fz] + \theta[y, z]]_F \\
&= [Fx, [Fy, z] + [y, Fz] + \theta[y, z]] + [x, F([Fy, z] + [y, Fz] + \theta[y, z])] \\
&\quad + \theta[x, [Fy, z] + [y, Fz] + \theta[y, z]] \\
&= [Fx, [Fy, z] + [y, Fz] + \theta[y, z]] + [x, [Fy, Fz]] \\
&\quad + \theta[x, [Fy, z] + [y, Fz] + \theta[y, z]].
\end{aligned} \tag{132}$$

$$\begin{aligned}
[[x, y]_F, z]_F &= [[Fx, y] + [x, Fy] + \theta[x, y], z]_F \\
&= [F([Fx, y] + [x, Fy] + \theta[x, y]), z] + [[Fx, y] + [x, Fy] + \theta[x, y], Fz] \\
&\quad + \theta[[Fx, y] + [x, Fy] + \theta[x, y], z] \\
&= [[Fx, Fy], z] + [[Fx, y] + [x, Fy] + \theta[x, y], Fz] \\
&\quad + \theta[[Fx, y] + [x, Fy] + \theta[x, y], z].
\end{aligned} \tag{133}$$

$$\begin{aligned}
[[x, z]_F, y]_F &= -[[Fx, z] + [x, Fz] + \theta[x, z], y]_F \\
&= -[F([Fx, z] + [x, Fz] + \theta[x, z]), y] - [[Fx, z] + [x, Fz] + \theta[x, z], Fy] \\
&\quad - \theta[[Fx, z] + [x, Fz] + \theta[x, z], y] \\
&= -[[Fx, Fz], y] - [[Fx, z] + [x, Fz] + \theta[x, z], Fy] \\
&\quad - \theta[[Fx, z] + [x, Fz] + \theta[x, z], y].
\end{aligned} \tag{134}$$

It follows from (132)-(134) that the Leibniz identity holds. ∎

We must note that if $\mathfrak{L}$ is a Leibniz-admissible algebra and F is a Rota-Baxter operator then

$$Fx \dashv Fy - F(Fx \dashv y + x \dashv Fy + \theta x \dashv y) = Fy \vdash Fx - F(Fy \vdash x + y \vdash Fx + \theta y \vdash x), \tag{135}$$

for all $x, y \in \mathfrak{L}$. In particular, when A is a Lie-admissible algebra the condition (5) can be written in the form

$$Fx \cdot Fy - F(Fx \cdot y + x \cdot Fy + \theta x \cdot y) = Fy \cdot Fx - F(Fy \cdot x + y \cdot Fx + \theta y \cdot x), \tag{136}$$

for every $x, y \in A$, which gives rise to a new type of Rota-Baxter operator even for associative algebras.

Let $(\mathfrak{L}, [.,.])$ be a Leibniz algebra, then for all $x, y, z \in \mathfrak{L}$ from the Leibniz identity we have

$$[x, [y, z]] = [[x, y], z] - [[x, z], y], \tag{137}$$

and also

$$[x, [z, y]] = [[x, z], y] - [[x, y], z], \tag{138}$$

it implies that $[x, [y, z] + [z, y]] = 0$. Thus, for the Leibniz algebra $\mathfrak{L}$ the subspace $\mathfrak{L}^{ann}$ generated by $\{[y, z] + [z, y] | y, z \in \mathfrak{L}\}$ plays an important role in the theory since it determines the possible non-Lie character of $\mathfrak{L}$. Note that a Leibniz algebra is a Lie algebra if and only if $\mathfrak{L}^{ann} = 0$. Also, we define the subspace $Z^r(\mathfrak{L})$ generated by $\{x \in \mathfrak{L} | [y, x] = 0 \ \forall y \in \mathfrak{L}\}$. Clearly $\mathfrak{L}^{ann} \subset Z^r(\mathfrak{L})$ and both subspaces are bilateral ideals. Let us assume that $\mathfrak{L} = J \oplus L$, such that J is a bilateral ideal and $\mathfrak{L}^{ann} \subset J \subset Z^r(\mathfrak{L})$, then we say that $\mathfrak{L}$ is a split Leibniz algebra. In this case L is necessarily a Lie algebra. For more details with relations to these concepts we recommend to see [9].

Let F be a Rota-Baxter operator for the Leibniz algebra $(\mathfrak{L}, [.,.])$ then $\mathfrak{L}^{ann}_F$ and $Z^r_F(\mathfrak{L})$ denote the respective ideal now with respect to the Leibniz bracket $[.,.]_F$. Clearly, we have $\mathfrak{L}^{ann}_F \subset \mathfrak{L}^{ann}$.

Proposition 32 *Let F be a Rota-Baxter operator for the Leibniz algebra $(\mathfrak{L}, [.,.])$. Assume that $y \in Z^r(\mathfrak{L})$, then in order to $y \in Z^r_F(\mathfrak{L})$ is necessary and sufficient that $Fy \in Z^r(\mathfrak{L})$.*

Proof. For all $x, \in \mathfrak{L}$ and all $y \in Z^r(\mathfrak{L})$ we

$$[x, y]_F = [Fx, y] + [x, Fy] + \theta[x, y] = [x, Fy].$$

Hence, the proposition follows. ■

Admissible dendriform algebras and admissible Rota-Baxter operators

In this section we introduce several new structures of dendriform type.

Definition 33 *An **admissible dendriform algebra** is a vector space D together with maps: $\succ D \times D \longrightarrow D$ and $\prec D \times D \longrightarrow D$ such that for all $x, y, z \in D$*

$$x \succ (y \succ z) - (x \prec y + x \succ y) \succ z = y \succ (x \succ z) - (y \prec x + y \succ x) \succ z, \tag{139}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{140}$$

$$(x \prec y) \prec z - x \prec (y \prec z + y \succ z) = (x \prec z) \prec y - x \prec (z \prec y + z \succ y). \tag{141}$$

Note that all dendriform algebra is an admissible dendriform algebra. The following result is obvious.

Proposition 34 *Let D be a vector space with two bilinear operations $\prec$ and $\succ$ such that $(D,\succ)$ is a left-symmetric algebra, $(D,\prec)$ is a right-symmetric algebra and*

$$x \succ (y \prec z) = (x \succ y) \prec z, \quad (x \prec y) \succ z = (y \prec x) \succ z,$$
$$x \prec (y \succ z) = x \prec (z \succ y),$$

for all $x, y, z \in D$. Then $(D, \prec, \succ)$ is an admissible dendriform algebra.

Proof. It is straightforward. ∎

Example 1 *Recently Bai, Liu and Ni introduced the concept of L-dendriform algebra [7] which we now recall: let A be a vector space with two bilinear operations denoted by $\succ$ and $\prec$: $A \times A \longrightarrow A$. $(A \succ, \prec)$ is called an L-dendriform algebra if for any $x, y, z \in A$,*

$$x \succ (y \succ z) - (x \prec y + x \succ y) \succ z = y \succ (x \succ z) - (y \prec x + y \succ x) \succ z, \tag{142}$$

$$x \succ (y \prec z) - (x \succ y) \prec z = y \prec (x \prec z + x \succ z) - (y \prec x) \prec z. \tag{143}$$

It is easy to show that all L-dendriform algebra in which for any $x, y, z \in A$

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z),$$

is an admissible dendriform algebra. In the mentioned paper [7] the authors proved that being $(A, \prec, \succ)$ an L-dendriform algebra then the two following bilinear operations

$$x \circ y = x \succ y + x \prec y, \quad x \bullet y = x \succ y - y \prec x, \tag{144}$$

define different structures of left-symmetric algebras on A. $(A, \circ)$ is called the associated horizontal left-symmetric algebra of $(A, \succ, \prec)$ and $(A, \bullet)$ is called the associated vertical left-symmetric algebra of $(A, \succ, \prec)$. Both $(A, \circ)$ and $(A, \bullet)$ have the same sub-adjacent Lie algebra $\mathfrak{g}(A)$ define by

$$[x, y] = x \succ y + x \prec y - y \succ x - y \prec x, \quad \forall x, y \in A.$$

From now on, any L-dendriform algebra in the sense of [7] will be called ***left L-dendriform algebra****, the motivation to make this could be exposed immediately. Observe that all dendriform algebra is a left L-dendriform algebra.*

Watching closely the definition 33 one can arrive at the following

Definition 35 *let A be a vector space with two bilinear operations denoted by $\succ$ and $\prec$: $A \times A \longrightarrow A$. $(A \succ, \prec)$ is called* ***right L-dendriform algebra*** *if for any $x, y, z \in A$,*

$$(x \prec y) \prec z - x \prec (y \prec z + y \succ z) = (x \prec z) \prec y - x \prec (z \prec y + z \succ y). \tag{145}$$

$$x \succ (y \prec z) - (x \succ y) \prec z = (x \prec z + x \succ z) \succ y - x \succ (z \succ y). \tag{146}$$

It is clear that a right L-dendriform algebra for which

$$x \succ (z \succ y) = (x \prec z + x \succ z) \succ y,$$

for all $x, y, z \in A$ is an admissible dendriform algebra.

Theorem 36 *Let* $(A, \succ, \prec)$ *be a right L-dendriform algebra. Define for all* $x, y \in A$ *a new product by* $x \bullet y = x \prec y - y \succ x$*, then* $(A, \bullet)$ *is a right-symmetric algebra.* $(A, \bullet)$ *is called the associated vertical right-symmetric algebra of* $(A, \succ, \prec)$*.*

Proof. Despite its simplicity we prefer to give the proof of the theorem

$$\begin{aligned}
L &= (x \bullet y) \bullet z - x \bullet (y \bullet z) - (x \bullet z) \bullet y + x \bullet (z \bullet y) \\
&= (x \prec y - y \succ x) \bullet z - x \bullet (y \prec z - z \succ y) \\
&\quad - (x \prec z - z \succ x) \bullet y + x \bullet (z \prec y - y \succ z) \\
&= (x \prec y) \prec z - z \succ (x \prec y) - (y \succ x) \prec z + z \succ (y \succ x) \\
&\quad - x \prec (y \prec z) + (y \prec z) \succ x + x \prec (z \succ y) - (z \succ y) \succ x \\
&\quad - (x \prec z) \prec y + y \succ (x \prec z) + (z \succ x) \prec y - y \succ (z \succ x) \\
&\quad + x \prec (z \prec y) - (z \prec y) \succ x - x \prec (y \succ z) + (y \succ z) \succ x.
\end{aligned}$$

Denote by T_i the i-th term of L including its sign, for instance $T_3 = -(y \succ x) \prec z$. Then, one can see that under the hypotheses of the theorem

$$T_1 + T_5 + T_7 + T_9 + T_{13} + T_{15} = 0,$$

$$T_2 + T_4 + T_8 + T_{11} + T_{14} = 0,$$

and

$$T_3 + T_6 + T_{10} + T_{12} + T_{16} = 0.$$

It implies that $L = 0$. ■

Given a right L-dendriform algebra A, if for any $x, y, z \in A$ the relation $(x \prec z + x \succ z) \succ y - x \succ (z \succ y) = 0$ holds, then A is an admissible dendriform algebra. Also observe that all dendriform algebra is a right L-dendriform algebra A.

It is time to study the relationship between admissible dendriform algebras and left-symmetric algebras. In this sense, we have

Theorem 37 *Let* $(A, \prec, \succ)$ *be an admissible dendriform algebra. Define the new operation* $x \bullet y = x \succ y - y \prec x$ *for all* $x, y \in A$*, then* $(A, \bullet)$ *is a left-symmetric algebra.*

Proof. For all $x, y, z \in A$ it is easy to see that

$$(x \bullet y) \bullet z - x \bullet (y \bullet z) - (y \bullet x) \bullet z + y \bullet (x \bullet z) = T_1 + T_2 + T_3 + T_4 = 0,$$

because

$$\begin{aligned}
T_1 &= x \succ (z \prec y) - (x \succ z) \prec y = 0, \\
T_2 &= (y \succ z) \prec x - y \succ (z \prec x) = 0,
\end{aligned}$$

$$\begin{aligned}
T_3 = {} & (x \succ y) \succ z - (y \prec x) \succ z - x \succ (y \succ z) - (y \succ x) \succ z \\
& + (x \prec y) \succ z + y \succ (x \succ z) = 0,
\end{aligned}$$

and

$$\begin{aligned}
T_4 = {} & -z \prec (x \succ y) + z \prec (y \prec x) - (z \prec y) \prec x + z \prec (y \succ x) \\
& - z \prec (x \prec y) + (z \prec x) \prec y = 0. \qquad (147)
\end{aligned}$$

■

Definition 38 *Let $(A,\cdot)$ be an algebra. We say that F is an* ***admissible Rota-Baxter operator of weight θ on*** *A if*

$$Fx\cdot Fy - F(Fx\cdot y + x\cdot Fy + \theta x\cdot y) = Fy\cdot Fx - F(Fy\cdot x + y\cdot Fx + \theta y\cdot x),$$

for all $x,y\in A$. In this case A is called an ***admissible Rota-Baxter algebra of weight*** *θ and it will be denoted by $(A,\cdot,F)$.*

Now, we extend a result mentioned before and obtained by Aguiar in [2] when $\theta=0$. We must indicate that this result of Aguiar was proved by Ebrahimi-Fard for $\theta\neq 0$ in [16].

Theorem 39 *Let $(A,\cdot)$ be an associative algebra and let F be an admissible Rota-Baxter operator of weight 0. Define new multiplications on A by*

$$x\succ y = Fx\cdot y,\qquad x\prec y = x\cdot Fy,$$

for all $x,y\in A$. Then $(A,\prec,\succ)$ is an admissible dendriform algebra.

Proof. We check only (141); the others proofs are similar. For all $x,y,z\in A$ we have

$$\begin{aligned}(x\prec y)\prec z - x\prec(y\prec z + y\succ z) &= (x\prec y)\cdot Fz - x\prec(y\cdot Fz + Fy\cdot z)\\ &= x\cdot(Fy\cdot Fz) - x\cdot(F(y\cdot Fz + Fy\cdot z))\\ &= x\cdot(Fy\cdot Fz - F(y\cdot Fz + Fy\cdot z))\\ &= x\cdot(Fz\cdot Fy - F(z\cdot Fy + Fz\cdot y))\\ &= (x\prec z)\prec y - x\prec(z\prec y + z\succ y).\end{aligned}$$

■

Proposition 40 *Let $(A,\cdot,F)$ be an admissible Rota-Baxter algebra of weight θ. Then, $(A,\circ,F)$ is also an admissible Rota-Baxter algebra of weight θ where $\circ$ is defined as*

$$x\circ y = Fx\cdot y + x\cdot Fy + \theta x\cdot y,$$

for all $x,y\in A$.

Proof. Let $x,y\in A$ be two arbitrary vectors, then we have

$$\begin{aligned}Fx\circ Fy &= F(Fx)\cdot Fy + Fx\cdot F(Fy) + \theta Fx\cdot Fy,\\ Fx\circ y &= F(Fx)\cdot y + Fx\cdot Fy + \theta Fx\cdot y,\\ x\circ Fy &= Fx\cdot Fy + x\cdot F(Fy) + \theta x\cdot Fy.\end{aligned}$$

Denote $L(x,y;F) = Fx\circ Fy - F(Fx\circ y + x\circ Fx + \theta x\circ y)$ then

$$\begin{aligned}L(x,y;F) = {} & [F(Fx)\cdot Fy - F(F(Fx)\cdot y + Fx\cdot Fy + \theta Fx\cdot y)]\\ & + [Fx\cdot F(Fy) - F(Fx\cdot Fy + x\cdot F(Fy) + \theta x\cdot Fy)]\\ & + \theta[Fx\cdot Fy - F(Fx\cdot y + x\cdot Fy + \theta x\cdot y)].\end{aligned}$$

Hence, taking into account that F is an admissible Rota-Baxter operator it follows that $L(x,y;F) = L(y,x;F)$, that is

$$Fx\circ Fy - F(Fx\circ y + x\circ Fx + \theta x\circ y) = Fy\circ Fx - F(Fy\circ x + y\circ Fy + \theta y\circ x).$$

■

Proposition 41 *Let $(A,\cdot,F)$ be an admissible Rota-Baxter algebra of weight θ and suppose that $(A,\cdot)$ is a Lie-admissible algebra. Then, $(A,[.,.]_*)$ is a Lie-admissible algebra, where the product $*$ is defined as $x*y = Fx\cdot y - y\cdot Fx - \theta y\cdot x$ and $[x,y]_* = x*y - y*x$ for all $x,y \in A$.*

Proof. The proof is a simple calculation and therefore it will not be presented. ■

Definition 42 *Let $(D,\dashv,\vdash)$ be a vector space provided of two bilinear products $\dashv$ and $\vdash$. We say that a linear operator $F : D \longrightarrow D$ is an* ***admissible Rota-Baxter operator of weight θ on*** *D if by definition*

$$Fx \dashv Fy - F(Fx \dashv y + x \dashv Fy + \theta x \dashv y) = Fy \vdash Fx - F(Fy \vdash x + y \vdash Fx + \theta y \vdash x),$$

for all $x,y \in D$. D is called an ***admissible Rota-Baxter dialgebra*** *which is denoted by $(D,\vdash,\dashv,F,\theta)$. Let us say that $(D,\dashv,\vdash,F,\theta)$ is a* ***restrictive admissible Rota-Baxter dialgebra*** *if F satisfies*

$$Fx \dashv Fy = F(Fx \dashv y + x \dashv Fy + \theta x \dashv y), \tag{148}$$

$$Fx \vdash Fy = F(Fx \vdash y + x \vdash Fy + \theta x \vdash y), \tag{149}$$

for all $x,y \in D$. In this case, the operator F is called a ***restrictive admissible Rota-Baxter operator of weight θ on*** *D.*

It is clear that any restrictive admissible Rota-Baxter dialgebra is also a admissible Rota-Baxter dialgebra.

We remember that if $(D,\dashv,\vdash)$ is an associative dialgebra, D^{ann} denotes the bilateral ideal of D spanned by the elements of the form $x \dashv y - x \vdash y$ for all $x,y \in D$. Observe that $x \dashv z = 0 = z \vdash x$ for all $x \in D$ and every $z \in D^{ann}$. Let D be a dialgebra and we define the subsets $Z_\vdash = \{z \in D | z \vdash x = 0,\ \forall x \in D\}$, $Z_\dashv = \{z \in D | x \dashv z = 0,\ \forall x \in D\}$, and $Z_B = Z_\vdash \cap Z_\dashv$. It is immediately that $D^{\mathrm{ann}} \subset Z_B$. The reader interested in these concepts can consult [31].

Theorem 43 *Let $(D,\dashv,\vdash)$ be an associative dialgebra. Let F be an admissible Rota-Baxter operator of weight 0 on D. Introduce two new operations on D in the form*

$$x \prec y = x \dashv Fy, \qquad x \succ y = Fx \vdash y,$$

for all $x,y \in D$. Suppose that $F(D^{ann}) \subset Z_B$ then D is an admissible dendriform algebra.

Proof. First of all note that $x \dashv F(Fy \dashv z) = x \dashv F(Fy \dashv z \pm Fy \vdash z) = x \dashv F(Fy \vdash z)$. Of the similar manner one can see that $x \dashv F(z \vdash Fy) = x \dashv F(z \dashv Fy)$. Hence

$$\begin{aligned}(x \prec y) \prec z = (x \dashv Fy) \prec z &= (x \dashv Fy) \dashv Fz = x \dashv (Fy \dashv Fz)\\ &= x \dashv F(Fy \dashv z + y \dashv Fz) + x \dashv (Fz \vdash Fy) - x \dashv F(Fz \vdash y + z \vdash Fy)\\ &= x \prec (Fy \vdash z + y \dashv Fz) + x \dashv (Fz \dashv Fy) - x \prec (Fz \vdash y + z \dashv Fy)\\ &= x \prec (y \succ z + y \prec z) + (x \dashv Fz) \dashv Fy - x \prec (z \succ y + z \prec y)\\ &= x \prec (y \succ z + y \prec z) + (x \prec z) \prec y - x \prec (z \succ y + z \prec y),\end{aligned}$$

for all $x,y,z \in D$. It shows (141). The proofs of (139) and (140) are similar and these can be omitted. ■

Proposition 44 *Let $(D, \dashv, \vdash)$ be a Leibniz-admissible algebra and F an admissible Rota-Baxter operator of weight θ on D such that $F(D^{ann}) \subset Z_B$. Define new products by*

$$x \lhd y = Fx \dashv y + x \dashv Fy + \theta x \dashv y, \quad x \rhd y = Fx \vdash y + x \vdash Fy + \theta x \vdash y,$$

for all $x, y \in D$. Then $(D, \lhd, \rhd)$ is also a Leibniz-admissible algebra.

Proof. To prove that the bracket $\{x, y\} = x \lhd y - y \rhd x$ satisfies the Leibniz identity we have into account that $[x, y] = x \dashv y - y \vdash x$ equips D with a structure of Leibniz algebra; moreover that $\{x, y\} = [Fx, y] + [x, Fy] + \theta[x, y]$ and $[Fx, Fy] = F([Fx, y] + [x, Fy] + \theta[x, y])$ for all $x, y \in D$. The rest is a simple calculation. ■

Relation between the generalized dendriform algebras and the left-symmetric dialgebras

In the present section we propose a structure of dendriform type which has been called for us generalized dendriform algebra. The generalized dendriform algebras turn out to be related to left-symmetric dialgebras in a similar way to as the dendriform algebras are related to left-symmetric algebras.

Definition 45 *Let S be a vector space in which we have defined three bilinear multiplications: $\succ, \prec, \circ$. We say that these products endow S with a structure of* ***0-generalized dendriform algebra*** *if these operations satisfy the following axioms:*

$$(x \prec y) \prec z = x \prec (y \prec z + y \succ z), \tag{150}$$

$$(x \succ y) \prec z = x \succ (y \prec z), \tag{151}$$

$$x \succ (y \succ z) = (x \prec y + x \succ y) \succ z, \tag{152}$$

$$x \succ (y \circ z) + x \circ (y \circ z) = (y \circ z) \prec x, \tag{153}$$

$$(x \circ y) \succ z = z \prec (x \circ y). \tag{154}$$

We say that a 0-generalized dendriform algebra is a ***generalized dendriform algebra of type** I, if additionally we have*

$$x \succ (y \circ z) = (x \succ y + x \prec y) \circ z, \tag{155}$$

$$x \circ (y \succ z) - x \circ (z \prec y) = (y \circ z) \prec x - (x \circ z) \prec y, \tag{156}$$

$$x \circ (y \circ z) = (x \circ y) \circ z, \tag{157}$$

for all $x, y, x \in D$.

We call a 0-generalized dendriform algebra, ***generalized dendriform algebra of type** II if also we have*

$$(x \circ y) \circ z = x \circ (y \succ z + y \circ z), \tag{158}$$

$$(x \circ y) \prec z = x \circ (y \prec z) + (z \circ y) \prec x. \tag{159}$$

Remark 46 *Observe that the category of dendriform algebras can be identified with the subcategory of objects in the category of generalized dendriform algebras of any type with* $\circ = 0$.

We now turn to explain how one can obtain the axioms (150)-(157) and (150)-(154) together with (158), (159). We know that if $(S, \succ, \prec)$ is a dendriform algebra then $(S, \vdash)$ is a left-symmetric algebra with the operation $\vdash$ defined by $x \vdash y = x \succ y - y \prec x$, that is, the axiom (4) holds with respect to the product $\vdash$. Under the same hypothesis if we also put $x \dashv y = x \prec y - y \succ x$, all the axioms (1)-(4) hold and the left-symmetric dialgebra reduces to a left-symmetric algebra. On other hand, if we would like to define $\dashv$ different from $\vdash$ but such that $(S, \vdash, \dashv)$ remains a left-symmetric dialgebra, we do this through a third bilinear operation, that is, since $x \dashv y - x \vdash y = x \circ y$ defines $\circ$ as a bilinear operation different of zero, then we have $x \dashv y = x \succ y - y \prec x + x \circ y$ for all $x, y \in S$. Next, in order to check that (153)-(157) and (153), (154), (158), (159) constitute a good choice, we introduce

$$x \vdash y = x \succ y - y \prec x, \quad x \dashv y = x \succ y - y \prec x + x \circ y = x \vdash y + x \circ y,$$

in (1)-(3) allowing us to see that these relations are sufficient to obtain a left-symmetric dialgebra. This procedure is summarized in the following theorem.

Theorem 47 *Let* $(S, \succ, \prec, \circ)$ *be a generalized dendriform algebra of type* I *or* II. *Define*

$$x \vdash y = x \succ y - y \prec x, \quad x \dashv y = x \succ y - y \prec x + x \circ y = x \vdash y + x \circ y, \tag{160}$$

for all $x, y, z \in S$, *then* $(S, \vdash, \dashv)$ *is a left-symmetric dialgebra.*

Proof. Let us make the proof in the case for which $(S, \succ, \prec, \circ)$ is a generalized dendriform algebra of type I. Notice that we have already indicated that the axioms (150)-(152) imply (4). From (153) we have for all $x, y, z \in S$

$$x \dashv (y \dashv z) = x \dashv (y \vdash z + y \circ z) = x \dashv (y \vdash z).$$

It shows that (1) holds. On other hand, of (154) follows that

$$(x \dashv y) \vdash z = (x \vdash y + x \circ y) \vdash z = (x \vdash y) \vdash z,$$

for all $x, y, z \in S$. Thus, we have verified the axiom (2). Now,

$$\begin{aligned} N &= x \dashv (y \dashv z) - (x \dashv y) \dashv z - y \vdash (x \dashv z) + (y \vdash x) \dashv z \\ &= x \dashv (y \vdash z + y \circ z) - (x \vdash y + x \circ y) \dashv z \\ &\quad - y \vdash (x \vdash z + x \circ z) + (y \vdash x) \vdash z + (y \vdash x) \circ z \\ &= x \vdash (y \vdash z + y \circ z) + x \circ (y \vdash z + y \circ z) - (x \vdash y + x \circ y) \vdash z \\ &\quad - (x \vdash y + x \circ y) \circ z - y \vdash (x \vdash z + x \circ z) + (y \vdash x) \vdash z + (y \vdash x) \circ z. \end{aligned}$$

Taking into account that $(S, \vdash)$ is a left-symmetric algebra follows

$$\begin{aligned} N &= x \vdash (y \circ z) + x \circ (y \vdash z) + x \circ (y \circ z) - (x \circ y) \vdash z - (x \vdash y) \circ z \\ &\quad - (x \circ y) \circ z - y \vdash (x \circ z) + (y \vdash x) \circ z. \end{aligned}$$

Finally, (154)-(157) imply that $N = 0$. The proof for a generalized dendriform algebra of type II is similar and therefore this will not be presented. ■

We already know that in any admissible dendriform algebra $(A, \prec, \succ)$ we can provide A of a structure of left-symmetric algebra with the product defined by $x \vdash y = x \succ y - y \prec x$ for all $x, y \in A$. This fact is also hold for any left L-dendriform algebra. On other hand, a left L-dendriform algebra is not necessarily an admissible dendriform algebra. Thus, the process that we have followed above to construct generalized dendriform algebras of type I and II can be repeated if one replaces (150)-(152) by (139)-(141) or (142)-(143) and keeps the operations $\vdash$ and $\dashv$ defined by (160). Therefore, the following two definitions and theorems do not need more justification.

Definition 48 *Let S be a vector space in which we have defined three bilinear multiplications: $\succ, \prec, \circ$. We say that these products equip S with a structure of* 0-***generalized admissible dendriform algebra*** *if these operations satisfy the following axioms:*

$$x \succ (y \succ z) - (x \prec y + x \succ y) \succ z = y \succ (x \succ z) - (y \prec x + y \succ x) \succ z, \tag{161}$$

$$x \succ (y \prec z) = (x \succ y) \prec z, \tag{162}$$

$$(x \prec y) \prec z - x \prec (y \prec z + y \succ z) = (x \prec z) \prec y - x \prec (z \prec y + z \succ y), \tag{163}$$

$$x \succ (y \circ z) + x \circ (y \circ z) = (y \circ z) \prec x, \tag{164}$$

$$(x \circ y) \succ z = z \prec (x \circ y). \tag{165}$$

We say that a 0*-generalized admissible dendriform algebra is a* ***generalized dendriform algebra of type*** III *if (155)-(157) hold. On other hand, by a* ***generalized dendriform algebra of type*** IV *we will understand a* 0*-generalized admissible dendriform algebra in which (158),(159) hold.*

Theorem 49 *Let $(S, \succ, \prec, \circ)$ be a generalized admissible dendriform algebra of type III or IV. If we define two new operations by*

$$x \vdash y = x \succ y - y \prec x, \quad x \dashv y = x \succ y - y \prec x + x \circ y = x \vdash y + x \circ y, \tag{166}$$

for all $x, y, z \in S$, then $(S, \vdash, \dashv)$ is a left-symmetric dialgebra.

Definition 50 *Let S be a vector space with three bilinear operations: $\succ, \prec, \circ$. We say that these products equip S with a structure of* 0-***generalized left*** L-***dendriform algebra*** *if these operations satisfy the following axioms:*

$$x \succ (y \succ z) - (x \prec y + x \succ y) \succ z = y \succ (x \succ z) - (y \prec x + y \succ x) \succ z, \tag{167}$$

$$x \succ (y \prec z) - (x \succ y) \prec z = y \prec (x \prec z + x \succ z) - (y \prec x) \prec z. \tag{168}$$

$$x \succ (y \circ z) + x \circ (y \circ z) = (y \circ z) \prec x, \tag{169}$$

$$(x \circ y) \succ z = z \prec (x \circ y). \tag{170}$$

We say that a 0*-generalized left L-dendriform algebra is a* ***generalized dendriform algebra of type*** V *if (155)-(157) hold. On other hand, by a* ***generalized dendriform algebra of type*** VI *we will understand a* 0*-generalized left L-dendriform algebra in which (158),(159) hold.*

Theorem 51 *Let $(S, \succ, \prec, \circ)$ be a generalized dendriform algebra of type V or VI. Then, the two new operations defined on S by*

$$x \vdash y = x \succ y - y \prec x, \quad x \dashv y = x \succ y - y \prec x + x \circ y = x \vdash y + x \circ y, \tag{171}$$

for all $x, y, z \in S$, transform $(S, \vdash, \dashv)$ in a left-symmetric dialgebra.

Remark 52 *Le $(S, \succ, \prec, \circ)$ be a generalized dendriform algebra of any type. We already know that $(S, \vdash, \dashv)$ is a left-symmetric dialgebra, where $x \vdash y = x \succ y - y \prec x$ and $x \dashv y = x \vdash y + x \circ y = x \succ y - y \prec x + x \circ y$. Hence, $(S, [.,.])$ is a Leibniz algebra with the bracket defined as $[x, y] = x \dashv y - y \vdash x$ for all $x, y \in S$. Define $x \triangleright y = x \succ y + x \prec y$ and $x \triangleleft y = x \triangleright y + x \circ y = x \succ y + x \prec y + x \circ y$, then*

$$\begin{aligned} [x, y] = x \dashv y - y \vdash x &= (x \succ y - y \prec x + x \circ y) - (y \succ x - x \prec y) \\ &= x \triangleleft y - y \triangleright x, \end{aligned}$$

for all $x, y \in S$. It shows that $(S, \triangleright, \triangleleft)$ is a Leibniz-admissible algebra with the same Leibniz algebra that $(S, \vdash, \dashv)$.

Some open problems

We give the following definition

Definition 53 *Let $(A, \cdot)$ be an associative algebra. We say that A is of* ***dendriform type****, if on A are defined two bilinear product $\succ$ and $\prec$ such that $\cdot = \succ + \prec$ and $(A, \prec, \succ)$ is an associative dendriform algebra. We say that $(A, \prec, \succ)$ is the associative dendriform algebra associated to $(A, \cdot)$.*

Problem 1: To give necessary and sufficient conditions under which an associative algebra can be of dendriform type.

Other open problem is the following

Problem 2: We propose the question of embedding an arbitrary admissible dendriform algebra into a Rota-Baxter algebra. The same problem is valid for any left L-dendriform algebra and all right L-dendriform algebra.

The construction of dendriform algebras, dendriform trialgebras and Rota-Baxter operators has been addressed in several works within which we have chosen only a few: [16], [17], [18], [15], [23], [25].

Problem 3: To represent the different structures of dendriform type defined for us through the space of trees and the space of words of some alphabet (of course, the description of these specific spaces should be an important part of the research). To find examples of Rota-Baxter operators for flexible and alternative algebras.

Problem 4: Is it possible to implement a Caley-Dickson process directly for dendriform algebras?.

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