FOLIATIONS WITH A KUPKA COMPONENT ON ALGEBRAIC MANIFOLDS.

Omegar CALVO–ANDRADE

October 1994

Abstract. We consider codimension one holomorphic foliations in complex projective manifolds of dimension at least 3, having a compact Kupka component and represented by integable holomorphic sections ω of the bundle $TM^* \otimes L$, where L denotes a very ample holomorphic line bundle. We will show that, if the transversal type is not the radial vector field and $H^1(M,\mathbb{C}) = 0$, then the foliation has a meromorphic first integral

0. Introduction.

Let M be a compact complex manifold of dimension ≥ 3. By a **complex open book foliation** of M, we mean a codimension one holomorphic foliation with singularities $\mathcal{F}$ on M such that

(1) The foliation $\mathcal{F}$ is represented by an integrable section ω of the bundle $T^*M \otimes L$, where L is a **very ample** holomorphic line bundle.
(2) The singular set of the foliation has a compact connected component K of the Kupka singular set.

Recall that an "open book" foliation of a real manifold M, consists of a codimension two submanifold $K \subset M$, and a fibration $\pi : M - K \to S^1$, such that each fiber of π is a manifold with boundary K. The submanifold K is called the **binding**.

A complex open book foliation of the projective space $\mathbb{P}^n \quad n \geq 3$, is given by the fibers of a rational map if, and only if, the Kupka component K is a complete intersection [C–L], and then, the foliation looks like a "fibration" $\pi : \mathbb{P}^n - K \to \mathbb{P}^1$, and the binding is precisally the Kupka set.

A codimension one holomorphic foliation with singularities on M, admits an **affine transversal struvture** if there exists an analityc subset $\Lambda \subset M$ such that, in $M - \Lambda$ there exists an open cover $\{U_\alpha\}$, and submersions $f_\alpha : U_\alpha \to \mathbb{C}$ which defines the foliation, such that

$$f_\alpha = c_{\alpha\beta} \cdot f_\beta + d_{\alpha\beta} \quad c_{\alpha\beta} \in \mathbb{C}^*, \quad d_{\alpha\beta} \in \mathbb{C}.$$

Supported by Conacyt: 3398–E 9307

In this note, we are going to prove the following theorem:

2.5 Theorem. *Let $\mathcal{F}$ be a complex open book decomposition of M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$. If the transversal type of K is not the radial vector field, then the foliation has at least one compact leaf, moreover, in the complement of this leaf, the foliation admits an affin transversal structure.*

In particular, when $H^1(M;\mathbb{C}) = 0$, one can say more:

2.6 Theorem. *Let $\mathcal{F}$ be as in 2.5. If $H^1(M;\mathbb{C}) = 0$, then the foliation has a meromorhic first integral*

And the foliation looks like the fibers of a "fibration" over $\mathbb{P}^1$.

1 Positive Vector Bundles.

In this section, we will discus the notion of positivity of holomorphic vector bundles.

Let M be a compact, complex manifold and let $E \to M$ be a holomorphic vector bundle; we denote by $H^0\big(M, \mathcal{O}(E)\big)$ or by $\Gamma(E)$, the (finite dimensional) complex vector space of holomorphic sections of the vector bundle E.

1.1 Definition. A holomorphic vector bundle $E \to M$ is called **positive** if there exists an hermitian metric in E whose *curvature tensor* $\Theta = (\Theta^{\rho}_{\sigma i j})$ has the property that the hermitian quadratic form

$$\Theta(\zeta, \eta) = \sum_{\rho,\sigma,i,j} \Theta^{\rho}_{\sigma i j} \zeta^{\sigma} \overline{\zeta}^{\rho} \eta^{i} \overline{\eta}^{j}$$

is positive definite in the variables ζ, η.

Recall that the positivity of a holomorphic line bundle is a topological property, that is, a holomorphic line bundle L is positive if and only if its Chern class may be represented by a positive $(1,1)$–form in $H^2_{DR}(M)$ [G–H].

A holomorphic line bundle L is called **very ample**, if the sections of L gives an embedding into projective spaces. If a line bundle L is positive, then there exists $n_0 \in \mathbb{N}$ such that $L^{\otimes n}$ is very ample for $n \geq n_0$.

A complex manifold X of complex dimension n is called is k–**convex**, if there exists a $\mathcal{C}^2$ function $\varphi : X \to \mathbb{R}$ such that its Levy form

$$L(\varphi) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right),$$

has at least $n - k + 1$ positive eigenvalues, and φ is an **exhustion function**, that is

$$X = \bigcup_{a \in \mathbb{R}} \varphi^{-1}(a, \infty).$$

Observe that if X is an n–dimensional manifold 1–convex, then the function φ is **plurisubharmonic**, and then, it is a Stein manifold.

The following theorem, relates the geometric properties of a smooth submanifold $X \subset M$ and the convexity properties of the complement $M - X$, the proof may be found in [S].

1.2 Theorem. *Let M be a compact, complex manifold and let $X \subset M$ be a compact, k–codimensional smooth submanifold. If the normal bundle ν_X is positive, then $M - X$ is k–convex.*

The main property that we will use, is the following generalization of Hartog's extension Theorem:

1.3 Corollary. *Let M be a compact, complex manifold and let $X \subset M$ be a compact, k–codimensional smooth submanifold, where $k \geq 2$ and such that $M - X$ is k–convex. Let $\mathcal{E}$ be a locally free analytic sheaf. Then the restriction map*

$$\Gamma(M;\mathcal{E}) \to \Gamma(U;\mathcal{E})$$

is surjective

The main idea in the proof of theorem (2.5), consists in the following: First, we will prove the positivity of the normal bundle of a Kupka component, this implies, by theorem (1.2), that $M - K$ is 2–convex, and then, we will use the corollary (1.3), in order to find an integrating factor as in [C–L].

2. Kupka Type Singularities.

As usual, by a codimension one holomorphic foliation with singularities in a complex manifold M, we mean a family of holomorphic 1–forms ω_α defined on an open cover $\{U_\alpha\}$ of M, satisfying the **integrability condition** $\omega_\alpha \wedge d\omega_\alpha = 0$, and the **cocycle condition** $\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta$ when $U_\alpha \cap U_\beta \neq \emptyset$, where $\lambda_{\alpha\beta}$ are never vanishing holomorphic functions defined on $U_\alpha \cap U_\beta$. Let L be the holomorphic line bundle on M obtained with the cocycles $\{\lambda_{\alpha\beta}\}$, then the 1–forms $\{\omega_\alpha\}$ glue to a holomorphic section of the vector bundle $T^*M \otimes L$.

2.1 Definition. A **codimension one holomorphic foliation** $\mathcal{F}$ with singularities in the complex manifold M is an equivalence class of sections $\omega \in \Gamma(M, T^*M \otimes L)$ where L is a holomorphic line bundle such that ω does not vanishes on any connected component of M and satisfies the integrability condition $\omega \wedge d\omega = 0$.

The **singular set** of the foliation represented by the section ω, is the set of points $S_\omega = \{p \in M \,|\, \omega(p) = 0\}$. The **leaves of the foliation** are the leaves of the non–singular foliation defined on $M - S_\omega$. When a leaf $\mathcal{L}$ of the foliation $\mathcal{F}$ is such that its closure $\overline{\mathcal{L}}$ is a closed analytic subspace of M of codimension 1, we will also call $\overline{\mathcal{L}}$ a leaf of the foliation, and by abuse of language, we will say that $\overline{\mathcal{L}}$ is a **compact leaf** of the foliation.

Recall that an integrating factor of a foliation represented by a section $\omega \in \Gamma(M; T^* \otimes L)$ is a holomorphic section $\varphi \in H^0\big(M;\mathcal{O}(L)\big)$ such that the meromorphic 1–form

$$\Omega = \frac{\omega}{\varphi}$$

is closed.

We will use the following result, whose proof may be found in [C], under the hypotheses $H^1(M;\mathbb{C}) = 0$, if the section $\varphi = \varphi_1^{n_1} \cdots \varphi_k^{n_k} \in H^0\big(M;\mathcal{O}(L)\big)$ is an integrating factor of ω, then

$$\frac{\omega}{\varphi} = \Omega = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d\left(\frac{\psi}{\varphi_1^{n_1-1}\cdots\varphi_k^{n_k-1}}\right) \tag{1}$$

and ψ is a holomorphic section of the line bundle associated to the divisor

$$\sum_{i=1}^{k}(n_i-1)\cdot[\varphi_i=0].$$

We will say that a codimension one holomorphic foliation $\mathcal{F}$ on M is **positive** (resp. **very ample**), if it is represented by an integrable section $\omega \in \Gamma(T^*M \otimes L)$, and L is a **positive** (resp. **very ample**) holomorphic line bundle.

2.2 Definition. Let $\omega \in \Gamma(M, T^*M \otimes L)$ be a codimension one holomorphic foliation. The **Kupka singular set**, $K_\omega \subset S_\omega$ is defined by:

$$K_\omega = \{p \in S_\omega \mid \quad d\omega(p) \neq 0\}.$$

We say that $\omega \in H^0\big(M, \Omega^1(L)\big)$ has a **Kupka component** if the Kupka set has a compact connected component.

It is known, (see [Me] for details), that the Kupka set is well defined and it is independent of the choice of the section ω. The main property of the Kupka set, is the **local product property**, namely, for every connected component $K \subset K_\omega$ there exist a holomorphic 1-form η, called the **transversal type** at the component K, defined in a neighborhood V of $0 \in \mathbb{C}^2$ and vanishing only at 0, an open covering $\{U_\alpha\}$ of a neighborhood of K in M, and a family of submersions $\varphi_\alpha : U_\alpha \to \mathbb{C}^2$ such that $\varphi_\alpha^{-1}(0) = K \cap U_\alpha$, In particular, the Kupka set, is a codimension 2 locally closed, smooth submanifold of M, futhermore, the family of 1–forms $\omega_\alpha = \varphi_\alpha^*\eta$ defines the foliation in the open set U_α.

The transversal type η, is well defined up to biholomorphism and multiplication by non–vanishing holomorphic function, as well its dual vector field X, since $d\omega \neq 0$ we have that $\mathrm{Div}X(0) \neq 0$, thus the linear part $\mathbf{L} = DX(0)$, which is well defined up to linear conjugation and multiplication by scalars, has at least one non–zero eigenvalue. We can normalize, by multiplaying by a non–zero complex number, in orther to have eigenvalues $1, \lambda$. We will say that $\mathbf{L}$ is the **linear type** at the component K.

Let K be a connected component of the Kupka set of a foliation $\mathcal{F}$, and let $\mathbf{L}$ be the linear type of K, with eigenvalues $1, \lambda$, we say that the linear transversal type is **Poincaré**, if $\lambda \notin (-\infty, 0)$, **Siegel**, if $\lambda \in (-\infty, 0)$ and degenerate if $\lambda = 0$. When $\lambda = 1$, and the canonical Jordan form of $\mathbf{L}$ is non semisimple, we say that the linear type is non–semisimple.

The local behaviour of the leaves near the Kupka set, which dependes on the transversal type of the component, and the non–triviality of the embedding of $K \subset M$, which is mesured by the topological properties of the normal bundle, are strongly related, namely, [GM–L] have shown that if the first Chern class of the normal bundle $\nu_K(M)$ of a compact connected component K of the Kupka set is non–zero, then the linear transversal type must be semisimple, and with eigenvalues (μ,λ) where $\mu/\lambda\in\mathbb{Q}$.

As a consequence of the local product structure, a Kupka component of a foliation represented by a section $\omega\in\Gamma(M;T^*M\otimes L)$, is **subcanonically embedded** that is, its canonical bundle, and the second exterior power of the normal bundle, can be extended to a global holomorphic line bundle on M. In fact, we have the following formulaes:

$$\Omega_K^{n-2}=\Omega_M^n\otimes L|_K \quad \text{and} \quad \wedge^2\,\nu_K=L|_K,$$

where Ω_K^{n-2} and Ω_M^n denote the canonical bundle of K and M respectively.

Now, let $\mathcal{F}$ be a foliation represented by a section $\omega\in\Gamma(M;T^*M\otimes L)$, with a compact conected component K of the Kupka set. When the line bundle L is very ample, the above formulae implies that the first Chern class of the normal bundle of $K\subset M$, does not vanishes, because it is given by the formulae

$$c_1(\nu_K)=i^*c_1(L)\in H^2(K;\mathbb{C}),$$

and then, the linear transversal must be diagonalizable with eigenvalues $1,\lambda\in\mathbb{Q}$, furthermore, if the linear transversal type belongs to the Poincaré domain, then the transversal type is linearizable and semisimple, [GM–L]. This remark gives the following result.

2.3 Proposition. *Let $\omega\in H^0\big(M;\Omega^1(L)\big)$ be a very ample foliation with a Kupka component K. If the transversal type is not the radial vector field, then the normal bundle of K is positive.*

Proof. The above remark implies that the linear transversal type of the Kupka component K_ω is given by the 1–form

$$\eta=px\,dy-qy\,dx \quad \text{with} \quad p,q\in\mathbb{Z} \quad (p,q)\neq 1.$$

We are going to prove, that the linear transversal type belongs to the Poincaré domain:

In [C–S] it is shown that there exists a pair (E,σ), where $E\to M$ is rank two holomorphic vector bundle over M, and σ is a holomorphic section of the bundle E, whose zero locus is the Kupka set. The total Chern class of the bundle E is given by

$$c(E)=\left(1+\frac{p\cdot c_1(L)}{p+q}\right)\cdot\left(1+\frac{q\cdot c_1(L)}{p+q}\right)\in H^*(M;\mathbb{C}),$$

whenever the linear transversal type is $px\,dy-qy\,dx$.

Now, consider the embedding induced by the line bundle L, the degree of the Kupka set, which is a positive integrer, is given essentially by the formulae

$$\int_M c_2(E)\wedge c_1(L)^{n-2}=\frac{p\cdot q}{(p+q)^2}\int_M c_1(L)^n>0,$$

then, $p \cdot q > 0$, thus, the linear part belongs to the Poincaré domain.

Now, in [GM–L] it is shown, that the normal bundle ν_K splits in a direct sum of holomorphic line bundles, that is: $\nu_K = L_p \oplus L_q$, and its Chern classes satisfy the relation:

$$p \cdot c(L_q) - q \cdot c(L_p) = \mathbf{0} \in H^2(K; \mathbb{C}).$$

Since $\wedge^2 \nu_K = L_p \otimes L_q = L|_K$, we have that both line bundles are positive, and then, the normal bundle of the Kupka component $K \subset M$ is positive. $\square$

Remark. If the transversal type is given by the form $\eta = x\,dy - y\,dx$, and the normal bundle admits the exact sequence of vector bundles

$$0 \to L_1 \to \nu_K \to L_1 \to 0,$$

then the normal bundle of K is positive.

In fact, since $L_1 \otimes L_1 = L_K$, which is positive, implies that L_1 is positive. This condition is true, if for example, there exists a smooth separatrix of the Kupka set.

Now, the main consequence of the proposition (2.3) is the following result:

2.4 Theorem. *Let $\mathcal{F}$ be a complex open book decomposition of M, represented by a section $\omega \in H^0\big(M; \Omega^1(L)\big)$. Let K be the compact Kupka component of $\mathcal{F}$. Then, if the transversal type of K is not the radial vector field, then $M - K$ is 2–convex.*

Proof. The normal bundle $\nu_K(M)$ is positive, now, because the codimension of K in M is two, by proposition (1.2), we have that $M - K$ is 2–convex. $\square$

A codimension one foliation without singularities $\mathcal{F}$ on a manifold M has an **transversal affine structure**, if there exits an open cover $\{U_\alpha\}$ of M, and a family of submersions $f_\alpha : U_\alpha \to \mathbb{C}$ which define the foliation on U_α, such that whenever $U_\alpha \cap U_\beta \neq \emptyset$ we have

$$f_\beta = a_{\alpha\beta} \cdot f_\beta + b_{\alpha\beta} \quad a_{\alpha\beta} \in \mathbb{C}^*, \quad b_{\alpha\beta} \in \mathbb{C}.$$

In our case, we consider codimension one holomorphic foliations such that, outside of an algebraic set, the foliation admits a transversal affine structure. In this case, it is shown in [Sc], that a codimension one holomorphic foliation with singularities on a complex manifold M has a transversal affine structure, if and only if, the following holds:

Let Ω be a meromorphic section which defines the foliation out of its poles, then there exists a closed meromorphic 1–form θ, such that:

$$d\Omega = \Omega \wedge \theta, \quad \text{and} \quad \theta_\infty = \Omega_\infty.$$

Now, when a foliation has an transversal affine structure, it could be defined by a holomorphic section $\omega \in T^*M \otimes L_0$, where L_0 is a holomorphic line bundle with vanishing first Chern class and locally defined by closed forms, in fact, we have

$$df_\alpha = a_{\alpha\beta} \cdot df_\beta,$$

which implies that the families of 1–forms $\{df_\alpha\}$ glue to a holomorphic section of the bundle $T^*M \otimes L_0$, and the line bundle L_0 is defined by the cocycle $\{a_{\alpha\beta}\} \in H^1(M;\mathbb{C}^*)$.

Reciprocally, let M be a compact Kähler manifold and $\omega \in H^0\big(M, \Omega^1(L_0)\big)$, be a holomorphic section locally represented by closed forms, and L_0 a line bundle with zero Chern class, then the foliation has an transversal affine structure.

We are going to use the fact that, on a compact Kähler manifold, any holomorphic line bundle with Chern class zero, can be given by constant transition functions [G–H] p. 313. Now, let $U = \{U_\alpha\}$ be an open covering of M, and ω_α be a family of meromorphic, closed 1–forms, which defines the foliation out of its poles, and such that $\omega_\alpha = c_{\alpha\beta} \cdot \omega_\beta$, where $c_{\alpha\beta} \in \mathbb{C}^*$.

Now let Ω be a meromorphic 1–form, which defines the foliation. In U_α, we have

$$\rho_\alpha \cdot \Omega_\alpha = \omega_\alpha \Rightarrow \frac{d\rho_\alpha}{\rho_\alpha} \wedge \Omega_\alpha = d\Omega_\alpha$$

which implies that:

$$\rho_\alpha = c_{\alpha\beta}\rho_\beta, \quad \theta_\alpha = \frac{d\rho_\alpha}{\rho_\alpha}.$$

Now, we are in a possition to prove our main result.

2.5 Theorem. *Let $\mathcal{F}$ be a complex open book decomposition of M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$. If the transversal type of K is not the radial vector field, then the foliation has at least one compact leaf, moreover, in the complement of this leaf, the foliation admits an affin transversal structure.*

Proof. We must consider two cases: (1) If the transversal type is $\eta = px\,dy - qy\,dx$ where $1 < p < q$ are relativelly prime integers, then the meromorphic 1–form

$$\eta_\alpha = \frac{\omega_\alpha}{x_\alpha y_\alpha},$$

defines the foliation on U_α out of its poles, moreover, as was pointed in [C–L], those meromorphic forms satisfied the equality $\eta_\alpha = \eta_\beta$ when $U_\alpha \cap U_\beta \neq \emptyset$.

This shows that $\{x_\alpha y_\alpha\}$ defines a holomorphic section of the line bundle L on a neighborhood of K, and then, it defines an integrating factor for the foliation ω, on this neighborhhod, since $M - K$ is 2–convex, this section can be extended to M. The compact leaves are extensions of the divisors $x_\alpha = 0$ and $y_\alpha = 0$.

(2) In the second case, i. e., if the transversal type is of the form $\eta = x\,dy - qy\,dx$, we can applied the same argument as before to show that $\{x_\alpha^{q+1} = 0\}$ defines a divisor invariant by the foliation, but in this case, we have the folowing equation:

$$\frac{\omega_\alpha}{x_\alpha^{q+1}} = c_{\alpha\beta}\frac{\omega_\beta}{x_\beta^{q+1}}, \quad c_{\alpha\beta} \in \mathbb{C}^*.$$

The line bundle associated to the divisor $\{x_\alpha^{q+1} = 0\}$ has only the same Chern class of L, and the foliation may be described the meromorphic section

$$\Omega \in H^0\big(M; \Omega^1(L_0)\big) \quad \Omega_\alpha = \frac{\omega_\alpha}{x_\alpha^{q+1}},$$

and the line bundle L_0 has zero Chern class, and the meromorphic 1–form Ω is defined, locally by closed forms. $\square$

Now, we can state our final result.

2.6 Theorem. *Let $\mathcal{F}$ be a complex open book decomposition of M, with Kupka component K, and represented by a section $\omega \in \Gamma(M; T^*M \otimes L)$. If the transversal type of K is not the radial vector field, and $H^1(M;\mathbb{C}) = 0$, then the foliation has a meromorhic first integral.*

Proof. The Hodge decomposition theorem implies that, if $H^1(M;\mathbb{C}) = 0$, any holomorphic line bundle on M, is determined by its Chern class, the proof of the theorem above shows that, with this aditional hypothesis, the foliation has an integrating factor, the conclusion follows from the normal form of foliations with integrating factors. $\square$

Remark.

(1) With the hypothesis of theorem 2.6, the foliation is locally structural stable near the Kupka component K.

(2) When the transversal type is the radial vector field, and the Kupka set has a smooth separatrix, then the foliation admits an affin transvesal structure, this is given as follows: set $x_\alpha = 0$ be the equation of the separatrix on the open set U_α, as in the proof of the theorem 2.5, we can show that

$$\Omega_\alpha = \frac{\omega_\alpha}{x_\alpha^2},$$

defines a meromorphic section of $T^*M \otimes L_0$, where $L_0 \in Pic_0(M)$, and having a order two pole along a divisor D which is invariant by the foliation, and which is a separatrix of the Kupka set, moreover, this section is locally represented by closed forms.

Acknowledgements. Part of this work were done when the author was a guest at IMPA, he would like to thank this institution for their support and hospitality.

The author wishes to thank to Jesús Muciño, for his sugestions and comentaries, and Bruno Scardua.

References

[C] Calvo, O., *Deformations of Branched Lefschetz Pencils*, Bull. Bras. Math. Soc. **26 N. 1** (1995), 67–83.

[C–S] Calvo, O; Soares, M., *Chern Numbers of a Kupka component*, Ann. Inst. Fourier **44 N. 4** (1994), 1219–1236.

[C–L] D. Cerveau, A. Lins, *Codimension one Foliations in $\mathbb{C}P^n$, $n \geq 3$, with Kupka components*, Complex Analytic Methods in Dynamical Systems (C. Camacho, A. Lins, R. Mossou, P. Sad., eds.), vol. 222, Astérisque, 1994, pp. 93–133.

[G–H] Griffiths. Ph., Harris J, *Principles of Algebraic Geometry*, Pure & Applied Math., Wiley Intersc., New York, 1978.

[GM–L] Gómez-Mont X, Lins N. A., *Structural stability of foliations with a meromorphic first integral*, Topology **30,** (1990), 315-334.

[Me] Medeiros A., *Structural stability of integrable differential forms.*, LNM 597 (J. Palis, M. do Carmo, eds.), 1977, pp. 395-428.

[S] Schneider, M., *Über eine Vermutung von Hartshorne*, Math. Ann. **201** (1973), 221–229.

[Sc] Scárdua, B., *Transversely Affine Foliations on Complex Projective Spaces*, Ph. D. Thesis IMPA (1993).

CIMAT: Ap. Postal 402, Guanajuato, Gto. 36000 México.
E-mail address: calvo@mailer.main.conacyt.mx omegar@fractal.cimat.conacyt.mx