Stiefel–Whitney surfaces and the tri–genus of non–orientable 3–manifolds.

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0. Introduction

To a compact 3–manifold M there are associated several different numerical invariants that turn out to be closely related. If M is endowed with a (unique) differentiable structure and $f : M \to R$ is a smooth function, one obtains information about the structure of M from the critical points of f. The invariant $F(M)$ is the minimum of $\{\mu_M(f)\}$ over all f, where $\mu_M(f)$ denotes the number of critical points of f. Some other invariants of M that are defined without a differentiable structure are

$C(M)$ = the smallest number of open cells needed to cover M,

$cat(M)$ = the smallest number of sets, open and contractible in M needed to cover M (This is the Lusternik–Schnirelman category),

$N_0(M)$ = the minimal number of charts (i.e., open subsets of R^3) needed to cover M.

Work by Hempel and McMillan [HM] on $C(M)$ and by Takens [T] on $F(M)$ implies that $F(M) = C(M)$, and in [GG] it is shown that, modulo the Poincaré conjecture, that $C(M) = cat(M)$. Furthermore $cat(M)$ depends only on $\pi_1(M)$; in fact $cat(M) = 2$ if

$\pi_1(M) = 1$, $cat(M) = 3$ if $\pi_1(M)$ is free (non–trivial), and $cat(M) = 4$ otherwise.

For the study of $N_0(M)$ it turns out that a useful concept is that of categorical filling in the sense of Singhof [Si]. In particular, if M is closed and connected, $N_0(M)$ can be realized by orientable handlebodies; every M can be expressed as a union of three orientable handlebodies V_1, V_2, V_3 such that $V_i \cap V_j = \partial V_i \cap \partial V_j$ is a surface for $i \neq j$. Of course, if M is orientable we need only two orientable handlebodies (Heegaard splitting), so the interesting case is when M is non–orientable. Thus $N_0(M) = 2$ or 3. It is shown in [BE] and [GGH] that $N_0(M) = 2$ if and only if M is the union of two orientable 3–manifolds W_1, W_2. In fact each W_i can be choosen to be obtained from $F \times [0,1]$, where F is a closed orientable surface, by attaching 1–handles along $F \times \{0\}$.

$N_0(M)$ is related to the first Stiefel–Whitney class $w_1(M) \in H^1(M; Z_2)$. If $\beta : H^1(M; Z_2) \to H^2(M; Z)$ is the Bockstein connecting homomorphism induced by the exact coefficient sequence $0 \to Z \to Z \to Z_2 \to 0$, then (see[BE], [GGH]) $N_0(M) = 2$ if and only if $\beta w_1(M) = 0 \in H^2(M; Z)$.

If M is expressed as the union of three orientable handlebodies V_1, V_2, V_3 whose interiors are pairwise disjoint, and g_i denotes the genus of ∂V_i, and if $g_1 \leq g_2 \leq g_3$, then we say that M *has a splitting of type* (g_1, g_2, g_3). Ordering types lexicographically we consider splittings of minimal type (g_1, g_2, g_3) and call this ordered triple the *tri–genus* of M.

In [GHN] we showed that every M has tri–genus $(0, g_2, g_3)$ or $(1, g_2, g_3)$; the former if $\beta w_1(M) = 0$, the latter if $\beta w_1(M) \neq 0$. This was shown by considering *Stiefel–Whitney surfaces* of M, i.e., closed connected surfaces F in M that represent the non–trivial element of the torsion subgroup of $H_2(M; Z)$. The homology class $[F] \in H_2(M; Z_2)$ is the Poincaré dual of $w_1(M)$ and $M - F$ is an orientable 3–manifold. Such a Stiefel–Whitney surface may be 1–sided or 2–sided in M and it is shown in [GHN] (see also [BE]) that M contains a 2–sided Stiefel–Whitney surface if and only if $\beta w_1(M) = 0$.

The genus g of a Stiefel–Whitney surface F gives useful information about g_2. In [GHN] it is shown that if F is 2–sided then M has a splitting of type $(0, 2g, g_3)$ and if F is 1–sided then M has a splitting of type $(1, 2g-1, g_3)$, for some g_3. Furthermore, in the case $\beta w_1(M) = 0$, the converse holds: If M has a splitting of type $(0, g_2, g_3)$, then M contains a 2–sided Stiefel–Whitney surface of genus g with $2g \leq g_2$. The idea of the proof is to start with a decomposition $M = B^3 \cup V_2 \cup V_3$ of type $(0, g_2, g_3)$ and to change the decomposition by cutting B^3 into subballs, that are attached to V_2 and V_3, to $M = W_1 \cup W_2$ where W_1, W_2 are orientable 3–manifolds and where the union is along two orientable connected surfaces F_1, F_2, at least one of which has genus $\leq g_2/2$.

The main goal of this paper is to prove the converse in the case $\beta w_1(M) \neq 0$. The proof is considerably more difficult than in the above proof in the case $\beta w_1(M) = 0$.

Theorem A. *Let $M = V_1 \cup V_2 \cup V_3$ correspond to a splitting of type $(1, g_2, g_3)$. If $\beta w_1(M) \neq 0$, then M contains a (1–sided) Stiefel–Whitney surface of genus g with $2g - 1 \leq g_2$.*

Corollary. *(a) If $\beta w_1(M) = 0$, then M has tri–genus $(0, g_2, g_3)$ where $g_2 = 2g$ and g is the minimal genus of a 2–sided Stiefel–Whitney surface in M.*

(b) If $\beta w_1(M) \neq 0$, then M has tri–genus $(1, g_2, g_3)$ where $g_2 = 2g - 1$ and g is the minimal genus of a (1–sided) Stiefel–Whitney surface in M.

The computation of g_3 has been carried out in [N] for all Seifert fiber spaces M that are fiber bundles over S^1 with fiber an orientable surface. In these cases g_3 is much bigger that the (non–orientable) Heegaard genus $h(M)$ of M. In the remaining cases, i.e., for the Seifert fiber spaces M that do not fiber over S^1 with fiber an orientable surface, it is shown in [N] that $g_3 \leq h(m) + 1$ if $\beta w_1(M) = 0$. Furthermore it follows from [N] and Theorem A above that for these Seifert Fiber spaces $g_3 \leq h(M)$ if $\beta w_1(M) \neq 0$. The question of whether in these cases $g_3 = h(M) + 1$, resp. $g_3 = h(M)$, is still open. If M is not a Seifert fiber space we do not know if there is a relation between $h(M)$ and g_3.

Outline of the paper.

In Section 3 we start with a decomposition of $M = V_1 \cup V_2 \cup V_3$ where V_1 is a solid torus, V_2, V_3 are orientable 3–manifolds (not necessarily handlebodies). The goal is to show that for a decomposition with minimal complexity $c = (1, g_2, g_3)$, where $g_i = g(\partial V_i)$, this decomposition is nice, i.e., $V_1 \cap V_2$ and $V_1 \cap V_3$ consists of alternating annuli A_i, B_i on ∂V_1 as in Figure 2, and $V_2 \cap V_3 = G' \cup G''$ consists of 2-manifolds G' and G'' (not necessarily connected) such that $M' = \overline{M - V_1}$ cut along G' and M' cut along G'' is connected (and orientable). Section 1 provides some preliminary lemmas needed in this construction.

In Section 4 we show that we may connect the boundary curves of the 2–manifold G' (or G'') by annuli in V_1 as in Figure 5 to obtain a Stiefel–Whitney surface for M. Section 2 provides the technical support for this construction.

In Section 5 we prove a more general version of Theorem A in that V_2 and V_3 need not to be handlebodies.

Finally in Section 6 we consider as an Example a graph manifold and show how the Theorem applies to compute its tri–genus.

1. Preliminaries

Throughout this paper we work in the PL–category and use the following notations.

For a submanifold F of an n–manifold M^n we denote by $N(F)$ a regular neighborhood of F in M. All n–manifolds in this paper are compact.

For $(n-1)$–submanifolds F and G of ∂M^n and a homeomorphism $h : F \to G$ let M/h denote the quotient manifold $M/(x \sim h(x))$.

For n–submanifolds W_1, W_2 of M^n with $W_1 \cap W_2 = \partial W_1 \cap \partial W_2$ and an $(n-1)$–submanifold C of $W_1 \cap W_2$ we denote by $W_1 \cup W_2/C$ the union of a copy of W_1 and a copy of W_2 along C; i.e., if $N(W_1 \cap W_2) = (W_1 \cap W_2) \times I$ then $W_1 \cup W_2/C = \overline{W_1 \cup W_2 - (W_1 \cap W_2 - C) \times I}$.

Lemma 1. *Let M be a connected oriented n–manifold, F and G $(n-1)$-submanifolds of ∂M with the orientations of F and G induced by the orientation of M. Let $h : F \to G$ be a homeomorphism. Then M/h is orientable if and only if h is not orientation preserving.*

Proof: Triangulate M with F and G as subcomplexes and h a simplicial map. Orient the n-simplexes of M coherently. M/h is orientable if and only if for any two n-simplexes σ and τ that have $(n-1)$–faces s and t such that $h(s) = t$, the orientations of σ and τ are coherent in M/h. This is the case if and only if $h|s : s \to t$ is orientation reversing, hence $h : F \to G$ is not orientation preserving. $\square$

Let W_1, W_2 be connected orientable n–manifolds and F_i' (resp. F_i''), $i = 1, \dots, k$, be disjoint connected $(n-1)$–manifolds in ∂W_1 (resp. ∂W_2). Let $h_i : F_i' \to F_i''$ be a homeomorphism and $M_{ij} = W_1 \cup W_2/h_i \cup h_j$ $(i \neq j,\, i, j = 1, \dots, k)$.

Lemma 2. *(a) If M_{12} and M_{23} are non-orientable then M_{13} is orientable.*

(b) If $M_{12}, M_{23}, \dots, M_{\ell,\ell+1}$, are non-orientable for some $1 \leq \ell \leq k-1$ then $W_1 \cup W_2/h_1 \cup h_3 \cup \dots \cup h_{2m+1}$ is orientable for all $2m+1 \leq \ell$.

(c) If M_{12} is non-orientable and M_{13} is orientable then M_{23} is non-orientable.

Proof: (a) Since F_i', F_j'' are connected, we may orient W_1 and W_2 so that with the induced orientations on F_i', F_j'' the homeomorphism $h_1 : F_1' \to F_1''$ is orientation reversing, i.e. $M_1 = W_1 \cup W_2/h_1$ has an orientation compatible with the orientations of W_1 and W_2. Then $M_{12} = M_1/h_2$, where h_2 is orientation preserving, by Lemma 1. Now reverse the orientation of W_2; then $M_2 = W_1 \cup W_2/h_2$ is oriented and h_3 is orientation preserving, since $M_{23} = M_2/h_3$ is non-orientable. Hence with the original orientations of W_1 and W_2 and the induced orientations on F_3', F_3'' the homeomorphism h_3 is orientation reversing and $M_{13} = M_2/h_3$ is orientable.

(b) As in (a) we see that $h_1, h_3, \ldots, h_{2m+1}$ are orientation reversing and the result follows from Lemma 1.

(c) The proof is this case is similar. $\square$

The genus $g(G)$ of a 2–manifold of components $G_1, \ldots, G_k$ is defined to be the sum of the genera $g(G_i)$, $i = 1, \ldots, k$.

For easy reference we state the following lemma in two parts, even though both parts follow from an easy computation of Euler characteristics.

Lemma 3. *Suppose G is a closed orientable 2–manifold of c components.*

(a) If G is obtained from 2-manifolds G', G'' of k', k'' components, resp., by identifying m boundary curves of G' with m boundary curves of G'', then $g(G) = (g(G') - k') + (g(G'') - k'') + c + m$.

(b) If G is obtained from connected surfaces $G_1, \ldots, G_k$ and m annuli by identifying boundary curves in pairs then

$$g(G) = \sum_{i=1}^{k} g(G_i) - k + c + m.$$

$\square$

2. Arcs with colored vertices in a disk.

In this section we show that we can find a collection of disjoint proper arcs in a disk joining colored vertices and satisfying certain properties.

For an even number m we consider $\frac{m}{2}$ arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ prperly embedded in a disk D^2 that connect (all) m given colored vertices on ∂D^2 in pairs. The colors of the vertices are labelled $1, 2, \ldots, k, \tau$. A *walk* of length n is an ordered sequence of n of the arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ such that for adjacent arcs α_i, α_j an endpoint of α_i carries the same color as an endpoint of α_j. A walk w is a *τ-avoiding* walk if no arc of w has an endpoint of color τ. A *reduced walk* is a walk of length n that visits $n+1$ different colors.

A *loop* is a walk of length n that visits n different colors and such that, if $n > 1$, the boundary of each arc in the loop has two different colors.

Note that if we remove an arc from a loop of length $n > 1$ we obtain (after reordering, if necessary) a reduced walk of length $n-1$. Also note that a reduced walk does not contain a loop.

Lemma 4. *Let $v_1, \ldots, v_m \in \partial D^2$ be an ordered set of vertices with k colors, where the ordering is induced by an orientation of ∂D^2 . If the set of vertices of color x has at least two elements for each $1 \leq x \leq k$, then there exist $k-1$ disjoint properly embedded arcs $\alpha_1, \ldots, \alpha_{k-1} \subset D^2$ such that for each i, α_i has its ends v_r, v_s of different colors, $r - s$ is odd, and $\alpha_1, \ldots, \alpha_{k-1}$ is a reduced walk.*

Proof: If $k \geq 2$, find adjacent vertices v_i, v_{i+1} of colors x and y resp., $x \neq y$ (here i is *mod* m). Find a properly embedded arc α_{k-1} with ends v_i, v_{i+1}. Then α_{k-1} splits D^2 into D' and D'', where $int(D'')$ contains no vertex; in D' delete vertices v_i, v_{i+1}, and color the vertices of color y with color x. By assumption this new set of color x has at least two elements. We find now the system $\alpha_1, \ldots, \alpha_{k-1}$ by induction on the number of colors applied to D' and the new colored vertices.

For each $1 \leq i \leq k-1$, α_i has its ends v_r, v_s of different colors, and, since we allways connect adjacent vertices in each subdisk, $r - s$ is odd. Then $\alpha_1, \ldots, \alpha_{k-1}$ is a walk of length $k-1$ that visits k colors, that is, $\alpha_1, \ldots, \alpha_{k-1}$ is a reduced walk. $\square$

Lemma 5. *Let $X_1, \ldots, X_\ell \subset \{v_1, \ldots, v_m\}$ be ℓ subsets such that for each $i \in \{1, \ldots, \ell\}$, X_i contains at least two elements and all elements of X_i have color i.*

Let $Y = \{v_1, \ldots, v_m\} - \bigcup_{i=1}^{\ell} X_i$ be a set of vertices of color $\tau \notin \{1, \ldots, \ell\}$. Then there is a set of properly embedded arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}} \subset D^2$ connecting the m vertices such that

(a) *If $y = \#Y \geq \sum_{i=1}^{\ell} \#X_i - 2(\ell - 1)$, then*

(1) there are no τ–avoiding loops,

(2) no arc joins two vertices of the same color $c \in \{1, \dots, \ell\}$, and

(3) for given colors $c_1, c_2 \in \{1, \dots, \ell\}$, there is at most one arc joining a vertex of color c_1 to a vertex of color c_2.

(b) *If $y = \#Y < \sum\limits_{i=1}^{\ell} \#X_i - 2(\ell - 1)$, then*

(1) there are $\ell - 1$ arcs $\alpha_1, \dots, \alpha_{\ell-1}$ that do not contain color τ and the system $\alpha_1, \dots, \alpha_{\ell-1}$ is a reduced walk,

(2) there are y arcs $\alpha_\ell, \dots, \alpha_{\ell+y}$ each of which has exactly one endpoint of color τ, and

(3) each of the remaining arcs does not contain color τ.

Proof: The proof is by induction on $\ell + y$.

In case (a) the induction starts with $\ell + y = 2$, i.e., $\ell = 0$, $y = 2$, since $\#Y \geq \sum\limits_{i=1}^{\ell} \#X_i - 2(\ell - 1) \geq 2\ell - 2(\ell - 1) = 2$. Then $m = 2$, and we find and arc $\alpha_1 \subset D^2$ with $\partial \alpha_1 = \{v_1, v_2\}$.

In case (b) the induction starts with $\ell + y = 1$, i.e., $\ell = 1$ $y = 0$. We find here a set of arcs joining adjacent vertices.

Assume $\ell \geq 1$.

First case: There is a pair of vertices v_i, v_{i+1} of colors $c_1, c_2 \in \{1, \dots, \ell\}$, resp., and $c_1 \neq c_2$. An arc $\alpha \subset D^2$ which joins v_i and v_{i+1} splits D^2 into D' and D'', where $int(D'')$ contains no vertex.

In D' define

$$X'_j := X_j, \text{ for } j \in \{1, \dots, \ell\} - \{c_1, c_2\},$$
$$X'_{c_1} := X_{c_1} \cup X_{c_2} - \{v_i, v_{i+1}\},$$
$$Y' := (\{v_1, \dots, v_m\} - \{v_i, v_{i+1}\}) - \bigcup X'_j,$$
$$\ell' := \ell - 1,$$

and paint all the vertices of X'_{c_1} with color c_1. Then $\ell' + \#Y' < \ell + \#Y$.

In case (a),

$$\#Y' = \#Y \geq \sum \#X_j - 2(\ell - 1) = \sum \#X'_j + 2 - 2(\ell - 1) = \sum \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}$ satisfying (1), (2) and (3). Clearly $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}, \alpha$ satisfies (2) and (3) for D^2. A τ–avoiding loop would be of the form $\alpha, \alpha_{i_1}, \ldots, \alpha_{i_r}$ and the colors c_1 and c_2 appear in $\alpha_{i_1}, \ldots, \alpha_{i_r}$. Then $\alpha_{i_1}, \ldots, \alpha_{i_r}$ would be a reduced walk visiting $r+1$ colors. Since in D' color c_1 equals color c_2, $\alpha_{i_1}, \ldots, \alpha_{i_r}$ would be a loop in D', a contradiction. Hence $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}, \alpha$ satisfies also (1).

In case (b),

$$\#Y' = \#Y < \sum \#X_j - 2(\ell-1) = \sum \#X'_j + 2 - 2(\ell-1) = \sum \#X'_j - 2(\ell'-1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\ell-2}, \alpha_\ell, \ldots, \alpha_{\frac{m}{2}}$ satisfying (1), (2) and (3). Since $\alpha_1, \ldots, \alpha_{\ell-2}$ is a reduced walk that visits $\ell-1$ colors in D', and since color c_2 was changed to color c_1 in D', it follows that $\alpha_1, \ldots, \alpha_{\ell-2}, \alpha$ visits ℓ colors in D, hence it is a reduced walk in D^2. Clearly (2) and (3) hold for the system $\alpha_1, \ldots, \alpha_{\ell-2}, \alpha, \alpha_\ell, \ldots, \alpha_{\frac{m}{2}}$ in D^2.

Second case: For each i, either v_i and v_{i+1} have the same color $c \in \{1, \ldots, \ell\}$, or one of them has color τ. Find a pair v_i, v_{i+1} one with color τ and the other with color $c \in \{1, \ldots, \ell\}$ (possible, for $\ell \geq 1$). We assume, without loss of generality, that v_i has color τ, and v_{i+1} has color c. Find an arc $\alpha_{\frac{m}{2}} \subset D^2$ connecting v_i with v_{i+1}; then $\alpha_{\frac{m}{2}}$ splits D^2 into D' and D'', where $int(D'')$ contains no vertex.

First subcase: $\#X_c > 2$. In D' we define

$$X'_j := X_j, \text{ for } j \in \{1, \ldots, \ell\} - \{c\},$$

$$X'_c := X_c - \{v_{i+1}\},$$

$$Y' := (\{v_1, \ldots, v_m\} - \{v_i, v_{i+1}\}) - \bigcup X'_j,$$

$$\ell' := \ell.$$

Then $\ell' + \#Y' < \ell + \#Y$.

In case (a),

$$\#Y' = \#Y - 1 \geq \sum \#X_j - 2(\ell-1) - 1 = \sum \#X'_j + 1 - 2(\ell-1) - 1 = \sum \#X'_j - 2(\ell'-1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}$ satisfying (1), (2) and (3). Clearly $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3) for D^2.

In case (b) we have

$$\#Y' = \#Y - 1 < \sum \#X_j - 2(\ell-1) - 1 = \sum \#X'_j + 1 - 2(\ell-1) - 1 = \sum \#X'_j - 2(\ell'-1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\ell-1}, \alpha_{\ell+1}, \ldots, \alpha_{\ell+y}, \alpha_{\ell+y+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfying (1), (2) and (3) in D'. It is easy to see that then $\alpha_1, \ldots, \alpha_{\ell-1}, \alpha, \alpha_{\ell+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3) in D^2.

Second subcase: $\#X_c = 2$. Paint vertex v_r with color τ, where $X_c = \{v_{i+1}, v_r\}$. In D' define

$$X'_j := X_j, \text{ for } j \in \{1, \ldots, \ell\} - \{c\},$$

$$Y' := (\{v_1, \ldots, v_m\} - \{v_i, v_{i+1}\}) - \bigcup_{j \neq c} X'_j,$$

$$\ell' := \ell - 1.$$

Then $\ell' + \#Y' < \ell + \#Y$.

In case (a) we have

$$\#Y' = \#Y \geq \sum \#X_j - 2(\ell - 1) = \sum_{j \neq c} \#X'_j + 2 - 2(\ell - 1) = \sum_{j \neq c} \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}-1}$ satisfying (1), (2) and (3), and is easy to see that $\alpha_1, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3) for D^2.

In case (b) we have

$$\#Y' = \#Y < \sum \#X_j - 2(\ell - 1) = \sum_{j \neq c} \#X'_j + 2 - 2(\ell - 1) = \sum_{j \neq c} \#X'_j - 2(\ell' - 1),$$

and by induction we find in D' a system of $\dfrac{m}{2} - 1$ arcs $\alpha_1, \ldots, \alpha_{\ell-2}, \beta$, $\alpha_{\ell+1}, \ldots, \alpha_{\ell+y}$, $\alpha_{\ell+y+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfying (1), (2) and (3) in D'. Here β is an arc that connects vertex v_r to a vertex of a color different from τ. Then the system $\alpha_1, \ldots, \alpha_{\ell-2}, \beta$ is a reduced walk in D^2 and the arcs $\alpha, \alpha_{\ell+1}, \ldots, \alpha_{\ell+y}$ satisfy (2) in D^2, and the system $\alpha_1, \ldots, \alpha_{\ell-2}, \beta, \alpha, \alpha_{\ell+1}, \ldots, \alpha_{\frac{m}{2}}$ satisfies (1), (2) and (3). □

3. Minimal decompositions of M.

In this section M is a closed non-orientable 3-manifold that admits a decomposition (V_1, V_2, V_3) as follows:

$$(*) \qquad M = V_1 \cup V_2 \cup V_3$$

V_1 is a solid torus, V_2, V_3 are orientable 3-manifolds, $V_i \cap V_j = \partial V_i \cap \partial V_j$ is a 2-manifold for $i \neq j$.

Let $g_i = g(\partial V_i)$ be the sum of the genera of the boundary surfaces of V_i. We assume that $g_1 = 1 \leq g_2 \leq g_3$ and define the *complexity* c of the decomposition (V_1, V_2, V_3) is the ordered triple $c = (g_1, g_2, g_3) = (1, g_2, g_3)$.

We first choose a decomposition (V_1, V_2, V_3) for M for which c is minimal. Let d be the sum of the number of components of $V_1 \cap V_2$ and $V_1 \cap V_3$. Among all decompositions (V_1, V_2, V_3) of M with complexity c we choose one for which d is minimal.

The goal in this section is to show, step by step, that the given decomposition of M into (V_1, V_2, V_3) has certain additional properties.

Claim (1): *The intersections $V_1 \cap V_2$ and $V_1 \cap V_3$ contain no disks.*

Proof: For suppose a component of $V_1 \cap V_2$ is a disk D. Let $N = D \times I$ be a regular neighborhood of D in V_2 such that $N \cap V_1 = D$, $N \cap V_3 = \partial D \times I$, see Fig. 1. For $V_2' = \overline{V_2 - N}$, $V_3' = V_3 \cup N$ we have $g(\partial V_2') = g_2$ and $g_3' = g(\partial V_3') \leq g_3$, since V_3' is obtained from V_3 by attaching a 2-handle to ∂V_3. Note that $g_2 \leq g_3'$, since otherwise M would have a decomposition (V_1, V_3', V_2') with complexity $c' < c$, which contradicts the minimality of c. For the decomposition (V_1, V_2', V_3') the total number of disk components of $V_1 \cap V_i'$ has not increased $(i = 2, 3)$, the complexity $c' \leq c$, and the number of components of the intersections $d' < d$, which contradicts the minimality of d.

A corresponding argument holds if $V_1 \cap V_3$ contains disk components. $\square$

By Claim 1 $V_1 \cap V_i$ consists only of annuli $(i = 2, 3)$, all parallel and essential in ∂V_1. Note that an annulus A on ∂V_i is *incompressible* in V_i if and only if ∂A is not contractible in V_i.

Claim (2): *If $\beta w_1(M) \neq 0$, then all annuli of $V_1 \cap V_i$ are incompressible in V_1 and $V_i (i = 2, 3)$.*

Proof: For if one annulus of $V_1 \cap V_i$ is compressible in V_1 then there is a decomposition

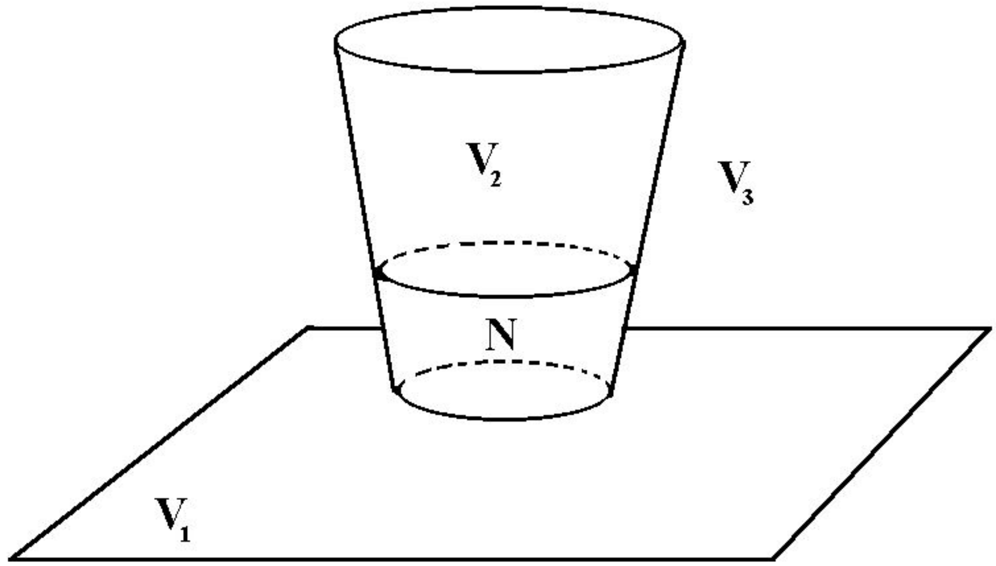

Figure 1.

of V_1 into 3-balls $B_1, B_1', \ldots, B_m, B_m'$ such that $B_i \cap B_j'$ is empty or a meridian disk and $(B_1 \cup B_2 \cup \cdots \cup B_m) \cap V_2 = V_1 \cap V_2$, $(B_1' \cup B_2' \cup \ldots \cup B_m') \cap V_3 = V_1 \cap V_3$ are the annuli of $V_1 \cap V_2$ and $V_1 \cap V_3$. Then $M = W_2 \cup W_3$ where $W_2 = B_1 \cup \cdots \cup B_m \cup V_2$ and $W_3 = B_1' \cup \cdots \cup B_m' \cup V_3$ are orientable 3-manifolds. It follows from Lemma 8.1 of [BE] that $\beta w_1(M) = 0$, a contradiction.

If an annulus A of $V_1 \cap V_2$ is compressible in V_2 then there are two disjoint disks D_1, D_2 in V_2 such that $(D_1 \cup D_2) \cap V_1 = (\partial D_1 \cup \partial D_2) \cap V_1 = \partial A$ and $D_1 \cup D_2 \cup A$ bounds a 3-ball H in V_2. Then $V_2' = \overline{V_2 - H}$ is orientable and $V_1' = V_1 \cup H$ is a punctured lensspace $\overline{L - B_1^3}$. It follows that $\partial(V_2' \cup V_3) = \partial V_1'$ is a 2-sphere and $M = V_1' \cup V_2' \cup V_3 = L\#(B^3 \cup V_2' \cup V_3)$. As in the proof of Lemma 1.5 of [GHN] we cut B^3 into sub-balls $B_1, \ldots, B_s, B_1', \ldots, B_t'$ such that $B_i \cap V_2'$ are connected planar surfaces that meet V_3 only in their boundaries, $B_j' \cap V_3$ are connected planar surfaces meeting V_2' only in their boundaries, and such that $B^3 \cup V_2' \cup V_3 = W_2 \cup W_3$, where $W_2 = B_1 \cup \cdots \cup B_s \cup V_2'$, $W_3 = B_1' \cup \cdots \cup B_t' \cup V_3$ are orientable 3-manifolds. Hence $\beta w_1(M) = \beta w_1(B^3 \cup V_2' \cup V_3) = 0$ which violates the hypothesis.

The same argument shows that the annuli of $V_1 \cap V_3$ are incompressible in V_3. $\square$

We now assume that $\beta w_1(M) \neq 0$ and the decomposition (V_1, V_2, V_3) is such that

$$V_1 \cap V_2 = A_1 \cup \cdots \cup A_m$$

$$V_1 \cap V_3 = B_1 \cup \cdots \cup B_m,$$

where the annuli A_i, B_j are incompressible in V_1 and all A_i are incompressible in V_2, all B_j are incompressible in V_3.

Recall that among all such decompositions with a fixed $g_2 = g(\partial V_2)$ we are assuming that we have chosen one for which m is minimal.

V_1 admits a Seifert fibration such that the annuli A_i, B_j are fibered. In the schematic pictures that follow, V_1 is represented by the orbit surface of this Seifert fibration. We chose an oriented crossing curve c on ∂V_1 (i.e. c intersects each fiber exactly once) and label the annuli A_i, B_j so that, following c, they are encountered in the sequence $A_1, B_1, A_2, B_2, \ldots, A_m, B_m$, see Fig. 2.

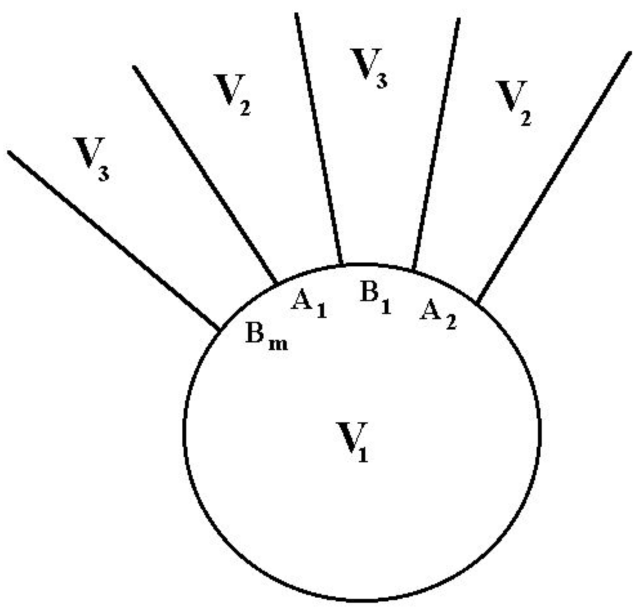

Figure 2.

Claim (3): *$V_1 \cup V_2/(A_i \cup A_{i+1})$ and $V_1 \cup V_3/(B_i \cup B_{i+1})$ are non-orientable.*

Proof: Let $N = (A_i \cup B_i \cup A_{i+1}) \times I$ be a regular neighborhood of $int(A_i \cup B_i \cup A_{i+1})$ in V_1 such that $N \cap V_2 = A_i \cup A_{i+1}$, $N \cap V_3 = B_i$, see Fig. 3. Note that $V_1 \cup V_2/(A_i \cup A_{i+1})$ is homeomorphic to the submanifold $V_2' = V_2 \cup N$ of M. If V_2' is orientable, then for $V_1' = \overline{V_1 - N}$ we have a decomposition (V_1', V_2', V_3) for M of the same complexity c, since V_1' is homeomorphic to V_1, and $g(\partial V_2') = g(\partial V_2)$, because the incompressible annuli A_i and A_{i+1} of ∂V_2 are replaced by the essential annuli B_i and $\overline{\partial N - A_i \cup B_i \cup A_{i+1}}$ of $\partial V_2'$. In this new decomposition the two annuli A_i, A_{i+1} of $V_1 \cap V_2$ are replaced by one annulus $A = \overline{\partial N - A_i \cup B_i \cup A_{i+1}}$ of $V_1' \cap V_2'$. Since ∂A is not contractible in V_2' and V_3 this contradicts the minimality of m. $\square$

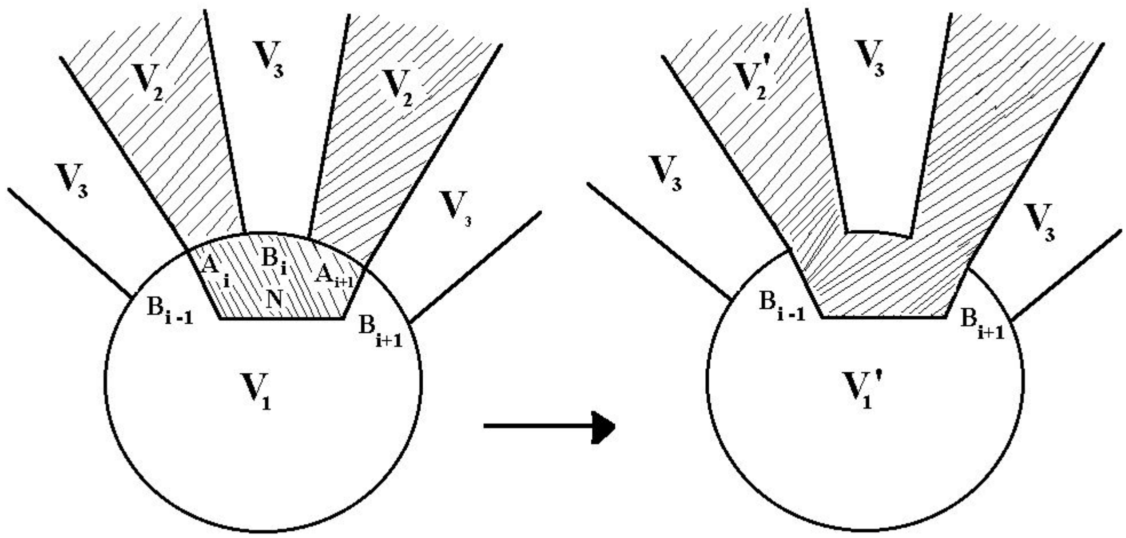

Figure 3.

Claim (4): *Let $A_{i_1}, \ldots, A_{i_l}$ be a subcollection of the annuli $A_1, \ldots, A_m$. Then $V_1 \cup V_2/(A_{i_1} \cup \cdots \cup A_{i_l})$ is orientable if and only if all indices are even or all indices are odd. An analogous statement holds if we identify V_1 and V_3 along a subcollection of $B_1, \ldots, B_m$. In particular m is even.*

Proof: Think of $V_1 \cup V_2/A_i$ as being obtained from $V_1 \cup V_2$ by gluing a copy of A_i on ∂V_1 to a copy of A_i on ∂V_2 with a homeomorphism h_i; i.e. $V_1 \cup V_2/A_i = V_1 \cup V_2/h_i$. Relabelling the annuli we may assume $i_1 = 1$. First we show that $V_1 \cup V_2/(A_1 \cup A_3 \cup \cdots \cup A_{1+2\ell})$

is orientable for each ℓ. By Claim (3) $M_{12} = V_1 \cup V_2/(h_1 \cup h_2), M_{23} = V_1 \cup V_2/(h_2 \cup h_3), \ldots, M_{2\ell,2\ell+1} = V_1 \cup V_2/(h_{2\ell} \cup h_{2\ell+1})$ are non-orientable and Lemma 2 implies that $V_2 \cup V_3/(A_1 \cup A_3 \cup \cdots \cup A_{2\ell+1}) = V_1 \cup V_2/(h_1 \cup h_3 \cup \cdots \cup h_{2\ell+1})$ is orientable.

Now it suffices to show that $V_1 \cup V_2/(A_i \cup A_{i+2k+1})$ is non-orientable for each k. But this follows from Lemma 2(c), since $V_1 \cup V_2/(A_i \cup A_{i+2k})$ is orientable and $V_1 \cup V_2/(A_{i+2k} \cup A_{i+2k+1})$ is non-orientable by step 3. $\square$

We denote components of $V_2 \cap V_3$ that contain a curve of $B_{i-1} \cap A_i$, $A_i \cap B_i$ by G'_t, G''_s, resp. When looking at Fig. 2 from a point in V_1, G'_t is to the left of V_2, and G''_s is to the right of V_2, accordingly we call the curve $c''_i = A_i \cap B_i$ the *right* boundary curve of A_i and $c'_i = A_i \cap B_{i-1}$ the *left* boundary curve of A_i.

Claim (5): *Let G'_t, G''_s be components of $V_2 \cap V_3$ that contain the left curve $c'_i = A_i \cap B_{i-1}$ and right curve $c''_j = A_j \cap B_j$, resp. Then G'_t and G''_s are distinct.*

Proof: For suppose $G'_t = G''_s$. For annuli A and B let $A \times I$ and $B \times I$ be solid tori in V_1 such that $A \times I \cap \partial V_1 = A \times \partial I$, $B \times I \cap \partial V_1 = B \times \partial I$, $A \times 0 = A_i$, $A \times 1 = A_j$, $B \times 0 = B_{i-1}$, $B \times 1 = B_j$, and $A \times I \cap B \times I$ is an anulus C in V_1 with boundary curves c'_i and c''_j (see Fig. 4). Here we assume that i and $i-1$ are different from j since a similar argument holds in these cases. Note that $V_2 \cup A \times I$ and $V_3 \cup B \times I$ are homeomorphic to $V_1 \cup V_2/A_i \cup A_j$ and $V_1 \cup V_3/B_{i-1} \cup B_j$, resp. By Claim (4) one is orientable, the other is non-orientable. Suppose that $V_2 \cup A \times I$ is non-orientable and think of $V_2 \cup A \times I$ as V_2/h, where $h : A_i \to A_j$ is a homeomorphism with $h(c'_i) = c''_j$. By Lemma 1, h is orientation preserving (where A_i, A_j have the orientations induced from V_2). Hence $h' = h|c'_i : c'_i \to c''_j$ is orientation preserving (where c'_i, c''_j receive their orientations from $G'_t = G''_s$) and $G'_t \cup C \cong G'_t/h'$ is a non-orientable surface in the boundary of the orientable manifold $V_3 \cup B \times I$, a contradiction. $\square$

We now call the components G'_t (resp. G''_s) of $V_2 \cap V_3$ that contain a left (resp. right) boundary curve of some A_j, a *left* (resp. *right*) *component.* We let

$$G' = G'_1 \cup \cdots \cup G'_{k'},$$

$$G'' = G''_1 \cup \cdots \cup G''_{k''}$$

where k' (resp. k'') is the number of left (resp. right) components of $V_2 \cap V_3$.

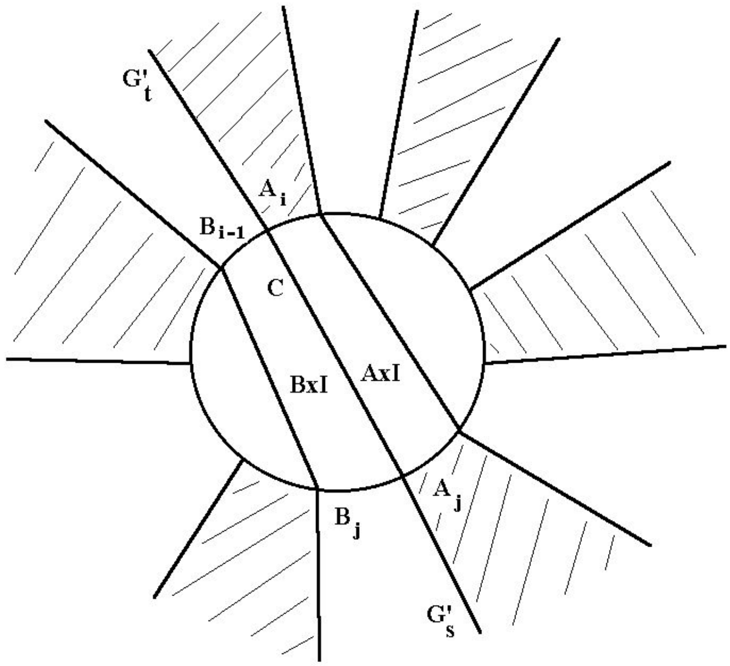

Figure 4.

Thus, $V_2 \cap V_3 = G' \cup G''$ and the left and right components alternate around the crossing curve c. Note that $V_2 \cup V_3/G''$ is obtained by cutting the submanifold $V_2 \cup V_3$ of M along G'.

Claim (6): *$V_2 \cup V_3/G''$ and $V_2 \cup V_3/G'$ are orientable.*

Proof: We show the first part. Clearly $V_2 \cup V_3/G''$ is orientable if G'' is connected. If $k'' > 1$, suppose that running around the crossing curve c we encounter different components $G''_{i_1}, G'_{j_2}, G''_{i_2}$. Now $V_2 \cup V_3/(G''_{i_1} \cup G'_{j_2})$ is non-orientable, since it contains a submanifold homeomorphic to $V_1 \cup V_2/(A_i \cup A_{i+1})$ which is non-orientable by step(3). Similarly $V_2 \cup V_3/(G'_{j_2} \cup G''_{i_2})$ is non-orientable. Hence by Lemma 2, $V_2 \cup V_3/(G''_{i_1} \cup G''_{i_2})$ is orientable. Think of $V_2 \cup V_3/(G''_{i_1} \cup G''_{i_2})$ as $V_2 \cup V_3/(h''_{i_1} \cup h''_{i_2})$, where h''_i is a homeomorphism from a copy of G''_i in V_2 to a copy of G''_i in V_3. Then h''_{i_1} and h''_{i_2} are orientation reversing by Lemma 1. Now if $k'' > 2$, continuing along c, we let G''_{i_3} be the first component of G'' distinct from G''_{i_1}, G''_{i_2} and let G'_j be the component of G' just before G''_{i_3}. Thus we encounter along c the distinct components $G''_{i_1}, G'_j, G''_{i_3}$ or $G''_{i_2}, G'_j, G''_{i_3}$, say the former. By the above argument $V_2 \cup V_3/(G''_{i_1} \cup G'_j) = V_2 \cup V_3/(h''_{i_1} \cup h'_j)$ is nonorientable, and since h''_{i_1} is orientation reversing, h'_j is orientation preserving; also $V_2 \cup V_3/(G'_j \cup G''_{i_3}) = V_2 \cup V_3/(h'_j \cup h''_{i_3})$ is non-orientable and hence h''_{i_3} is orientation reversing. Therefore $V_2 \cup V_3 / G''_{i_1} \cup G''_{i_2} \cup G''_{i_3} = V_2 \cup V_3 / h''_{i_1} \cup h''_{i_2} \cup h''_{i_3}$ is orientable. Now the argument can be continued for the remaining components of G''. $\square$

4. Construction of a Stiefel-Whitney Surface.

We now show how to construct a Stiefel-Whitney surface F from the given decomposition (V_1, V_2, V_3) of M obtained in section 3.

Recall that $G' = G'_1 \cup \cdots \cup G'_{k'}$, $G'' = G''_1 \cup \cdots \cup G''_{k''}$ are the components of $V_2 \cap V_3$ such that $G' \cap V_1, (G'' \cap V_1)$ consist of the left, (right,resp.) boundary components of the annuli $A_1, \ldots, A_m$.

Let $C_1, \ldots, C_{\frac{m}{2}}$ be any collection of properly embedded disjoint annuli in V_1 such that $\partial(C_1 \cup \cdots \cup C_{\frac{m}{2}}) = \partial G'$ (see Fig. 5). Let $F' = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$.

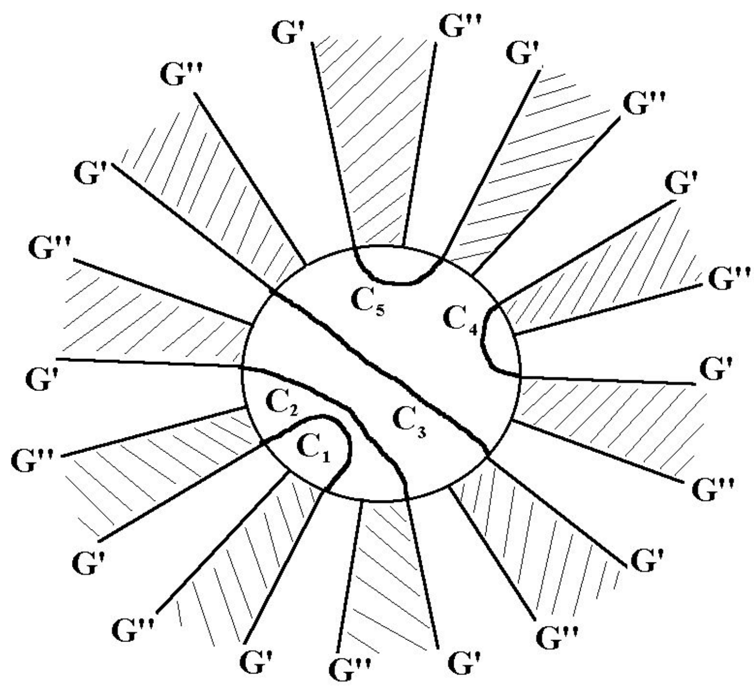

Figure 5.

Lemma 6. *$M - F'$ is orientable.*

Proof: By Claim 6 $M'' = V_2 \cup V_3/G''$ is orientable. The annuli $C_1, \ldots, C_{\frac{m}{2}}$ cut V_1 into disjoint solid tori $W_1, \ldots, W_\ell$ and $\overline{M - N(F')}$ can be represented as $M'' \cup W_1 \cup \cdots \cup W_\ell$, where $W_i \cap (M'')$ consists of annuli $K_{i_1}, \ldots, K_{i_\ell}$ and where $K_j = A_{u_j} \cup B_{u_j}$ (for each j). If $W_i \cap M''$ consists of only one annulus K_{i_1}, , then $W_i \cup M''$ is orientable. Thus assume that W_i meets M'' in at least two annuli. Then W_i meets M'' in annuli $K_{i_1}, \ldots, K_{i_\ell}$ where the indices $i_1, \ldots, i_\ell$ differ by even numbers, since for each annulus C_i with boundary curves c'_r, c'_s we have that $r - s$ is odd. Note that the manifold $M'' \cup W_i = M'' \cup W_i/K_{i_1} \cup \cdots \cup K_{i_j}$ is homeomorphic to $M'' \cup W_i/A_{u_1} \cup \cdots \cup A_{u_\ell}$, where $K_{i_j} = A_{u_j} \cup B_{u_j}$. By Claim 4 it follows that $M'' \cup W_i$ is orientable. Since $W_1, \ldots, W_\ell$ are disjoint, it follows that $M'' \cup W_1 \cup \cdots \cup W_\ell$ is orientable. $\square$

Note that the same construction can be done with G'' instead of G' to obtain F''.

Lemma 7. *Assume that each G'_i has a least two boundary components. There are disjoint properly embedded annuli $C_1, \ldots, C_{k'-1}$ in V_1 such that for each $1 \leq l \leq k'-1$, ∂C_l consists of two left components of $\partial A_i, \partial A_j$ with $i - j$ odd and such that $G' \cup C_1 \cup \cdots \cup C_{k'-1}$ is connected and has the same genus as G'.*

Proof: The orbit surface D^2 of V_1 is a disk. Assign colors $1, 2, \ldots, k'$ to the components $G'_1, \ldots, G'_{k'}$, resp. Then $G'_i \cap V_1$ projects to a set of colored vertices on ∂D^2 with at least two elements. Apply the Lemma 4 and obtain $\alpha_1, \ldots, \alpha_{k'-1} \subset D^2$ a set of disjoint properly embedded arcs with ends of different colors. The lifts of these arcs to V_1 gives a system of disjoint annuli as required. $\square$

Lemma 8. *Assume that each G'_i has at least two boundary components. Then there are annuli $C_1, \ldots, C_{\frac{m}{2}}$ in V_1 such that $\partial(C_1 \cup \cdots \cup C_{\frac{m}{2}}) = \partial G'$ and $F' = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a connected Stiefel-Whitney surface.*

For if we choose the first $k'-1$ annuli C_i as in Lemma 7, we can complete this collection to a system $C_1, \ldots, C_{\frac{m}{2}}$ as required. $\square$

Now assume that $G'_1, \ldots, G'_l$ each have at least two boundary components and the surfaces $G'_{l+1}, \ldots, G'_{k'}$ each have connected boundary ($l < k'$). Let b_i denote the number of boundary components of G'_i and let $b = \sum_{i=1}^{\ell} b_i - 2(\ell - 1)$.

Lemma 9. *If $\beta w_1(M) \neq 0$, then $k' - \ell < b$.*

Proof: Assume $k' - \ell \geq b$. We will show that $\beta w_1(M) = 0$. For each $i \in \{1, \ldots, \ell\}$, assign color i to the component G'_i, and color each component $G'_{\ell+1}, \ldots, G'_{k'}$ with color τ. Then $G'_i \cap V_1$ projects to a set X_i of vertices of color i on ∂D^2, with $\#X_i \geq 2$ $(1 \leq i \leq \ell)$, and $G'_{\ell+1} \cap V_1, \ldots, G'_{k'} \cap V_1$ projects to a set Y of vertices with color τ, where D^2 is the orbit surface of V_1. We have that $Y = \{v_1, \ldots, v_m\} - \bigcup_{i=1}^{\ell} X_i$. By hypotesis, $\sum \#X_i - 2(\ell - 1) = b \leq k' - \ell = \#Y$, and we can apply Lemma 5 (a) to find disjoint arcs $\alpha_1, \ldots, \alpha_{\frac{m}{2}} \subset D^2$ such that the lifts of these arcs gives a system of disjoint annuli $C_1, \ldots, C_{\frac{m}{2}} \subset V_1$ in such a way that the 2-manifold $F' = G'_1 \cup \cdots \cup G'_{k'} \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a 2-manifold with empty boundary and, is obtained by joining different components $G'_1 \cup \cdots \cup G'_{k'}$ together. Since all G'_i are 2-sided in $V_2 \cup V_3$, it follows that F' is orientable and is 2–sided in M. By Lemma 6, $M - F'$ is orientable. Let $W_1 = N(F')$ be a regular neighborhood of F' in M, and $W_2 = \overline{M - W_1}$. Then W_1, and W_2 are both orientable 3–manifolds, and $M = W_1 \cup W_2$. By [BE, Lemma 8.1] we conclude that $\beta w_1(M) = 0$. □

Lemma 10. *If $\beta w_1(M) \neq 0$, there are disjoint annuli $C_1, \ldots, C_{\frac{m}{2}}$ in V_1 such that $F' = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a connected Stiefel-Whitney surface and such that*

(a) $F_l = G'_1 \cup \cdots \cup G'_l \cup C_1 \cup \cdots \cup C_{l-1}$ is connected.

(b) Each of $C_l, \ldots, C_{k'-1}$ joins a component G'_i to F_l where $l + 1 \leq i \leq k'$.

(c) Each of $C_{k'}, \ldots, C_{\frac{m}{2}}$ joins two boundary components of F_l.

Proof: As in the proof of Lemma 9, we paint the components $G'_1, \ldots, G'_\ell$ with colors $1, \ldots, \ell$, resp., and we paint the components $G_{\ell+1}, \ldots, G'_{k'}$ all with color τ. Since $\beta w_1(M) \neq 0$, we have that $k' - \ell < b$, and we can apply Lemma 5 (b) to the projections of $\{G'_i \cap V_1\}$ into the orbit space D^2 of V_1. Hence there is a collection of disjoint arcs in D^2 which interconnect the colored vertices in D^2 . Conditions (1), (2) and (3) imply that the surface F' constructed from the G'_i and the lifts C_i of the α_i is connected and satisfies (a),(b), and (c). □

5. Proof of the Theorem

Theorem. *Let* (V_1, V_2, V_3) *be a decomposition of* M *as in* $(*)$ *of Section 3 with* $g_1 = 1 \leq g_2 \leq g_3$. *If* $\beta w_1(M) \neq 0$ *then there is a Stiefel-Whitney surface* F *in* M *such that* $g_2 \geq 2g(F) - 1$.

Proof: We may assume that (V_1, V_2, V_3) has the properties of Section 3. We look at the surfaces $G' = G'_1 \cup \cdots \cup G'_{k'}$ and $G'' = G'' \cup \cdots \cup G''_{k''}$ obtained in Section 3. Note that $\partial V_2 = G' \cup G'' \cup A_1 \cup \cdots \cup A_m$, hence by Lemma 3(a), $g_2 = (g' - k') + (g'' - k'') + m + c$, where $g' = g(G'), g'' = g(G'')$, and c is the number of components of ∂V_2. Without loss of generality we assume that

$$(**) \qquad g' - k' \leq g'' - k''$$

Case I: Each G'_i has at least two boundary components.

By Lemma 8 there are annuli $C_1, \ldots, C_{\frac{m}{2}}$ in V_1 such that $F = G' \cup C_1 \cup \cdots \cup C_{\frac{m}{2}}$ is a connected Stiefel-Whitney surface. By Lemma 3 (b) we have

$$g(F) = g' - k' + 1 + \frac{m}{2},$$

and by $(**)$

$$\begin{aligned} g_2 =&(g' - k') + (g'' + k'') + m + c \\ \geq&2(g' - k') + m + c \\ \geq&2(g' - k') + m + 1 \\ =&2(g(F) - 1 - \frac{m}{2}) + m + 1 \\ =&2g(F) - 1 \end{aligned}$$

Case II: There are l components $G'_1, \ldots, G'_l$ each with disconnected boundary and $k' - l$ components $G'_{l+1}, \ldots, G_{k'}$ each with connected boundary.

Let $F = F'$ of Lemma 10. Then F is obtained from F_l by gluing all the components G'_i (for $l + 1 \leq i \leq k'$) to F_l and identifying any remaining boundary components in pairs. In particular, F is connected and as in Case I we obtain $g_2 \geq 2g(F) - 1$. $\square$

Corollary. *Let M be a non–orientable 3–manifold. Then*

(a) $\beta w_1(M) = 0$ if and only if M has tri–genus $(0, 2g, g_3)$, where g is the minimal genus of a 2–sided Stiefel–Whitney surface for M.

(b) $\beta w_1(M) \neq 0$ if and only if M has tri–genus $(1, 2g-1, g_3)$, where g is the minimal genus of a 1–sided Stiefel–Whitney surface for M.

Proof: Part (a) is shown in Corollary (a) of [GHN]. There it is shown that M contains a 2–sided Stiefel–Whitney surface if and only if $\beta w_1(M) = 0$ and that M has splitting of type $(1, 2g-1, g_3)$ if M contains a 1–sided Stiefel–Whitney surface of genus g (Theorem 2.3 of [GHN]). Hence if $\beta w_1(M) \neq 0$, for the tri–genus $(1, g_2, g_3)$ we have $g_2 \leq 2g - 1$. By the Theorem, $g_2 \geq 2g - 1$. □

6. An application.

We show how the Theorem can be applied to compute the tri–genus of a certain graph manifold.

Let $\hat{M}_1$ and $\hat{M}_2$ be closed Seifert fiber spaces of type $NnII$. $\hat{M}_i$ can be described as follows (with the notations as in [S]). A simple closed curve α on the orbit surface $\hat{G}_i$ of $\hat{M}_i$ has valuation $\kappa(\alpha) = +1$ or -1, depending on whether translation of a fiber h_i in $\hat{M}_i$ along α preserves or reverses the orientation of h_i. The orbit surface $\hat{G}_i$ is a non–orientable surface represented as a connected sum of $k_i \geq 2$ projective planes with generators $v_1^i, \ldots, v_{k_i}^i$ and $\kappa(v_1^i) = +1$, and $\kappa(v_j^i) = -1$ for $j = 2, \ldots, k_i$ $(i = 1, 2)$. Now $p_i^{-1}(v_1^i)$ is a 1–sided torus T_i and $M - T_i$ is orientable, where $p_i : \hat{M}_i \to \hat{G}_i$ is the projection to the orbit surface. It is shown in [GHN] that the tri–genus of $\hat{M}_i$ is $(1, 1, g_{i3})$ for some $g_{i3} \geq 1$.

Let N_i be a regular neighborhood of a regular fiber h_i in $\hat{M}_i$; let $M_i = \overline{\hat{M}_i - N_i}$ and let $\varphi : \partial M_1 \to \partial M_2$ be a homeomorphism that is not fiber preserving. Consider the graph manifold

$$M = M_1 \cup_\varphi M_2.$$

We will compute the tri–genus of M. Notice that without the Theorem there seems to be no easy way to compute the tri–genus of M.

Proposition. *M has tri–genus $(1, 3, g_3)$, for some $g_3 \geq 3$.*

Proof. The proof of the proposition is in three steps.

(1) First we show that $\beta w_1(M) \neq 0$, hence the tri–genus of M is $(1, g_2, g_3)$ for some $1 \leq g_2 \leq g_3$.

(2) Then we show that there is a (1–sided) Stiefel–Whitney surface of genus $g = 2$ in M; hence by [GHN] the tri–genus of M is $(1, g_2, g_3)$, where $g_2 \leq 2g - 1 = 3$.

(3) Finally we show that there is no Stiefel–Whitney surface of genus less than 2 in M. Therefore by the Corollary, the tri–genus of M is $(1, 3, g_3)$ for some $g_3 \geq 3$.

(1). By [GGH] it suffices to show that M contains an orientation reversing curve t of finite order in $H_1(M)$. Now $H_1(M_i)$ has generators

$$v_1^{(i)}, \ldots, v_{k_i}^{(i)}, q_1^{(i)}, \ldots, q_{r_i}^{(i)}, h^{(i)}, d^{(i)},$$

where $d^{(i)}$ corresponds to a crossing curve on ∂N_i, and relations (besides being commutative)

$$\begin{aligned} d^{(i)} = -b^{(i)} h^{(i)} + \sum_{j=1}^{k_i} q_j^{(i)} - 2(v_1^{(i)} + \cdots + v_{k_i}^{(i)}), \\ \alpha_j^{(i)} q_j^{(i)} + \beta_j^{(i)} h^{(i)} =& 0, \\ 2h^{(i)} =& 0 \end{aligned} \tag{a}$$

(see [S]).

The homeomorphism $\varphi : \partial M_1 \to \partial M_2$ is determined, ut to isotopy, by the induced homomorphism $\varphi_* : H_1(\partial M_1) \to H_2(\partial M_2)$ which is described in terms of the basis $d^{(i)}, h^{(i)}$ of $H_1(\partial M_i)$ by

$$\varphi_*(d^{(1)}) = pd^{(2)} + qh^{(2)},$$

$$\varphi_*(h^{(1)}) = rd^{(2)} + sh^{(2)},$$

where p, q, r, s are integers such that $ps - qr = \pm 1$, and $r \neq 0$, since φ is not fiber preserving.

Then $H_1(M)$ is generated by the generators of $H_1(M_1)$ and $H_1(M_2)$ and has the relations of $H_1(M_1)$ and $H_1(M_2)$ and the additional relations

$$d^{(1)} = pd^{(2)} + qh^{(2)}, \tag{b}$$

$$h^{(1)} = rd^{(2)} + sh^{(2)}. \tag{c}$$

Note that $t = v_1^{(1)} + \cdots + v_{k_i}^{(1)}$ is an orientation reversing curve in M_1, hence in M. Since $h^{(i)}$ has finite order in $H_1(M)$ $(i = 1, 2)$, it follows from (c) that $d^{(2)}$ has finite order (since $r \neq 0$), and then from (b) we see that $d^{(1)}$ has finite order. From (a) it follows that $q_j^{(1)}$ has finite order for $j = 1, \ldots, k_1$. The relation $d^{(1)} = -b^{(1)}h^{(1)} + \sum_{j=1}^{k_1} q_j^{(1)} - 2(v_1^{(1)} + \cdots + v_{k_1}^{(1)})$ now shows that t has finite order.

(2). The vertical torus $T_i \subset M_i \subset \hat{M}_i$ is a Stiefel–Whitney surface for M_i. We show that there is a Stiefel–Whitney surface F in M which is homeomorphic to $T_1 \# T_2$. This implies that M has a decomposition of type $(1, 2g-1, g_3)$, where $g = g(F) = 2$.

To obtain F, join T_1 and T_2 by an arc α in M such that $\alpha \cap (T_1 \cup T_2) = \partial\alpha$, let $N(\alpha)$ be a neighborhood of α, rel $T_1 \cup T_2$, such that $N(\alpha) \cap T_i$ is a disk D_i, and let $F = T_1 \cup T_2 \cup \partial N(\alpha) - (int(D_1) \cup int(D_2))$. Note that F is homeomorphic to $T_1 \# T_2$ and $[F] = [T_1] + [T_2]$ in $H_2(M)$. Also F is not homologous to 0, since $T_1 \cup T_2$ does not separate M. In M_i we have $[\partial N(T_i)] = 2[T_i]$ in $H_2(M_i)$ and since $\partial(\overline{M_i - N(T_i)}) = T \cup \partial N(T_i)$ we have $2[T_i] = \pm[T]$ in $H_2(M_i)$, hence in $H_2(M)$. Orient T such that $[T] = 2[T_1]$ in $H_2(M)$; then $[T] = -2[T_2]$. Hence $2[F] = 2([T_1] + [T_2]) = [T] - [T] = 0$ in $H_2(M)$. It follows that $[F]$ generates the torsion of $H_2(M)$ and hence F is a Stiefel–Whitney surface of M.

(3). Let F be a (1–sided) Stiefel–Whitney surface in M. Then $g(F) \neq 0$, since a 2–sphere is 2–sided in M.

Suppose $g(F) = 1$.

Put F into general position with respect to T and let c be an intersection curve of $F \cap T$. Suppose c bounds a disk on T. If c also bounds a disk D in F then, since M is irreducible, we can deform D across c and decrease the number of intersection curves of $F \cap T$. If c does not bound a disk in F then we surger F along c and get a surface F' with $[F'] = [F]$ in $H_2(M)$, i.e., F' is a Stiefel–Whitney surface of M. But this can not be since F' is a 2–sphere.

So assume that all intersections of $F \cap T$ are essential in T. Then they are also essential in F, since T is incompressible in M.

The intersection curves partition F into annuli $A_1, \ldots, A_k, B_1, \ldots, B_k$ where $F \cap M_1 = A_1 \cup \cdots \cup A_k$, $F \cap M_2 = B_1 \cup \cdots \cup B_k$ and the annuli A_i, B_j are π_1–injective in M_1 and in M_2, resp.

The annulus A_1 is 2–sided in M, since otherwise the boundary curves of A_1 would be cores of Moebius bands in T, which is impossible. By [W] (see [H]), we may assume that A_1 is vertical or horizontal in M_1. If A_1 is horizontal, then $p_1 : M_1 \to G_1$ restricts to a branched covering of A_1 over the orbit surface G_1 of M_1. Since G_1 is not a disk, annulus, or a Moebius band, this cannot be. Hence A_1 is vertical in M_1. The same argument shows that ∂B_1 consists of fibers of M_2. But this contradicts the assumption that $\varphi : \partial M_1 \to \partial M_2$ is not fiber preserving. Hence $g(F) \neq 1$. $\square$

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