SELF-INTERSECTION LOCAL TIME OF THE FLUCTUATION LIMIT OF A MULTITYPE BRANCHING PARTICLE SYSTEM

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Abstract

We give a necessary and sufficient condition for existence and continuity of self-intersection local time (SILT) of the fluctuation limit of a multitype branching particle system in R^d, with particles evolving according to symmetric stable motions with different stability parameters. We show that existence of SILT is determined by the "most mobile" of the stable motions.

1 Introduction

Let $X = (X(t))_{t\in[0,1]}$ be a centered, continuous, Gaussian process with values in the space $\mathcal{S}'(R^d)$ of tempered distributions on R^d. An intuitive definition of self-intersection local time (SILT) of X up to time $t \in [0,1]$ is given by the formal expression

$$\int_0^t \int_0^t \langle X(s) \otimes X(r), \delta(x-y)\varphi(x)\rangle \, ds \, dr, \tag{1.1}$$

where $\otimes$ denotes the tensor product in $\mathcal{S}'(R^d)$, $\varphi \in \mathcal{S}(R^d)$ (the C^∞ rapidly decreasing functions), δ is the Dirac distribution, and $\langle\cdot,\cdot\rangle$ stands for duality. Since $\delta(x-y)\varphi(x) \notin \mathcal{S}(R^{2d})$, the first question is how to give a rigorous meaning to (1.1). This problem was studied by Adler, Feldman and Lewin (1991), and Adler and Rosen (1993), when X is the Brownian and the α-stable density process, respectively. In Adler et al. (1991) and Adler and Rosen (1993) it is shown that the SILT of X exists if and only if $d < 2\alpha$, and for $\alpha = 2$ the SILT process, when it exists, has cadlag paths.

Motivated by their work, Bojdecki and Gorostiza (1995) gave a rigorous definition of (1.1) for a wide class of $\mathcal{S}'(R^d)$-valued Gaussian processes and a criterion for existence and continuity of SILT in terms of the covariance of a given process. They applied this criterion to various examples of generalized processes, and in particular, they proved that the SILT of the α-stable density process exists and is continuous if and only if $d < 2\alpha$. The methods of proof in Adler, Feldman and Lewin (1991), Adler and Rosen (1993), and Bojdecki and Gorostiza (1995,1996,1997), make use in a fundamental way of the Chapman-Kolmogorov formula and the self-similarity of α-stable processes.

Todorova (1997), and Gorostiza and Todorova (1998) obtained an existence and continuity result for SILT of a more general density process that corresponds to a system of particles of two types, where the partices of type i, $i = 1, 2$, move according to a symmetric α_i-stable process, and each particle switches back and forth between the two types with respective exponential waiting times with parameters V_i. It is shown that the SILT exists and is continuous if and only if $d < 2\min\{\alpha_1, \alpha_2\}$, thus generalizing the above results. The intuitive interpretation of this result is that the existence of SILT is determined by the "most mobile" of the stable motions. The main difficulty in determining the existence and continuity of SILT comes from the presence of different types of particle motions, which does not allow a direct application of the criterion in Bojdecki and Gorostiza (1995) because this density process is not Markovian nor self-similar.

In this paper we consider the SILT of a high density fluctuation limit of a system of particles of k types, $k \geq 2$, in which a particle of type i follows a symmetric α_i-stable motion for an exponential lifetime of parameter V_i, after which it branches according to a finite variance, multitype branching law. Due to the presence of a genuine multitype branching (not only of type mutation, as in the above work), the covariance functional of the process is more complicated and the implementation of the criteria quoted above is more difficult.

Under usual conditions for the Markov chain of types, we show that the SILT exists and is continuous if and only if $d < 2\min\{\alpha_1, ..., \alpha_k\}$; that is, again the " most mobile" motion determines existence of SILT. Clearly, in the case $k = 2$ and type mutation, this result reduces to the previous one.

The organization of this paper is as follows. In Section 2 we recall the definition of SILT of generalized Gaussian processes given by Bojdecki and Gorostiza (1995), as well as their criterion for existence and continuity of SILT. In Section 3 we introduce the particle system which we consider here, and state our main result on the existence of SILT (Theorem 3.2) which is proved in Section 4.

2 A general criterion on existence and continuity of SILT for $\mathcal{S}'(R^d)$-valued processes

We will condense the main result from Bojdecki and Gorostiza (1995). We refer the reader to that paper for additional information and details.

Let $\mathcal{F}$ denote the class of non-negative symmetric C^∞ functions f on R^d with bounded support, and such that $f(0) > 0$ and $\int f(x)\,dx = 1$. For $f \in \mathcal{F}$, $\varepsilon > 0$ and $\varphi \in \mathcal{S}(R^d)$ we define

$$f_\varepsilon(x) = \varepsilon^{-d} f\left(\frac{x}{\varepsilon}\right), \quad x \in R^d,$$

and

$$\Phi^f_{\varepsilon,\varphi}(x, y) = f_\varepsilon(x - y)\varphi(x), \quad x,\, y \in R^d.$$

Note that $\Phi^f_{\varepsilon,\varphi} \in \mathcal{S}(R^{2d})$ and $\Phi^f_{\varepsilon,\varphi}$ approximates $\delta(x - y)\varphi(x)$ as $\varepsilon \to 0$. In order to give a meaning to (1.1), the idea is to replace $\delta(x - y)\varphi(x)$ by $\Phi^f_{\varepsilon,\varphi}$, so that it makes sense, and take the limit as $\varepsilon \to 0$. But this is not enough; for existence of a limit it is also necessary to replace $X(s) \otimes X(r)$ by the Wick product $: X(s) \otimes X(r) :$, which is an $\mathcal{S}'(R^{2d})$-valued random field such that

$$\langle : X(s) \otimes X(r) :, \varphi \otimes \psi\rangle = \langle X(s), \varphi\rangle\langle X(r), \psi\rangle - E(\langle X(s), \varphi\rangle\langle X(r), \psi\rangle).$$

This leads to defining an approximate SILT $L^f_\varepsilon(t)$ by

$$\langle L^f_\varepsilon(t), \varphi\rangle = \int_0^t \int_0^t \langle : X(s) \otimes X(r) :, \Phi^f_{\varepsilon,\varphi}\rangle\, ds\, dr, \quad t \in [0, 1], \quad \varphi \in \mathcal{S}(R^d),$$

which is a continuous $\mathcal{S}'(R^d)$-process.

We can now give a precise meaning to (1.1).

Definition 2.1. For a given continuous centered Gaussian $\mathcal{S}'(R^d)$-valued process $X = (X(t))_{t\in[0,1]}$, if there exists an $\mathcal{S}'(R^d)$-valued process $L = (L(t))_{t\in[0,1]}$ such that for any $t \in [0, 1], \varphi \in \mathcal{S}(R^d)$ and $f \in \mathcal{F}$,

$$\left\langle L^f_\varepsilon(t), \varphi\right\rangle \to \langle L(t), \varphi\rangle$$

in L^2 as $\varepsilon \to 0$, then L is called the *self-intersection local time* (SILT) of X.

Let K denote the covariance functional of X :

$$K(s, \varphi; t, \psi) = E\left(\langle X(s), \varphi\rangle \langle X(t), \psi\rangle\right), \quad \varphi, \psi \in \mathcal{S}(R^d), \quad s, t \in [0, 1].$$

For test functions $\Phi^{(1)}, \Phi^{(2)} \in \mathcal{S}(R^d) \otimes \mathcal{S}(R^d)$ of the form

$$\Phi^{(1)} = \sum_{i=1}^{n} \varphi_i^{(1)} \otimes \psi_i^{(1)}, \quad \Phi^{(2)} = \sum_{j=1}^{m} \varphi_j^{(2)} \otimes \psi_j^{(2)}, \ \varphi_i^{(1)}, \psi_i^{(1)}, \varphi_j^{(2)}, \psi_j^{(2)} \in \mathcal{S}(R^d), \tag{2.1}$$

and $s, r, u, v \in [0,1]$, we consider the functional

$$\begin{aligned} &J_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}) \qquad (2.2)\\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} \left(K(s, \varphi_i^{(1)}; u, \varphi_j^{(2)}) K(r, \psi_i^{(1)}; v, \psi_j^{(2)}) + K(s, \varphi_i^{(1)}; v, \psi_j^{2}) K(r, \psi_i^{(1)}; u, \varphi_j^{(2)}) \right), \end{aligned}$$

which is the covariance functional of the random field $: X(s) \otimes X(r) :$ for test functions of the given form.

We now state the existence and continuity criterion for SILT:

Theorem 2.2. *1. Given a continuous centered Gaussian $\mathcal{S}'(R^d)$-process X, assume that $J_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)})$ has a well defined extension on $\mathcal{S}(R^{2d}) \times \mathcal{S}(R^{2d})$ such that*

(i) The functional

$$(\Phi^{(1)}, \Phi^{(2)}) \mapsto \int_{[0,t]^4} J_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)})\, ds\, dr\, du\, dv$$

is continuous on $\mathcal{S}(R^{2d}) \times \mathcal{S}(R^{2d})$ for each $t \in [0,1]$.

(ii) $J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi})$ converges to a finite limit as $\varepsilon, \delta \to 0$, for each $f, g \in \mathcal{F}$, $\varphi \in \mathcal{S}(R^d)$, $s, r, u, v \in [0,1]$, and this limit does not depend on f, g.

(iii)

$$\left| J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \right| \leq G_\varphi(s, r, u, v)$$

for some measurable function G_φ on $[0,1]^4$ which depends on φ but is independent of $\varepsilon, \delta, f, g$, and such that

$$\int_{[0,1]^4} G_\varphi(s, r, u, v)\, ds\, dr\, du\, dv < \infty$$

for each $\varphi \in \mathcal{S}(R^d)$.

Then the SILT L of the process X exists.

Assume in addition that

(iv) There exists a non-decreasing continuous function F on $[0,1]$ and a number $\gamma > 0$ such that for all $t_1, t_2 \in [0,1]$, $t_1 < t_2$, $\varphi \in \mathcal{S}(R^d)$,

$$\int_{[0,1]^4} \left(1_{[0,t_2]^2}(s,r) - 1_{[0,t_1]^2}(s,r) \right) \left(1_{[0,t_2]^2}(u,v) - 1_{[0,t_1]^2}(u,v) \right) G_\varphi(s,r,u,v)\, ds\, dr\, du\, dv$$

$$\leq C(\varphi)\left(F(t_2)-F(t_1)\right)^{1+\gamma},$$

where $C(\varphi)$ is a positive constant depending only on φ.

Then the SILT L is a continuous $\mathcal{S}'(R^d)$-process, and moreover L^f_ε converges weakly to L in $C([0,1],\mathcal{S}'(R^d))$ as $\varepsilon \to 0$.

2. Suppose that $J_{s,r,u,v}$ satisfies condition (i) but

$$\lim_{\varepsilon\to 0}\int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi},\Phi^f_{\varepsilon,\varphi})\,ds\,dr\,du\,dv=\infty$$

for some $\varphi\in\mathcal{S}(R^d)$ and $f\in\mathcal{F}$. Then X does not have SILT.

3 The particle system and the result on existence of SILT for the fluctuation limit process

We consider a particle system of k types in R^d, $k\geq 2$, that correspond to a symmetric α_i-stable motions, $\alpha_i\in(0,2]$, $i=1,2,...,k$. At time $t=0$ the particles of type i are distributed according to a Poisson random field with intensity measure $K\gamma_i\Lambda$, where $K>0$, $\lambda_i\geq 0$, and Λ denotes the Lebesgue measure on R^d. Each particle of type i moves according a symmetric α_i-stable process with exponential lifetime with parameter $V_i>0$, and then branches into particles of k types, following a branching law p_i ($p_i(j_1,...,j_k)$ is the probability that the particle of type i produces j_s particles of type s, $s=1,...,k$). The offspring particles appear at the same place of branching of their progenitor, and move and branch following the same rules.

We define the first branching moments

$$m_i^{(1)}(h)=\sum_{j_1,\dots,j_k\geq 0} j_h p_i(j_1,\dots j_k),\ i,h=1,...,k$$

and denote by $M^{(1)}$ the matrix

$$M^{(1)}=(m_i^{(1)}(h))_{i,h=1,2,\dots,k}.$$

We define the second factorial moments

$$m_i^{(2)}(h,l)=\sum_{j_1,\dots,j_k\geq 0} j_h(j_l-\delta_{hl})p_i(j_1,\dots,j_k),\ i,h,l=1,...,k,$$

where $\delta_{hl}$is the Kronecker δ, the functions

$$M_i^{(2)}(x,y)=\sum_{h,l=1}^{k} m_i^{(2)}(h,l)x_h y_l,\ x=(x_1,\dots,x_k),y=(y_1,\dots,y_k)\in R^k,\ i=1,\dots,k, \quad (3.1)$$

and the vector

$$M^{(2)} = (M_1^{(2)}, \ldots, M_k^{(2)})^t$$

(A^t means the transpose matrix of A).

We denote

$$V = diag(V_1, \ldots, V_k)$$

and

$$\Delta_\alpha = diag(\Delta_{\alpha_1}, \ldots \Delta_{\alpha_k}),$$

where Δ_{α_i} is the generator of the symmetric α_i-stable motion, $i = 1, ..., k$.

We define the semigroup $U(t),\ t \geq 0$, in $(L^1(R^d))^k$ with generator $U = \Delta_\alpha + V(M^{(1)} - I)$.

We assume the following conditions on the branching law:

(A) The mean matrix $M^{(1)}$ es irreducible and apperiodic.

(B) The second factorial moments are finite.

We denote by $N_i^K(t, A)$ the number of particles of type i which belong to the set $A \in B(R^d)$ at time t, $i = 1, ..., k$ ($N_i^K(t)$ is a random point measure), and by M_i^K the corresponding fluctuation process: $M^K = (M_1^K, ..., M_k^K)$, where

$$M_i^K(t) = K^{-1/2}(N_i^K(t) - EN_i^K(t)),\ t \in [0, 1],\ i = 1, ..., k.$$

We will use the notation

$$\Phi \odot \Psi = (\varphi_1 \psi_1, ..., \varphi_k \psi_k),$$

where $\Phi = (\varphi_1, ..., \varphi_k), \Psi = (\psi_1, ..., \psi_k), \phi_i, \psi_i \in \mathcal{S}(R^d)$, and

$$\langle \Gamma, \Phi \rangle = \sum_{i=1}^{k} \gamma_i \int_{R^d} \varphi_i(x) dx$$

where $\Gamma = (\gamma_1 \Lambda, ..., \gamma_k \Lambda)$.

The following result is known (López-Mimbela, 1992):

Theorem 3.1. *When $K \to \infty$, M^K converges weakly to the process $M = (M_1, ..., M_k)$ in the Skorohod space $D([0, 1], (\mathcal{S}'(R^d))^k)$, where M is a continuous centered Gaussian Markov process with covariance functional $K(s, \Phi; t, \Psi) = E(\langle M(s), \Phi \rangle \langle M(t), \Psi \rangle)$ given by*

$$\begin{aligned} K(s, \Phi; t, \Psi) &= \langle \Gamma, U(s)(\Phi \odot U(t-s)\Psi) \rangle \\ &+ \langle \Gamma, \int_0^s U(v) V M^{(2)}(U(s-v)\Phi, U(t-v)\Psi)\, dv, \rangle \\ & \quad s, t \in [0, 1],\ s \leq t, \Phi, \Psi \in (\mathcal{S}(R^d))^k. \end{aligned} \tag{3.2}$$

The process M is the high-density fluctuation limit of the given branching particle system.

Remark. The first term in (3.2) is due to the type mutation, while the second term is due to the genuine branching. This is the reason for the more complicated expression of the covariance functional here than in the model of type mutation in Gorostiza and Todorova (1998).

Our goal is to find conditions for the existence and continuity of the SILT of the process

$$X = M_1 + ... + M_k,$$

which allows the study the influence of the branching in the particle system. Note that $M_1, ..., M_k$ are Markovian processes, but X is not Markovian (from Lemma 4.1 we see that the covariance functional of X does not satisfy the condition $Cov(\langle X_r, \varphi\rangle, \langle X_t, \psi\rangle) = Cov(\langle X_r, \varphi\rangle, \langle X_{s,t}, \psi\rangle)$ for some random variable $X_{s,t}$ in $\mathcal{S}'(R^d)$, $r \leq s < t$, which is the condition that caracterizes the Markov property of Gaussian processes in $\mathcal{S}'(R^d)$ (e.g. Fernández, 1991).

Our main result is the following theorem.

Theorem 3.2. *We assume conditions (A) and (B). Then*

a) If $d < 2\min\{\alpha_1, ..., \alpha_k\}$, the SILT of X exists and is a continuous process.

b) If $d \geq 2\min\{\alpha_1, ..., \alpha_k\}$, the SILT of X does not exist.

Remarks. 1. When $\alpha_1 = \alpha_2 = ... = \alpha_k$, this result coincides with those of Adler, Feldman and Lewin (1991), Adler and Rosen (1993) (without branching), and Bojdecki and Gorostiza (1995,1996) (with branching).

2. We can prove analogously that the SILT of the process M_i exists if and only if $d < 2\alpha_i$, $i = 1, ..., k$, and it is a continuous process.

4 Proof of Theorem 3.1.

For simplicity in the expressions we assume $k = 2$. In the general case the proof is analogous.

We will prove that the conditions of Theorem 2.2 are satisfied using the methods of Gorostiza and Todorova (1998). First we prove the theorem for the case of stationary distribution of the Markov chain of types, and then we consider the general case by comparing with the stationary case. We introduce first some definitions.

Let denote by $B \equiv (\xi(t), \tau(t))_{t\geq 0}$ the *basic process,* or the *"expectation process,"* of our branching particle system. B is a Markov process in $R^d \times \{1, 2\}$ where the first component corresponds to the position and the second component corresponds to the type of the particle at time t. The type component (τ_t) is a Markov process in $\{1, 2\}$ with generador $A =$

$V(M^{(1)}-I)$, $A=(A_{ij})_{i,j=1,2}$, and the position component (ξ_t) follows a symmetric α_i-stable motion while $\tau_t = i$.

For $U(t) = (U_{ij}(t))_{i,j=1,2}$ acting on funtions of the form $F(x,i) = \varphi(x)1_{\{j\}}(i)$, $\varphi \in \mathcal{S}(R^d)$, $x \in R^d$, we write

$$U_{ij}(t)\varphi(x) = U(t)F(x,i). \tag{4.1}$$

Thus

$$U_{ij}(t)\varphi(x) = \int \varphi(y)h_t^{(ij)}(x,y)\,dy, \tag{4.2}$$

where

$$h_t^{(ij)}(x,y)\,dy = P\left[\xi(t)\in dy\,|\xi(0)=x, \tau(0)=i,\ \tau(t)=j\right]P\left[\tau(t)=j\,|\tau(0)=i\right]. \tag{4.3}$$

In particular the operators $U_{ij}(t)$, $i,j=1,2$, $t\geq 0$, are positive. For all $t>0$ the functions $h_t^{(ij)}(x,\cdot)$ are continuous and integrable, and satisfy

$$h_t^{(ij)}(x,y) = h_t^{(ij)}(x-y) \text{ and } \int h_t^{(ij)}(x)\,dx \leq 1. \tag{4.4}$$

We define the operators

$$U_i(t) = \gamma_1 U_{1i}(t) + \gamma_2 U_{2i}(t),\ i=1,2. \tag{4.5}$$

Hence

$$U_i(t)\varphi(x) = \int \varphi(y)h_t^{(i)}(x,y)\,dy,\ i=1,2. \tag{4.6}$$

where

$$h_t^{(i)}(x,y) = \gamma_1 h_t^{(1i)}(x,y) + \gamma_2 h_t^{(2i)}(x,y),\ i=1,2. \tag{4.7}$$

For $t>0$ the funtions $h_t^{(i)}(x,\cdot)$ are continuous and integrable.

We introduce the functions $p_t^{(i)}(x,y)$, $i=1,2$:

$$p_t^{(i)}(x,y) = h_t^{(i1)}(x,y) + h_t^{(i2)}(x,y); \tag{4.8}$$

hence

$$(U_{i1}(t)+U_{i2}(t))\varphi(x) = \int \varphi(y)p_t^{(i)}(x,y)\,dy. \tag{4.9}$$

From the above we have that for all $t>0$ the functions $p_t^{(i)}(x,\cdot)$ are continuous and integrable.

From (4.3) we have that

$$\int \varphi(y)p_t^{(i)}(x,y)\,dy = E\left[\varphi(\xi(t))|\xi(0)=x, \tau(0)=i\right],$$

therefore, by a slight abuse of language we refer to $p_t^{(i)}(x,y)$ as the transition densities of the position component of the basic process (altough it is not a Markov process).

Lemma 4.1. *The covariance functional of the process X is given by*

$$K(s,\varphi;t,\psi) = \sum_{i,j=1}^{2} \int U_i(s)\left[\varphi(x)\int p_{t-s}^{(i)}(x,y)\psi(y)\,dy\right](x)\,dx$$

$$+\sum_{j,l=1}^{2}\int_0^s\int L_{jl}(s)\left[\varphi(y)\int p_{s-v}^{(j)}(x,y)\,dy\int\psi(w)p_{t-v}^{(l)}(x,w)\,dw\right](x)\,dx\,dv, \qquad (4.10)$$

where

$$U_i(s) = \gamma_1 U_{1i}(s) + \gamma_2 U_{2i}(s)$$

and

$$L_{jl}(s) = \sum_{i=1}^{2}\gamma_i\left(V_1 m_1^{(2)}(j,l)U_{i1}(s) + V_2 m_2^{(2)}(j,l)U_{i2}(s)\right),\ j,l=1,2.$$

Proof. We apply formula (3.2) with $\Phi = (\varphi,\varphi)$ and $\Psi = (\psi,\psi)$, using that

$$\begin{aligned} K(s,\varphi;t,\psi) &= E(\langle M_1(s)+M_2(s),\varphi\rangle\langle M_1(t)+M_2(t),\psi\rangle) \\ &= E(\langle (M_1(s),M_2(s),(\varphi,\varphi)\rangle\langle (M_1(t),M_2(t),(\psi,\psi)\rangle). \end{aligned}$$

From Lemma 5.1 in Gorostiza and Todorova (1998), we obtain the expression for the first term of (3.2), which is the first term of (4.10).

We will obtain the expression for the second term of (3.2). First we compute the expression for $M^{(2)}(U(s-v)\Phi, U(t-v)\Psi))$. Using (4.9) we obtain

$$\begin{aligned} M^{(2)}(U(s-v)\Phi,U(t-v)\Psi)) &= \begin{bmatrix} M_1^{(2)}\left(U(s-v)\Phi,U(t-v)\Psi\right) \\ M_2^{(2)}\left(U(s-v)\Phi,U(t-v)\Psi\right)\end{bmatrix} \\ &= \begin{bmatrix} M_1^{(2)}\left(\begin{bmatrix}(U_{11}+U_{12})(s-v)\varphi \\ (U_{21}+U_{22})(s-v)\varphi\end{bmatrix},\begin{bmatrix}(U_{11}+U_{12})(t-v)\psi \\ (U_{21}+U_{22})(t-v)\psi\end{bmatrix}\right) \\ M_2^{(2)}\left(\begin{bmatrix}(U_{11}+U_{12})(s-v)\varphi \\ (U_{21}+U_{22})(s-v)\varphi\end{bmatrix},\begin{bmatrix}(U_{11}+U_{12})(t-v)\psi \\ (U_{21}+U_{22})(t-v)\psi\end{bmatrix}\right)\end{bmatrix} \\ &= \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} A_1 &= \sum_{i,j=1}^{2} m_1^{(2)}(i,j)[(U_{i1}+U_{i2})(s-v)\varphi][(U_{j1}+U_{j2})(t-v)\psi], \\ A_2 &= \sum_{i,j=1}^{2} m_2^{(2)}(i,j)[(U_{i1}+U_{i2})(s-v)\varphi][(U_{j1}+U_{j2})(t-v)\psi]. \end{aligned}$$

Hence,

$$U(v)VM^{(2)}(U(s-v)\Phi, U(t-v)\Psi) = \left[\begin{array}{c} V_1U_{11}(v)A_1 + V_2U_{12}(v)A_2 \\ V_1U_{21}(v)A_1 + V_2U_{22}(v)A_2 \end{array}\right].$$

We substitute the above expression in the second term of (3.2). From (4.9) we have for $i, j = 1, 2$,

$$\begin{aligned} (U_{i1}+U_{i2})(s-v)\varphi(x) &= \int \varphi(y)p^{(i)}_{s-v}(x,y)\,dy, \\ (U_{j1}+U_{j2})(t-v)\psi(x) &= \int \psi(w)p^{(j)}_{t-v}(x,w)\,dw,\ i,j=1,2. \end{aligned}$$

Using this and grouping the terms which contain $\int \varphi((y)p^{(i)}_{s-v}(x,y)\,dy \int \psi(w)p^{(j)}_{t-v}(x,w)\,dw$, for $i, j = 1, 2$, we obtain the formula (4.10). □

Under the conditions (A) and (B), there exists an invariant measure $(\overline{\gamma}_1, \overline{\gamma}_2)$ for the Markov chain of types (τ_t). Analogously to Todorova (1997), Appendix, we obtain that the measure $\overline{\Gamma} = (\overline{\gamma}_1\Lambda, \overline{\gamma}_2\Lambda)$ is invariant for the semigroup U:

$$\langle \Lambda, U(t)\Phi\rangle = \langle \Lambda, \Phi\rangle,\ \Phi \in (\mathcal{S}(R^d))^2.$$

In this case, from the proof of the above lemma we obtain a more simple expression for the covariance functional.

Corollary 4.2. *For $\Gamma = \overline{\Gamma}$, $0 \le s \le t$, the covariance $\overline{K}$ of X is given by*

$$K(s,\varphi;t,\psi) = \sum_{i,j=1}^{2} \int \overline{\gamma}_i \left[\varphi(x)\int p^{(i)}_{t-s}(x,y)\psi(y)\,dy\right](x)\,dx$$

$$+\sum_{j,l=1}^{2}\int_0^s \int a_{jl}\left[\varphi(y)\int p^{(j)}_{s-v}(x,y)\,dy \int \psi(w)p^{(l)}_{t-v}(x,w)\,dw\right](x)\,dx\,dv,$$

where $a_{jl} = \sum_{i=1}^2 \overline{\gamma}_i V_i m_i^{(2)}(j,l)$.

Substituting $K(s,\varphi;t,\psi)$ in (2.2), first for Φ, Ψ of the form $\Phi = \varphi\otimes\psi, \Psi = \overline{\varphi}\otimes\overline{\psi}$, and then for Φ, Ψ of the general form, and using the linearity of the tensor product, we obtain analogously to Lemma 5.3 in Gorostiza and Todorova (1998) the following result.

Lemma 4.3. *For $s \le u, r \le v$ we have*

$$J_{s,r,u,v}(\Phi,\overline{\Phi}) = J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi}) + J^{(2)}_{s,r,u,v}(\Phi,\overline{\Phi}),$$

where

$J^{(1)}_{s,r,u,v}(\Phi,\overline{\Phi})$

$$
\begin{aligned}
= \quad & \sum_{i,j=1}^{2} \int U_i(s)U_j(r) \left(\int \Phi(x,z)\overline{\Phi}(y,w) p_{u-s}^{(i)}(x,y) p_{v-r}^{(j)}(z,w)\, dy\, dw \right) \, dx\, dz \\
+ \quad & \sum_{i,p,q=1}^{2} \int_0^r \int U_i(s) L_{pq}(\overline{\omega}) \\
& \times \left(\int \Phi(x,\overline{y})\overline{\Phi}(y,\overline{w}) p_{u-s}^{(i)}(x,y) p_{r-\overline{\omega}}^{(p)}(\overline{x},\overline{y}) p_{v-\overline{\omega}}^{(q)}(\overline{x},\overline{w})\, dy\, d\overline{y}\, d\overline{w} \right) \, dx\, d\overline{x}\, d\overline{\omega} \\
+ \quad & \sum_{i,j,p,q=1}^{2} \int_0^s \int_0^r \int L_{ji}(\omega) L_{pq}(\overline{\omega}) \\
& \times \left(\int \Phi(y,\overline{y})\overline{\Phi}(w,\overline{w}) p_{s-\omega}^{(j)}(x,y) p_{u-\omega}^{(i)}(x,w) p_{r-\overline{\omega}}^{(p)}(\overline{x},\overline{y}) p_{v-\overline{\omega}}^{(q)}(\overline{x},\overline{w})\, dy\, dw\, d\overline{y}\, d\overline{w} \right) \\
& \times d\overline{x}\, dx\, d\overline{\omega}\, d\omega,
\end{aligned}
$$

and

$$
J_{s,r,u,v}^{(2)}(\Phi,\overline{\Phi}) = J_{s,r,u,v}^{(1)}(\Phi, S\overline{\Phi}),
$$

where $S\Phi(x,y) = \Phi(y,x)$.

Corollary 4.4. *For* $\Gamma = \overline{\Gamma}$, *we have*

$$
\overline{J}_{s,r,u,v}^{(1)}(\Phi,\overline{\Phi}) = \overline{J}_{(1),s,r,u,v}(\Phi,\overline{\Phi}) + \overline{J}_{(2),s,r,u,v}(\Phi,\overline{\Phi}) + \overline{J}_{(3),s,r,u,v}(\Phi,\overline{\Phi}),
$$

where

$$
\begin{aligned}
\overline{J}_{(1),s,r,u,v}(\Phi,\overline{\Phi}) \quad = \quad & \sum_{i,j=1}^{2} \int \overline{\gamma}_i \overline{\gamma}_j \left(\int \Phi(x,z)\overline{\Phi}(y,w) p_{u-s}^{(i)}(x,y) \right. \\
& \left. \times\; p_{v-r}^{(j)}(z,w)\, dy\, dw \right) \, dx\, dz \qquad (4.11) \\
\overline{J}_{(2),s,r,u,v}(\Phi,\overline{\Phi}) \quad = \quad & \sum_{i,p,q=1}^{2} \int_0^r \int \overline{\gamma}_i a_{pq} \left(\int \Phi(x,\overline{y})\overline{\Phi}(y,\overline{w}) \right. \\
& \left. \times\; p_{u-s}^{(i)}(x,y) p_{r-\overline{\omega}}^{(p)}(\overline{x},\overline{y}) p_{v-\overline{\omega}}^{(q)}(\overline{x},\overline{w})\, dy\, d\overline{y}\, d\overline{w} \right) \, dx\, d\overline{x}\, d\overline{\omega} \qquad (4.12) \\
\overline{J}_{(3),s,r,u,v}(\Phi,\overline{\Phi}) \quad = \quad & \sum_{i,j,p,q=1}^{2} \int_0^s \int_0^r \int a_{ji} a_{pq} \left(\int \Phi(y,\overline{y})\overline{\Phi}(w,\overline{w}) \right. \\
& \times\; p_{s-\omega}^{(j)}(x,y) p_{u-\omega}^{(i)}(x,w) p_{r-\overline{\omega}}^{(p)}(\overline{x},\overline{y}) p_{v-\overline{\omega}}^{(q)}(\overline{x},\overline{w}) \\
& \left. \times\; dy\, dw\, d\overline{y}\, d\overline{w} \right) d\overline{x}\, dx\, d\overline{\omega}\, d\omega, \qquad (4.13)
\end{aligned}
$$

and

$$
\overline{J}_{s,r,u,v}^{(2)}(\Phi,\overline{\Phi}) = \overline{J}_{s,r,u,v}^{(1)}(\Phi, S\overline{\Phi}).
$$

Having the expression for the function $J_{s,r,u,v}$, condition (1) of Theorem 2.2 is proved analogously to the Corollary 5.7 of Gorostiza and Todorova (1998).

Let $q_t^{(i)}(x,y)$, $i=1,2$, denote the transition density of the symmetric α_i-stable motion.

Using a renewal argument (see e.g. Athreya and Ney, 1972), conditioning on the first branching and using that the probability of a jump from type i to type $j \neq i$, for the Markov chain of types is $-A_{ij}/A_{ii}$, $i,j=1,2$, (see e.g. Grimmett and Stirzaker, 1982, Proposition 6.9.14), we obtain the following lemma.

Lemma 4.5. *The transition densities of the basic process satisfy the following system of equations:*

$$\begin{aligned} p_t^{(1)}(x,y) &= e^{-V_1 t} q_t^{(1)}(x,y) + A_1 V_1 \int_0^t e^{-V_1 s} \int q_s^{(1)}(x,z) p_{t-s}^{(2)}(z,y)\, dz\, ds, \\ p_t^{(2)}(x,y) &= e^{-V_2 t} q_t^{(2)}(x,y) + A_2 V_2 \int_0^t e^{-V_2 s} \int q_s^{(2)}(x,z) p_{t-s}^{(1)}(z,y)\, dz\, ds, \end{aligned} \tag{4.14}$$

where $A_1 = -A_{12}V_1/A_{11}, A_2 = -A_{21}V_2/A_{22}$. In particular, $p_t^{(i)}(x,y) \geq K q_t^{(i)}(x,y)$, $x,y \in R^d$, $i=1,2$, where $K = e^{-\max\{V_1,V_2\}}$.

Analogously to Lemma 5.9 of Gorostiza and Todorova (1998), we prove the following result.

Lemma 4.6. *(a) The densities $p_t^{(i)}(x,y)$, $i=1,2$, satisfy the inequalities*

$$p_t^{(i)}(x,y) \geq a q_t^{(i)}(x,y),\ x,y \in R^d, \tag{4.15}$$

$$p_t^{(i)}(x,y) \leq b t^{-d/\min\{\alpha_1,\alpha_2\}},\ x,y \in R^d, \tag{4.16}$$

for some positive constants a and b.

(b) $p_t^{(i)}(x,y) = p_t^{(i)}(x-y)$, $i=1,2$.

(c) For $i \neq j$,

$$p_t^{(i)}(x,y) \leq q_t^{(i)}(x,y) + \int_0^t \int q_s^{(i)}(x,z) q_{t-s}^{(j)}(z,y)\, dz\, \mu_t^{(i)}(ds),\ t \in [0,1],$$

where

$$\mu_t^{(i)}(ds) = P(\theta_t^{(i)} \in ds)$$

and $\theta_t^{(i)}$ the total time in the interval [0,t) during which the basic process moves according to α_i, starting with type i, on the event that there is a change of type in $(0,t]$.

(d) There is a positive constant $C = C(d,\alpha_1,\alpha_2)$ such that

$$\sup_{x,z \in R^d} \int q_r^{(1)}(x,z) q_s^{(2)}(z,y)\, dz \leq C(r+s)^{-d/\min\{\alpha_1,\alpha_2\}}$$

for all $s, r \in [0,1]$.

Remark. If we consider in (d) densities $q^{(\cdot)}$ of the same type, the inequality holds by the Chapman-Kolmogorov formula and the self-similarity. In the case of two types the inequality plays the role of Chapman-Kolmogorov and self-similarity.

We can now prove the condition (2) of Theorem 2.2 analogously to the Proposition 5.10 of Gorostiza and Todorova (1998), obtaining the following result.

Lemma 4.7. *The limit* $\lim_{\varepsilon,\delta\to 0} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi})$ *exists for all* $f, g \in \mathcal{F}$, $\varphi \in \mathcal{S}(R^d)$, $s, r, u, v \in [0,1]$, *and this limit does not depend on* f, g. *In the case of the invariant measure we have*

$$\begin{aligned}
&\lim_{\varepsilon,\delta\to 0} J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \\
&= \sum_{i,j=1}^{2} \int \overline{\gamma}_i \overline{\gamma}_j \varphi(x)\varphi(y) p^{(i)}_{u-s}(x,y) p^{(j)}_{v-r}(x,y)\, dx\, dy \\
&+ \sum_{i,p,q=1}^{2} \int_0^r \int \overline{\gamma}_i a_{pq} \varphi(x)\varphi(y) p^{(i)}_{u-s}(x,y) p^{(p)}_{r-\overline{\omega}}(\overline{x}, x) p^{(q)}_{v-\overline{\omega}}(\overline{x}, y)\, dy\, dx\, d\overline{x} d\overline{\omega} \\
&+ \sum_{i,j,p,q=1}^{2} \int_0^s \int_0^r \int a_{ji} a_{pq} \left(\int \varphi(y)\varphi(w) p^{(j)}_{s-\omega}(x,y) p^{(i)}_{u-\omega}(x,w) p^{(p)}_{r-\overline{\omega}}(\overline{x}, y) \right. \\
&\left. \times\, p^{(q)}_{v-\overline{\omega}}(\overline{x}, w) dy\, dw d\overline{x}\, dx \right) d\overline{\omega}\, d\omega. \qquad (4.17)
\end{aligned}$$

In the following lemma we obtain the covariance inequalities for the process X in the stationary and the general case, and for the functions J in both cases. The proof is analogous to the proof of Lemma 5.11 of Gorostiza and Todorova (1998).

Lemma 4.8. *(a) There exist positive constants* c_1 *and* c_2 *such that for any positive* $\varphi, \psi \in \mathcal{S}(R^d)$,

$$c_1 \overline{K}(s,\varphi;t,\psi) \le K(s,\varphi;t,\psi) \le c_2 \overline{K}(s,\varphi;t,\psi).$$

(b) There exist positive constants b_1 *and* b_2 *such that for any positive* $\Phi^{(1)}, \Phi^{(2)} \in \mathcal{S}(R^{2d})$,

$$b_1 \overline{J}^{(i)}_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}) \le J^{(i)}_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}) \le b_2 \overline{J}^{(i)}_{s,r,u,v}(\Phi^{(1)}, \Phi^{(2)}),\ i = 1, 2.$$

With these preliminary results we now proceed to prove Theorem 3.2.

Proof of Theorem 3.2.

(a) Case $d \ge 2\min\{\alpha_1, \alpha_2\}$.

We assume $\alpha_1 \le \alpha_2$. Let $\varphi > 0$. Using the Fatou lemma, Lemma 4.8 (b) and Corollary 4.4 we obtain

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \int_{[0,1]^4} J_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^f_{\varepsilon,\varphi})\, ds\, dr\, du\, dv \\
&\ge b_1 \liminf_{\varepsilon\to 0} \int_{[0,1]^4} \overline{J}^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^f_{\varepsilon,\varphi})\, ds\, dr\, du\, dv \\
&\ge b_1 \overline{\gamma}_1^2 \int_{[0,1]^4} \int p^{(1)}_{|u-s|}(x,y) p^{(1)}_{|v-r|}(x,y)\varphi(x)\varphi(y)\, dx\, dy\, ds\, dr\, du\, dv \\
&+ b_1 \overline{\gamma}_1^2 \int_{[0,1]^4} \int p^{(1)}_{|u-r|}(x,y) p^{(1)}_{|v-s|}(x,y)\varphi(x)\varphi(y)\, dx\, dy\, ds\, dr\, du\, dv.
\end{aligned}$$

But the last integral is ∞ (see Bojdecki and Gorostiza, 1995, Lemma 2.6 a)).

Using the part 2 of Theorem 2.2 we conclude that in this case the SILT does not exist.

(b) Case $d < 2\min\{\alpha_1, \alpha_2\}$.

We suppose $\alpha_1 \le \alpha_2$. We will verify conditions (iii) and (iv) of Theorem 2.2.

From the positivity of the operators U_i and L_{ij} and from Lemma 4.8 (b)

$$\left| J^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,\varphi}, \Phi^g_{\delta,\varphi}) \right| \le J^{(1)}_{s,r,u,v}(|\Phi^f_{\varepsilon,\varphi}|, |\Phi^g_{\delta,\varphi}|) \le b_2 \overline{J}^{(1)}_{s,r,u,v}(|\Phi^f_{\varepsilon,\varphi}|, |\Phi^g_{\delta,\varphi}|) = b_2 \overline{J}^{(1)}_{s,r,u,v}(\Phi^f_{\varepsilon,|\varphi|}, \Phi^g_{\delta,|\varphi|}).$$

Therefore it suffices to bound the terms of $\overline{J}^{(1)}_{s,r,u,v}$ in (4.12) and (4.13). From Gorostiza and Todorova (1998) we have a bound $G_\varphi(s,r,u,v)$ for (4.11) which satisfies the conditions (iii) and (iv) of Theorem 2.2.

For (4.12) we can find a bound $G_\varphi(s,r,u,v)$ which satisfies the conditions (iii) and (iv) of Theorem 2.2 proceeding as follows: first we apply the inequality of Lemma 4.6 (c) to the densities $p^{(i)}_t$ and then for the obtained terms we apply the inequality of Lemma 4.6 (d) to both factors $q^{(i)}_t$ which contain the variable $\overline{x}$ in (4.12). For instance, applying the inequalities

$$p^{(1)}_{|u-s|}(x,y) \le q^{(1)}_{|u-s|}(x,y) + \int_0^{|u-s|} \int q^{(1)}_m(x,z) q^{(2)}_{|u-s|-m}(z,y)\, dz\, \mu^{(1)}_{|u-s|}(ds) \tag{4.18}$$

$$p^{(p)}_{r-\overline{w}}(\overline{x},\overline{y}) \le q^{(p)}_{r-\overline{w}}(\overline{x},\overline{y}) + \int_0^{r-\overline{w}} \int q^{(p)}_n(\overline{x},\overline{z}) q^{(p')}_{r-\overline{w}-n}(\overline{z},\overline{y})\, d\overline{z}\, \mu^{(p)}_{r-\overline{w}}(ds) \tag{4.19}$$

$$p^{(q)}_{v-\overline{w}}(\overline{x},\overline{w}) \le q^{(q)}_{v-\overline{w}}(\overline{x},\overline{w}) + \int_0^{v-\overline{w}} \int q^{(q)}_l(\overline{x},\overline{t}) q^{(q')}_{v-\overline{w}-l}(\overline{t},\overline{w})\, d\overline{t}\, \mu^{(q)}_{v-\overline{w}}(ds) \tag{4.20}$$

in (4.12) and grouping the second term in (4.18) and the first terms in (4.19) and (4.20), we obtain the following term, for which we will obtain a bound $G^{(1)}_\varphi(s,r,u,v)$:

$$\int_0^{|u-s|} \int_0^r |\varphi(x)||\varphi(y)| f_\varepsilon(x-\overline{y}) g_\delta(y-\overline{w}) q^{(1)}_m(x,z) q^{(2)}_{|u-s|-m}(z,y) q^{(p)}_{r-\overline{w}}(\overline{x},\overline{y})$$

$$\times\, q^{(q)}_{|v-\overline{\omega}|}(\overline{x},\overline{w})dx\,dy\,dz\,d\overline{x}\,d\overline{y}\,dw\,d\overline{\omega}\,\mu^{(1)}_{|u-s|}(dm)$$

$$\leq \sup_x |\varphi(x)| \int_0^r (r-\overline{\omega}+|v-\overline{\omega}|)^{-d/\alpha_1}\,d\overline{\omega} \int_0^{|u-s|} |\varphi(y)| f_\varepsilon(x-\overline{y}) g_\delta(y-\overline{w})$$

$$\times\, q^{(1)}_m(x,z) q^{(2)}_{|u-s|-m}(z,y) d\overline{w}\,d\overline{y}\,dx\,dz\,dy\,\mu^{(1)}_{|u-s|}(dm)$$

$$= \sup_x |\varphi(x)| \int |\varphi(y)|\,dy \int_0^r (r-\overline{\omega}+|v-\overline{\omega}|)^{-d/\alpha_1}\,d\overline{\omega}$$

(the last equality is obtained by multiple integration in the indicated order, and using that $q^{(i)}_t$ are densities).

Using $x^2+y^2 \geq 2xy,\ x,y\geq 0$, and $\overline{\omega}\in[0,r]$ we obtain for $d<2\alpha_1$,

$$\int_0^r (r-\overline{\omega}+|v-\overline{\omega}|)^{-d/\alpha_1}\,d\overline{\omega} \leq |v-r|^{-d/2\alpha_1}\int_0^r (r-\overline{\omega})^{-d/2\alpha_1}\,d\overline{\omega} \leq c(|v-r|)^{-d/2\alpha_1},$$

where c is a constant. The function $G^{(1)}_\varphi(s,r,u,v) = c\sup_x|\varphi(x)|(\int|\varphi(y)|\,d)(|v-r|)^{-d/2\alpha_1}$ satisfies the conditions (iii) and (iv) of Theorem 2.2.

We bound analogously the other terms of (4.12) and those of (4.13).

We proceed analogously with the terms of $\overline{J}^{(2)}_{s,r,u,v}$.

This completes the proof of Theorem 3.2. $\Box$.

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References

R. J. Adler, R. Feldman and M. Lewin, *Intersection local times for infinite systems of Brownian motions and for the Brownian density process*, Ann. Probab. 19 (1991), 192-220.

R. J. Adler and J. S. Rosen, *Intersection local times of all orders for Brownian and stable density processes -construction, renormalization and limit laws*, Ann. Probab. 21 (1993), 1073-1123.

K. B. Athreya and P. E. Ney, Branching Processes, Springer-Verlag, 1972.

T. Bojdecki and L. G. Gorostiza, *Self-intersection local time for Gaussian $\mathcal{S}'(R^d)$-processes: Existence, path continuity and examples*, Stoch. Proc. Appl. 60 (1995), 191-226.

T. Bojdecki and L. G. Gorostiza, *Self-intersection local time for Gaussian $\mathcal{S}'$-processes, and application to fluctuation limits of branching particle systems*, in "Stochastic Analysis:

Random Fields and Measure-Valued Processes", Israel Mathematical Conference Proceedings 10 (1996), 49-55.

T. Bojdecki and L. G. Gorostiza, *Self-intersection local time for $\mathcal{S}'(R^d)$-Wiener processes and related Ornstein-Uhlenbeck processes* (preprint), (1997).

B. Fernández, *Markov properties of the fluctuation limit of a particle system with death*, J. Math. Anal. Appl. 155 (1991), 66-77.

L. G. Gorostiza and E. Todorova, *Self-intersection local time of an $\mathcal{S}'(R^d)$-valued process involving motions of two types*, (1998), preprint.

G. Grimmet and D. Stirzaker, Probability and Random Processes, Oxford, (1982).

J. A. López-Mimbela, *Fluctuation limits of multitype branching random fields*, J. Multiv. Anal. 40 (1992), 56-83.

E. Todorova, *Tiempo local de autointersección del proceso de densidad de un sistema de partículas de dos tipos*, Ph.D. thesis, Centro de Investigación y de Estudios Avanzados, (1997), Mexico.