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The negative Wiener-Hopf factor of two-sided jumps Lévy processes, and an application to Insurance Risk Theory

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Abstract

We consider a class of two-sided jumps Lévy processes whose positive jumps have a rational Laplace transform, and obtain an explicit expression for the probability density of their negative Wiener-Hopf factor. This formula is in terms of an infinite series of convolutions of known functions, which we present explicitly. This result, together with the corresponding quintuple law of the overshoots and undershoots of the process, allows us to obtain an expression for the corresponding Generalized Expected discounted penalty function, which was introduced in Biffis and Morales [2010].

Keywords and phrases: Two-sided jumps Lévy process, Wiener-Hopf factorization, Generalized discounted penalty function, First passage time, Risk process, Quintuple Law.

1   Introduction

The Wiener-Hopf factorization of Lévy processes is important in applied probability due to its applications in several branches, such as insurance mathematics, theory of branching processes and mathematical finance. In particular, the negative Wiener-Hopf factor can be used to compute several functionals in insurance mathematics, such as the Expected Discounted Penalty Function (EDPF), which was introduced in Gerber and Shiu [1998], and its generalized version, which was introduced and studied in Biffis and Morales [2010] and Biffis and Kyprianou [2009] for the case of spectrally negative Lévy processes. The EDPF represents one of the most relevant risk measures in insurance risk theory, since it allows one to obtain, as particular cases, the ruin probability, the Laplace transform of the time to ruin, the joint bivariate tail of the severity of ruin and the surplus prior to ruin, among many others. While the distribution of the positive Wiener-Hopf factor has been studied by many authors (see, for instance, Kuznetsov [2010a], Kuznetsov and Peng [2012] and Lewis and Mordecki [2008]), the distribution of the negative one has not received many attention.

In general, the negative Wiener-Hopf factor is harder to study in the case of Lévy processes when the negative jumps are given by pure jumps processes with unbounded variation. In our previous work, Kolkovska and Martín-González [2016], we considered the case of a two-sided jumps Lévy risk process without brownian component with negative jumps given by the dual of a compound Poisson process plus an α-stable process with only negative jumps.

In this communication, in Theorem 1 and Theorem 2 we present the Laplace transforms and explicit expressions, respectively, for the probability density of the negative Wiener-Hopf factor for a wide class of Lévy processes having two-sided jumps. Our formula for the density of the negative Wiener-Hopf factor is in terms of known functions and the derivative of a q-scale function of an associated spectrally negative Lévy process. These q-scale functions have recently been the object of study in many recent works (see, for instance, Cohen et al. [2012] and the references therein). Using these expressions for the negative Wiener-Hopf factor, in Theorem 3 we give an explicit expression for the Generalized Expected Discounted Penalty Function of the Levy process we consider.

Our results on the negative Wiener-Hopf factor complement the result in Lewis and Mordecki [2008], and our formula for the Generalized EDPF extends the results in Biffis and Morales [2010] and Biffis and Kyprianou [2009] to the case of two-sided jumps Lévy risk processes.

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2 Preliminaries and main results

In this section we give some definitions and preliminary results, and present the main results. The proofs will be given in the forthcomming Ph. D. thesis of Ehyter M. Martín-González.

Throughout this work we denote by $\widehat{f}(r)$ the Laplace transform of a nonnegative function f, for all r such that this Laplace transform exists and is finite.

The subclass of two-sided jumps Lévy processes we study consists of all two-sided jumps Lévy processes $\mathcal{X}$, defined by the equation

$$\mathcal{X}(t) = c_0 t + \gamma \mathcal{B}(t) + \mathcal{Z}(t) - \mathcal{S}(t), \ , t \geq 0, \tag{2.1}$$

where $c_0 \geq 0$ is a drift term, $\mathcal{Z} = \{\mathcal{Z}(t), t \geq 0\}$ is a compound Poisson process with Lévy measure $\lambda_1 f_1(x)$, where f_1 is a probability density with Laplace transform of the form

$$\widehat{f}_1(r) = \frac{Q(r)}{\prod\limits_{i=1}^{N}(q_i + r)^{m_i}} \tag{2.2}$$

where $N, m_i \in \mathbb{N}$ with $m_1 + m_2 + \cdots + m_N = m$ and $0 < q_1 < q_2 < \ldots q_m$ and $Q(r)$ is a polynomial function of degree $m - 1$ or less. $\mathcal{B} = \{\mathcal{B}(t), t \geq 0\}$ is a Brownian motion and $\mathcal{S} = \{\mathcal{S}(t), t \geq 0\}$ is a pure jumps Lévy process with positive jumps.

We set $\Psi_{\mathcal{S}}(r) = \int\limits_{0+}^{\infty} \left(1 - e^{-rx} - rx 1_{\{x<1\}}\right) \nu_{\mathcal{S}}(dx)$, which satisfies $\Psi_{\mathcal{S}}(r) = -\log \mathrm{E}\left[e^{-r\mathcal{S}(1)}\right]$, i.e. $\Psi_{\mathcal{S}}(r)$ is the Laplace exponent of the process $\mathcal{S}$. For the process $\mathcal{X}$, we set:

$$L_{\mathcal{X}}(r) = c_0 r + \gamma^2 r^2 + \lambda_1 \left(\frac{Q(-r)}{\prod\limits_{j=1}^{N}(q_j - r)^{m_j}} - 1\right) - \Psi_{\mathcal{S}}(r). \tag{2.3}$$

We refer to the equation $L_{\mathcal{X}}(r) - \delta = 0$ as the Generalized Lundberg Equation (GLE for short) associated to the process $\mathcal{X}$ and call $L_{\mathcal{X}}(r)$ the Generalized Lundberg Function (GLF for short) of $\mathcal{X}$. For $r < q_1$, this function coincides with $log E\left[e^{r\mathcal{X}(1)}\right]$. We let $\Psi_{\mathcal{X}}^{[c]}(r) = -log E\left[e^{ir\mathcal{X}(1)}\right]$ denote the characteristic exponent of $\mathcal{X}$. Hence, the GLF above is related to $\Psi_{\mathcal{X}}^{[c]}(r)$ by the equality

$$-\Psi_{\mathcal{X}}^{[c]}(-ir) = L_{\mathcal{X}}(r). \tag{2.4}$$

In the case when $\mathcal{S}$ is a subordinator, the GLF of $\mathcal{X}$ can be rewritten as:

$$L_{\mathcal{X}}(r) = cr + \gamma^2 r^2 + \lambda_1 \left(\frac{Q(-r)}{\prod\limits_{j=1}^{N}(q_j - r)^{m_j}} - 1\right) - G_{\mathcal{S}}(r). \tag{2.5}$$

where in this case $c = c_0 + \int\limits_{0+}^{1} x \nu_{\mathcal{S}}(dx)$, and $G_{\mathcal{S}}(r) = \int\limits_{0+}^{\infty} \left(1 - e^{-rx}\right) \nu_{\mathcal{S}}(dx)$.

We impose the condition that $E\left[\mathcal{X}(1)\right] > 0$, which implies that the process $\mathcal{X}$ drifts to infinity, hence in the case when $\mathcal{S}$ is a subordinator, we obtain $\int\limits_{0+}^{\infty} (x \wedge 1) \nu_{\mathcal{S}}(dx) < \infty$, meaning that the Lévy measure of $\mathcal{S}$ has a finite mean in this case. In the sequel, we denote this mean by $\mu_{\mathcal{S}} = \int\limits_{0+}^{\infty} x \nu_{\mathcal{S}}(dx)$.

When $\mathcal{S}$ is a subordinator we have $c_0 r - \Psi_{\mathcal{S}}(r) = cr - G_{\mathcal{S}}(r)$ where $c = c_0 + \mu_{\mathcal{S}}$. In view of this, in what follows we set $c = c_0$ when $\mathcal{S}$ is a general pure positive jumps Lévy process and $c = c_0 + \mu_{\mathcal{S}}$ when $\mathcal{S}$ is a subordinator.

Now we recall the definition of the Wiener-Hopf factors:

Let us define the random variables $S_t^{\mathcal{X}} = \sup_{0 \leq s \leq t} \mathcal{X}(s)$ and $I_t^{\mathcal{X}} = \inf_{0 \leq s \leq t} \mathcal{X}(s)$, and let e_q be an exponential random variable with mean $1/q$, independent of $\mathcal{X}$. The positive and negative Wiener-Hopf factors of $\mathcal{X}$ are given, respectively, by $S_{e_q}^{\mathcal{X}}$ and $I_{e_q}^{\mathcal{X}}$, and they can be identified through their characteristic functions, by the equality

$$\frac{q}{q + \Psi_{\mathcal{X}}^{[c]}(r)} = E\left[e^{ir S_{e_q}^{\mathcal{X}}}\right] E\left[e^{ir I_{e_q}^{\mathcal{X}}}\right].$$

We also recall the definition of a q-scale function associated to a spectrally negative Lévy process (i.e. a Lévy process with only negative jumps).

Let $\mathcal{Z}$ be a spectrally negative Lévy process with Laplace exponent $\Psi_{\mathcal{Z}}(r) = -\log E[e^{-r\mathcal{Z}(1)}]$. It is known (see, for instance, Kyprianou [2006]) that for any $q \geq 0$, there exists a function $\mathfrak{W}^{(q)}(x)$ (called the scale function of $\mathcal{Z}$) such that $\mathfrak{W}^{(q)}(x) = 0$ for $x < 0$ and $\mathfrak{W}^{(q)}$ is characterized on $[0, \infty)$ as the unique strictly increasing and right-continuous function whose Laplace transform satisfies:

$$\int_0^\infty e^{-rx} \mathfrak{W}^{(q)}(x) dx = \frac{1}{\Psi_{\mathcal{Z}}(r) - q}, \quad \text{for all } r > v(q). \tag{2.6}$$

where $v(q)$ is the biggest solution of $\Psi_{\mathcal{Z}}(r) - q = 0$. The case when $q = 0$ is denoted as $\mathfrak{W}(x)$.

We impose the condition that $E[\mathcal{X}(1)] > 0$ and $\int_{0+}^\infty (x^2 \wedge x) \, \nu_{\mathcal{S}}(dx) < \infty$. In the case when $\mathcal{S}$ is a subordinator, we denote its finite mean by $\mu_{\mathcal{S}} = \int_{0+}^\infty x \nu_{\mathcal{S}}(dx)$.

The extra condition $\int_{0+}^\infty (x^2 \wedge x) \, \nu_{\mathcal{S}}(dx) < \infty$ implies that we can rewrite $\Psi_{\mathcal{S}}(r)$ without using the indicator function $1_{\{x<1\}}$. Hence, when $\mathcal{S}$ is a subordinator we have $c_0 r - \Psi_{\mathcal{S}}(r) = cr - G_{\mathcal{S}}(r)$ where $c = c_0 + \mu_{\mathcal{S}}$. In view of this, in what follows we set $c = c_0$ when $\mathcal{S}$ is a general pure positive jumps Lévy process and $c = c_0 + \mu_{\mathcal{S}}$ when $\mathcal{S}$ is a subordinator.

In what follows we set $\mathbb{C}_+ = \{z \in \mathbb{C} : Re(z) \geq 0\}$ and $\mathbb{C}_{++} = \{z \in \mathbb{C} : Re(z) > 0\}$. For a given function $L : \mathbb{C} \to \mathbb{C}$, we recall that $s \in \mathbb{C}$ is a root of the function L, with multiplicity $m \geq 1$ if $L(s) = 0$, $\frac{d^j}{dr^j} L(r)|_{r=s} = 0$ for all $j = 1, 2, \ldots, m-1$, and $\frac{d^m}{dr^m} L(r)|_{s=r} \neq 0$.

Since the case when $c = \gamma = 0$ and both $\mathcal{Z}$ and $\mathcal{S}$ are compound Poisson processes has been studied in Labbé et al. [2011], we consider only the following three cases:

Case A. $c = \gamma = 0$ and $\mathcal{S}$ is a driftless subordinator (other than a compound Poisson process),

Case B. $c > 0$, $\gamma = 0$ and $\mathcal{S}$ is any subordinator,

Case C. Any other case except the case when $c = \gamma = 0$ and $\mathcal{S}$ is a compound Poisson process.

The following result holds.

Lemma 1 *If $\delta > 0$:*

a) In case A, $L_{\mathcal{X}}(r) - \delta = 0$ has m roots in $\mathbb{C}_{++}$,

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b) *In cases B and C,* $L_X(r) - \delta = 0$ *has* $m + 1$ *roots in* $\mathbb{C}_{++}$.

In all the cases above, there is exactly one root $\rho_{1,\delta}$ *in the interval* $(0, q_1)$. *This root satisfies the equality* $\lim_{\delta \downarrow 0} \rho_{1,\delta} = 0$, *and when* $\delta = 0$, $\rho = 0$ *is a simple root of* $L_X(r) = 0$ *in cases A, B and C.*

In what follows we denote by β_A the function whose value is 1 in case A, and 0 otherwise. We assume that the equation $L_X(r) - \delta = 0$ has R different roots in $\mathbb{C}_{++}$, denoted respectively by $\rho_{1,\delta}, \ldots, \rho_{R,\delta}$ with multiplicities $k_1, k_2, \ldots, k_R$ such that $\sum_{j=1}^{R} k_j = m + 1 - \beta_A$. We let $\rho_{1,\delta}$ be the real root such that $\rho_{1,\delta} \in [0, q_1)$, which implies $k_1 = 1$.

Since $L_X(r) - \delta \to L_X(r)$ as $\delta \downarrow 0$, for all j the roots $\rho_{j,\delta}$ have a limit when $\delta \downarrow 0$. We denote this limits as ρ_j, i.e. for all j we denote

$$\rho_j := \lim_{\delta \downarrow 0} \rho_{j,\delta} \tag{2.7}$$

For $a = 0, 1, \ldots, m + 1$, we define the operator $\mathcal{T}_{s,a}$ by the equation

$$\mathcal{T}_{s,a}f(u) = \int_u^\infty (y - u)^a e^{-s(y-u)} f(y) dy,$$

for each measurable f and complex s such that the integral above exists and is finite. Clearly this operator is linear. If ν is a measure such that $\int_u^\infty (y - u)^a e^{-s(y-u)} \nu(dy)$ exists, we define

$$\mathcal{T}_{s,a}\nu(u) = \int_u^\infty (y - u)^a e^{-s(y-u)} \nu(dy) \tag{2.8}$$

for $a = 0, 1, \ldots, m + 1$. We denote its Laplace transform as $\widehat{\mathcal{T}}_{s,a}f(r)$ for all $r \in \mathbb{C}_+$ such that this Laplace transform exists. Analogously, we denote the Laplace transform of $\mathcal{T}_{s,a}\nu(u)$ by $\widehat{\mathcal{T}}_{s,a}\nu(r)$

In the case when $a = 0$, we obtain the operator $T_s f(x)$ defined in Dickson and Hipp [2001] as $T_s f(u) = \int_u^\infty e^{-s(y-u)} f(y) dy$. In the case when ν is a measure, we have $\mathcal{T}_{s,0}\nu(u) = \int_u^\infty e^{-s(y-u)} \nu(dy)$.

Let us denote the Laplace transform of $T_s f(x)$ (respectively $T_s \nu(x)$) as $\widehat{T}_s f(x)$ (respectively $\widehat{T}_s \nu(x)$).
In order to simplify our notation, for each $j = 1, 2, \ldots, R$ and $\delta \geq 0$ we set

$$E(j, a, \delta) = \binom{k_j - 1}{a} \frac{(-1)^{1-k_j+a}}{(k_j - 1)!} \frac{\partial^{k_j-1-a}}{\partial s^{k_j-1-a}} \left[\frac{\prod_{l=1}^{N} (q_l - s)^{m_l} (\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R} (\rho_{l,\delta} - s)^{k_l}} \right]_{s=\rho_{j,\delta}},$$

$$E_*(j, a, \delta) = \binom{k_j - 1}{a} \frac{(-1)^{1-k_j+a}}{(k_j - 1)!} \frac{\partial^{k_j-1-a}}{\partial s^{k_j-1-i}} \left[\frac{\prod_{l=1}^{N} (q_l - s)^{m_l} (\rho_{j,\delta} - s)^{k_j}}{\prod_{l=1}^{R} (\rho_{l,\delta} - s)^{k_l}} s \right]_{s=\rho_{j,\delta}},$$

and define, for $\delta \geq 0$, the functions

$$\ell_\delta(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E(j, a, \delta) T_{\rho_{j,\delta},a} \nu_S(u), \tag{2.9}$$

$$\mathcal{L}_\delta(u) = \sum_{j=1}^{R} \sum_{a=0}^{k_j-1} E_*(j, a, \delta) T_{\rho_{j,\delta},a} \overline{\mathcal{V}}_S(u) \tag{2.10}$$

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Now we present an explicit expression for probability density of the negative Wiener-Hopf factor $I_{e_\delta}^{\mathcal{X}}$, corresponding to the process $\mathcal{X}$ as defined in (2.1).

We recall that the distribution of the positive Wiener-Hopf factor for the class of processes to which $\mathcal{X}$ belongs, was explicitly given in Lewis and Mordecki [2008]. Let us set $a_\delta = \delta \frac{\prod_{j=1}^{N} q_j^{m_j}}{\prod_{j=1}^{R} \rho_{j,\delta}^{k_j}}$ when $\delta > 0$ and $a_0 = E[\mathcal{X}(1)] \frac{\prod_{j=1}^{N} q_j^{m_j}}{\prod_{j=2}^{R} \rho_j^{k_j}}$ when $\delta = 0$. Using the notation from the previous section, we set $Q_1(r) = \prod_{i=1}^{N} (q_i - r)^{m_i}$ and $\widehat{e}_{0,\delta}^{+}(r) = \frac{\prod_{j=1}^{N} (q_j - r)^{m_j}}{\prod_{l=1}^{R} (\rho_{j,\delta} - r)^{k_l}}$, for $r \neq \rho_{j,\delta}, j = 1, 2, \ldots, R$.

Let us consider the following measures, for $j = 1, 2, 3$:

$$\chi_{j,\mathcal{S}}(dx; \delta) = \begin{cases} \nu_\mathcal{S}(dx) + \ell_\delta(x)dx & j = 1, \\[2mm] \ell_\delta(x)dx & j = 2, \\[2mm] \left[\overline{\mathcal{V}}_\mathcal{S}(x) - \mathcal{L}_\delta(x) \right] dx & j = 3. \end{cases} \tag{2.11}$$

defined for $u > 0$ and $\delta \geq 0$.

Definition 1 *For $j = 1, 2, 3$ we denote by $\mathcal{N}_{j,\delta} = \{\mathcal{N}_{j,\delta}(t), t \geq 0\}$, $\delta \geq 0$, the subordinator with Lévy measure $\chi_{j,\mathcal{S}}(dx; \delta)$, and define, for $\delta \geq 0$, the function $W_\delta(u), u > 0$ through its Laplace transform:*

$$\widehat{W}_\delta(r) = \frac{1}{[\delta - L_\mathcal{X}(r)] \widehat{e}_{0,\delta}^{+}(r)}. \tag{2.12}$$

Remark 1 *We note that $N_{2,\delta}$ is a compound Poisson process.*

We have the following main theorem.

Theorem 1 *The following assertions hold:*

a) *The random variables $-I_{e_\delta}^{\mathcal{X}}$ for $\delta > 0$ and $-I_\infty^{\mathcal{X}}$ satisfy the equalities in distribution:*

$$-I_{e_\delta}^{\mathcal{X}} \overset{d}{=} \begin{cases} \mathcal{N}_{1,\delta}(e_{a_\delta}) & \text{in case A} \\ \mathcal{N}_{2,\delta}(e_{a_\delta}) & \text{in case B} \\ \gamma^2 e_{a_\delta} + \mathcal{N}_{3,\delta}(e_{a_\delta}) & \text{in case C} \end{cases}, \quad -I_\infty^{\mathcal{X}} \overset{d}{=} \begin{cases} \mathcal{N}_{1,0}(e_{a_0}) & \text{in case A} \\ \mathcal{N}_{2,0}(e_{a_0}) & \text{in case B} \\ \gamma^2 e_{a_0} + \mathcal{N}_{3,0}(e_{a_0}) & \text{in case C} \end{cases} \tag{2.13}$$

where e_{a_δ} is an exponential random variable with mean $1/a_\delta$, for $\delta \geq 0$, independent of $\mathcal{N}_{j,\delta}$.

b) *The Laplace transforms of $-I_{e_\delta}^{\mathcal{X}}$ for $\delta > 0$ and $-I_\infty^{\mathcal{X}}$ satisfy, for $r \geq 0$, the equalities:*

$$\mathrm{E}\left[e^{-r\left[-I_{e_\delta}^{\mathcal{X}}\right]}\right] = a_\delta \widehat{W}_\delta(r) \text{ and } \mathrm{E}\left[e^{-r\left[-I_\infty^{\mathcal{X}}\right]}\right] = a_0 \widehat{W}_{\alpha,0}(r).$$

c) *For $\delta \geq 0$, the function $\widehat{W}_{\alpha,\delta}(r)$ satisfies the following equalities:*

$$a_\delta \widehat{W}_\delta(r) = \begin{cases} \dfrac{E\left[e^{-r\mathcal{S}(e_{a_\delta})}\right]}{1 + \frac{1}{a_\delta} E\left[e^{-r\mathcal{S}(e_{a_\delta})}\right]\left[\widehat{\ell}_\delta(0) - \widehat{\ell}_\delta(r)\right]}, & \text{in Case A}, \\[4mm] \dfrac{\frac{a_\delta}{a_\delta + \widehat{\ell}_\delta(0)}}{1 - \frac{1}{a_\delta + \widehat{\ell}_\delta(0)} \widehat{\ell}_\delta(r)}, & \text{in Case B}, \\[4mm] \dfrac{a_\delta \widehat{\mathfrak{W}}_\mathcal{Y}'(r)}{1 - \widehat{\mathfrak{W}}_\mathcal{Y}'(r)\left[\widehat{\mathcal{L}}_\delta(0) - \widehat{\mathcal{L}}_\delta(r)\right]}, & \text{in Case C}, \end{cases} \tag{2.14}$$

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where $\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r)$ is given by

$$\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r) = \frac{r}{a_\delta r + \gamma^2 r^2 - \Psi_{\mathcal{S}}(r)}, \ r, \delta \geq 0$$

In particular, $\widehat{\mathfrak{W}}'_{\mathcal{Y}}(r)$ is the Laplace transform of the derivative of the scale function (for $q = 0$) of the spectrally negative Lévy process $\mathcal{Y}_\delta = \{\mathcal{Y}_\delta(t), t \geq 0\}$ given by

$$\mathcal{Y}_\delta(t) = a_\delta t + \gamma \mathcal{B}(t) - \mathcal{S}(t), \ \delta \geq 0, \gamma \geq 0.$$

Let $\theta_\delta(x), x > 0, \delta \geq 0$ be the density of the random variable $\mathcal{S}(e_{a_\delta})$.

In the following theorem we invert the Laplace transform $\widehat{W}_\delta(r)$ for cases A, B and C.

Theorem 2 .

For $\delta \geq 0$, the function $W_\delta(u), u > 0$ is given by:

$$W_\delta(u) = \begin{cases} \frac{1}{a_\delta}\theta_\delta(u) + \frac{1}{a_\delta}\theta_\delta * \sum_{n=1}^{\infty}\left(-\frac{1}{a_\delta}\right)^n\left(\widehat{\ell}_\delta(0)\theta_\delta - \ell_\delta * \theta_\delta\right)^{*n}(u) & \text{in Case A,} \\ \\ \frac{1}{a_\delta + \widehat{\ell}_\delta(0)}\delta_0 + \frac{1}{a_\delta + \widehat{\ell}_\delta(0)}\sum_{n=1}^{\infty}\left(\frac{1}{a_\delta + \widehat{\ell}_\delta(0)}\right)^n\ell_\delta^{*n}(u), & \text{in Case B,} \\ \\ \mathfrak{W}'_{\mathcal{Y}}(u) + \mathfrak{W}'_{\mathcal{Y}} * \sum_{n=1}^{\infty}\left(\widehat{\mathcal{L}}_\delta(0)\mathfrak{W}'_{\mathcal{Y}} - \mathfrak{W}'_{\mathcal{Y}} * \mathcal{L}_\delta\right)^{*n}(u) & \text{in Case C,} \end{cases} \quad (2.15)$$

where δ_0 is the Dirac delta function. Moreover, when $\delta > 0$ the function $a_\delta W_\delta(u)$ is the density of the random variable $-I_{e_\delta}^{\mathcal{X}}$. If $\delta = 0$, the function $a_0 W_{\alpha,0}(u)$ is the density of the random variable $-I_\infty^{\mathcal{X}}$.

Now, using the Wiener-Hopf factors $S_{e_\delta}^{\mathcal{X}}$ and $I_{e_\delta}^{\mathcal{X}}$ we can derive formulae for the expected discounted penalty function (EDPF for short) for the class of Lévy risk processes $\mathcal{X}^u = \{\mathcal{X}^u(t), t \geq 0\}$ given by $\mathcal{X}^u(t) = u + \mathcal{X}(t), u \geq 0$, where $\mathcal{X}$ is defined in (2.1).

In this setting, the value $u \geq 0$ represents the initial capital of the risk process, the constant c and the process Z_1 represent, respectively, a fixed amount that the insurance company gets at each unit of time and a random amount of gains up to time t. The brownian component represents a perturbation due, for instance, to investment in the insurance market or random events which may mean either gains or losses for the insurance company. Finally, the process $\mathcal{S}$ represents the aggregate claim amount that the insurance company has to pay up to time t in the case when $\mathcal{S}$ is a subordinator. In the case when $\mathcal{S}$ does not have mononote paths, it can be interpreted as a the claims that the insurance company has to pay up to time t, perturbed by some random component whose interpretation is similar to that of the brownian component.

We recall the following generalized version of the Expected discounted penalty function (EDPF for short) associated to $\mathcal{X}^u$, considered in Biffis and Kyprianou [2009] and Biffis and Morales [2010].

Let us set $\tau_0^- = \inf\{t \geq 0 : \mathcal{X}^u(t) < 0\}$ and define the Generalized EDPF associated to $\mathcal{X}^u$ as

$$\phi(u; \delta, \omega) = \mathrm{E}\left[e^{-\delta\tau_0^-}\omega\left(|\mathcal{X}^u(\tau_0^-)|, \mathcal{X}^u(\tau_0^- -), I_{\tau_0^- -}^{\mathcal{X}^u}\right)1_{\{\tau_0^- < \infty\}}\Big|\mathcal{X}^u(0) = u\right], u \geq 0$$

where $\delta \geq 0$ represents a discounted force of interest, $\omega : \mathbb{R}_+^3 \to \mathbb{R}_+^3$ is a function known as penalty function such that $\omega_0 = \omega(0+, 0+, 0+)$ exists and it is positive, and the quantities $|\mathcal{X}^u(\tau_0^-)|, \mathcal{X}^u(\tau_0^- -)$ represent, respectively, the severity of ruin and the surplus immediate before ruin. The quantity $I_{\tau_0^- -}^{\mathcal{X}}$ is the last infimum before the ruin.

Let us define the process $\mathcal{X}^* = \{\mathcal{X}(t) = -\mathcal{X}(t), t \geq 0\}$ and set $\tau_u^+ = \{t \geq 0 : \mathcal{X}^*(t) > u\}$, for $u \geq 0$.

We define the following random variables associated to τ_u^+.

- $\overline{\mathcal{G}}_{\tau_u^+} = \sup\left\{s < \tau_u^+ : S_s^{\mathcal{X}^*} = \mathcal{X}^*(s)\right\}$: the time of the last maximum prior to first passage,

- $\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}$: the length of the excursion making the first passage,

- $\mathcal{X}^*(\tau_u^+) - u$: the overshoot at first passage,

- $u - \mathcal{X}^*(\tau_u^+ -)$: the undershoot at first passage,

- $u - S_{\tau_u^+ -}^{\mathcal{X}^*}$: the undershoot of the last maximum at first passage.

By definition of $\mathcal{X}^u$ and $\mathcal{X}^*$, we note that

$$\mathcal{X}^*(\tau_u^+) - u = |\mathcal{X}^u(\tau_0^-)|, \quad u - \mathcal{X}^*(\tau_u^+ -) = \mathcal{X}^u(\tau_0^- -), \quad u - S_{\tau_u^+ -}^{\mathcal{X}^*} = I_{\tau_0^- -}^{\mathcal{X}^u},$$

and $\overline{\mathcal{G}}_{\tau_u^+} + \tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+} = \tau_u^+ = \tau_0^-$. Hence we can rewrite ϕ as

$$\phi(u; \delta, \omega) = \mathrm{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)} \omega\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S_{\tau_u^+ -}^{\mathcal{X}^*}\right) 1_{\{\tau_u^+ < \infty\}}\right], u \geq 0.$$

To deal with the function above we consider the case in which the process $\mathcal{X}_\alpha^*$ passes above u by a jump or under the event $\{\mathcal{X}^*(\tau_u^+) = u\}$. This last possibility is known as **creeping**, and is due to the presence of the brownian component. It is known that, in this case, we have the equality

$$\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S_{\tau_u^+ -}^{\mathcal{X}^*}\right) = (0, 0, 0).$$

With this in mind, the above expression for $\phi(u)$ can be split as

$$\phi(u; \delta, \omega) = \mathrm{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)} \omega\left(\mathcal{X}^*(\tau_u^+) - u, u - \mathcal{X}^*(\tau_u^+ -), u - S_{\tau_u^+ -}^{\mathcal{X}^*}\right) 1_{\{\tau_u^+ < \infty, \mathcal{X}^*(\tau_u^+) - u > 0\}}\right]$$

$$+ \omega_0 \mathrm{E}\left[e^{-\delta\overline{\mathcal{G}}_{\tau_u^+} - \delta\left(\tau_u^+ - \overline{\mathcal{G}}_{\tau_u^+}\right)} 1_{\{\tau_u^+ < \infty, \mathcal{X}^*(\tau_u^+) = u\}}\right], \quad u \geq 0. \tag{2.16}$$

We recall cases A, B and C considered in the last section and the equation $\Psi_X(r) - \delta = 0$ and its R roots $\rho_{j,\delta} \in \mathbb{C}_{++}, j = 1, 2, \ldots, R$ with their multiplicities $k_1 \equiv 1, k_j, j = 2, 3, \ldots, R$, such that $\sum_{j=1}^R k_j = m + 1 - \beta_A$.

We now state the main result on the calculation of EDPF.

Theorem 3 *For $\delta \geq 0$, the EDPF ϕ associated to $\mathcal{X}^u = u + \mathcal{X}$ has the expression*

$$\phi(u; \delta, \omega) = \omega_0 \gamma^2 W_\delta(u) + H_{\delta,\omega} * W_\delta(u), \tag{2.17}$$

where $a_\delta W_\delta(u)$ is the density of the Wiener-Hopf factor $-I_{e_\delta}^{\mathcal{X}}$, given explicitly in Theorem 2, and $H_{\delta,\omega}$ is defined as

$$H_{\delta,\omega}(u) = K_\omega(u)\beta_A + \sum_{j=1}^R \sum_{a=0}^{k_j - 1} E(j, a, \delta) J_{\omega,\delta,a,j}(u), \tag{2.18}$$

where

$$K_\omega(y) = \int_{y+}^\infty \omega(x - y, y, y)\nu_{\mathcal{S}}(dx),$$

$$J_{\omega,\delta,a,j}(y) = \int_{y+}^\infty (v - y)^a e^{-\rho_{j,\delta}(v-y)} \int_v^\infty \omega(x - v, v, y)\nu_{\mathcal{S}}(dx)dv.$$

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