Robust Estimators of Openings and Closings in Mathematical Morphology

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Abstract

Morphological openings and closings form the cornerstones of a rich set of non–linear operators in signal processing. In this paper we show how robust extensions of these operators can be obtained within the framework of set–theory. Their dynamic behaviour is studied and the relationship with alternating rank filters established.

Key words: Openings, Closings, Mathematical Morphology, Rank filters.

1 Introduction

Mathematical morphology plays an important role in nonlinear signal and image processing; it has its roots in probing an image (signal) by translations of a small set (see [5] for a recent overview). From an operational point of view, its main characteristic is the use of maximum and minimum operators instead of linear ones (which on its own gave rise to interesting transpositions of several classical linear techniques to formulations using maxima and minima operators, cf. morphological scale-space [2], morphological wavelets [6] and morphological neural nets [9]).

In this paper we should like to focus on what we consider to be another fundamental feature of mathematical morphology: openings, closings and their compositions, are obtained by applying sequentially two dual operators, erosion and dilation. Apart of a nice geometric interpretation, this two-step approach allows one to have at one's disposal a powerful machinery for the manipulation of images. Given their sensitivity to noise, an intriguing question shows up: *How to construct a richer class of nonlinear morphological operators using a two-step approach which moreover is less sensitive to noise*?

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In recent literature, several proposals were published to improve the robustness of morphological operators. The majority aim to improve their performance as a filter.
One way is to make use of an explicit description of the statistical nature of the noise and to look for optimality with respect to a certain error measure (cf. [10] and [3]).
A second approach is to embed morphological operators in a parameterised family of operators where the extremes correspond to erosion and dilation. The simplest example is [7] where one substitutes the *max* operator by the L_p norm or a weighted variant. Yuille et al. [11] elaborate how to embed morphological operators in a statistical physics formulation and relate it with Bayesian reconstruction. Based on this result, Regazzoni et al. [8] introduce statistical erosion and dilation.

In this paper we follow a third path using a point of view based on set–theory, much closer to the basic principles of morphology and conserving the typical two–step approach of mathematical morphology. It will extend in a certain sense *Soft-Morphology* [1] and the resulting operators will be combinations of statistical erosions and dilations. A difference is that the main intention of this paper is to study operators as approximators of an opening/closing in the presence of noise (what can be used for the construction of a Pattern Spectrum, Top-Hat transforms, etc.).

In what follows, X will denote an image; in order to simplify the notation, depending on the context, X will represent the set of black pixels (as in the binary case) as well as the grey level function where X_x denotes the grey level value at position x. The cardinality of a set S is denoted by $|S|$. We will call S a symmetric set if $-S = S$. The translation of a binary image X by s, $\{x + s, x \in X\}$ is denoted by $X_{(s)}$.

2 Robust Openings for Binary Images

2.1 Construction

The starting point is the classical definition of an opening by a symmetric set S (called *structural element*): a pixel x belongs to the opening of a binary image X by the structural element S iff x belongs to at least *one* translation of S *totally* included in X.

A natural generalization is to weaken the total inclusion requirement to several partial ones, i.e. we require that there are at least r_2 translations of S covering x and that they have at least r_1 pixels in common with the image X.

For this purpose, we introduce the operator ρ:

$$\rho_{r,S}(X) = \{x : |S_{(x)} \cap X| \geq r\}. \tag{1}$$

Recalling the definition of erosion, $\varepsilon_S(X)$, and dilation, $\delta_S(X)$ by a set S:

$$\varepsilon_S(X) = \cap_{s \in S} X_{(s)}, \delta_S(X) = \cup_{s \in S} X_{(s)},$$

erosion resp. dilation corresponds to the case $\rho_{|S|,S}$ resp. $\rho_{1,S}$.

This allows us to formalize the above in the following way.

Definition 1 *For a given symmetrical structural element S, we define the (r_1, r_2)-opening, $R_S^{r_1,r_2}()$, as*

$$R_S^{r_1,r_2}(X) = \rho_{r_2,S}(\rho_{r_1,S}(X)). \tag{2}$$

The restriction to a symmetrical structural element is only for reasons of simplification and can be omitted (as in classical morphology). We observe that $R_S^{|S|,1}$ corresponds to a classical opening $X \circ S := \delta_S \varepsilon_S(X)$. On the other hand, a closing, $X \bullet S := \varepsilon_S \delta_S(X)$, coincides with $R_S^{1,|S|}$.

By making use of the equality:

$$\rho_{r,S}(X) = \cup_i \varepsilon_{B_i}(X),$$

where $\{B_i\}$ are all subsets of S with exactly r points, one obtains after some algebra that:

$$R_S^{r_1,r_2}(X) = \cap_j \cup_i \delta_{A_j}(\varepsilon_{B_i}(X)), \tag{3}$$

with $\{A_j\}$ resp. $\{B_i\}$ all subsets of S with exactly $|S| - r_2 + 1$ resp. r_1 elements.

Equation (3) shows that $X \cap R_S^{r_1,1}(X)$ coincides with the *rank-max opening* introduced in [5].

Among the main properties, we mention:

Proposition 2 *(1) If $X \subseteq Y$:$R_S^{r_1,r_2}(X) \subseteq R_S^{r_1,r_2}(Y)$*
(2) If $S \subseteq T$: $R_S^{r_1,r_2}(X) \subseteq R_T^{r_1,r_2}(X)$
(3) If $r_1 \leq s_1, r_2 \leq s_2$: $R_S^{s_1,s_2}(X) \subseteq R_S^{r_1,r_2}(X)$.
Consequently $\varepsilon_S(\varepsilon_S(X)) \subseteq R_S^{s_1,s_2}(X) \subseteq \delta_S(\delta_S(X))$.

(4) $R_S^{s_1,s_2}(X) = R_S^{|S|+1-s_1,|S|+1-s_2}(X^c)$

Contrary to the classical operators we lose the idempotent property. Nevertheless, as the following property shows, we will always obtain convergence under repetitive application.

Proposition 3 *Given a symmetrical structural element S, define recursively*

$$T^1(X) = R_S^{r_1,r_2}(X) \qquad T^t(X) = R_S^{r_1,r_2}(T^{t-1}(X)).$$

For any X, $T^t(X)$ will converge to a fixed image when t tends to infinity.

Proof

It is illustrative to describe the dynamical behaviour of $T^t(X)$ by means of a cellular automata. To this purpose, consider an image X of size $n \times m$ as one long vector of size $n.m$ with elements belonging to $\{0,1\}$. For a given image X, we define:

$$Y^1 = (Y_1^1, Y_2^1) = (X, X) \quad Y^t = (\rho_{S,r_1}(Y_2^{t-1}), \rho_{S,r_2}(Y_1^{t-1})). \tag{4}$$

Consequently, Y_1^{2t} corresponds to $T^t(X)$, the application of t times a robust opening on X. We construct:

$$M_S = \begin{pmatrix} 0 & N_S \\ N_S & 0 \end{pmatrix} \tag{5}$$

where N_S is the $n.m \times n.m$ matrix:

$$(N_S)_{x,y} = \begin{cases} 1 \text{ if } y \in S_{(x)} \\ 0 \text{ otherwise,} \end{cases}$$

and define the vector $\theta = (r_1, \cdots, r_1, r_2, \cdots, r_2)$ where r_1 and r_2 are repeated $n.m$ times.

All together, because of (4), one obtains:

$$Y^t = \mathbb{1}(M_S Y^{t-1} - \theta), \tag{6}$$

where $\mathbb{1}(x_i)$ equals 1 if x_i is positive and otherwise 0. In the sequel we suppose that θ eventually has been changed (without alternating the dynamics) in such a way that we can exclude the possibility that $M_S Y^{t-1} - \theta = 0$.

Following the derivation in [4], a Liapunov function is introduced, associating an energy with every state Y :

$$E(Y^t) = -\langle Y^t, M_S \cdot Y^{t-1}\rangle + \langle \theta, Y^t + Y^{t-1}\rangle.$$

As S is symmetric, M_S is a symmetric matrix. Hence

$$\Delta E := E(Y^t) - E(Y^{t-1}) = -\sum_x \{(Y_x^t - Y_x^{t-2})(\sum_y (M_S)_{x,y} Y_y^{t-1} - \theta_x)\}. \quad (7)$$

By construction each term in the outer sum in (7) is positive, hence ΔE is always non positive. As the state space is finite, from a certain moment on, E remains constant[4]. Because of (4), $\Delta E = 0$ implies that there exists a given t_0, such that for $t > t_0$:

$$Y^{2t} = Y^{2t_0}$$

2.2 Experiments

For implementation purposes, we first mention the fact that $\rho_{r,S}(X)$ is nothing else than thresholding the image $Z_x = |X \cap S_{(x)}|$ $(= \sum_y X_y S_{y-x})$ at level r. Consequently once Z being calculated, with a minimum effort $\rho_{r,S}(X)$ can be obtained for different values of r.

We present in Fig. 1(d), the results of the application of the robust opening $R_S^{22,7}(X)$ with S a disk of radius 3, to the image of Fig. 1(a) corrupted with 5% pepper and salt noise (Fig. 1(c)). This is to be compared with the opening of the original image (Fig. 1(b)). In Fig. 1(e) the classical opening of the noisy image is shown.

An important question concerns the influence of the values for r_1, r_2. For symmetry reasons we choose r_1 equal to $|S| + 1 - r_2$. Figure 2 shows a general picture of the impact of the parameter r_2. For two noise levels (5% resp. 10%) one shows in thin resp. fat lines, for different values of r_2 (x-axis) the percentage of pixels (y-axis) in which the robust opening differs from the opening of the original image (full lines) and from the original image itself (dotted lines).
As one observes, the higher the value of r_2 (i.e. the closer to the median filter), the more emphasis is put on noise removal and less on the calculation of the opening.

In Figure 3 the fixed points are shown under repeated application of $R_S^{r_1,r_2}(X)$ for different parameter values. The results suggest how they can be used to localize objects.

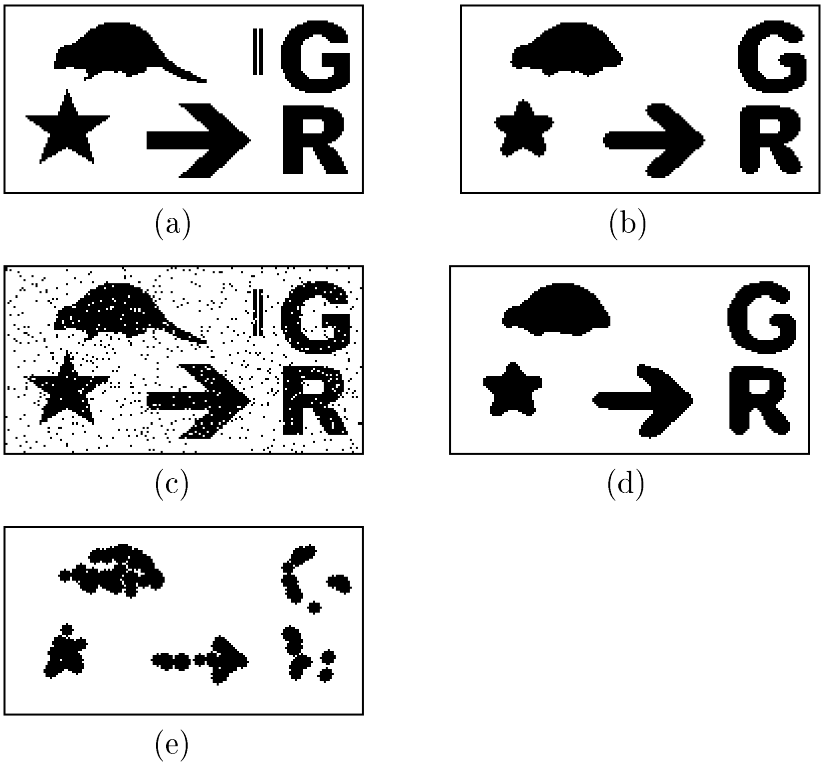


Fig. 1. Comparison of a classical and a robust opening: (a) original image X; (b) $X\ o\ S$; (c) $N(X)\!:= X$ corrupted with 5% pepper and salt noise; (d) $R_S^{25,5}(N(X))$; (e) $N(X)\ o\ S$.

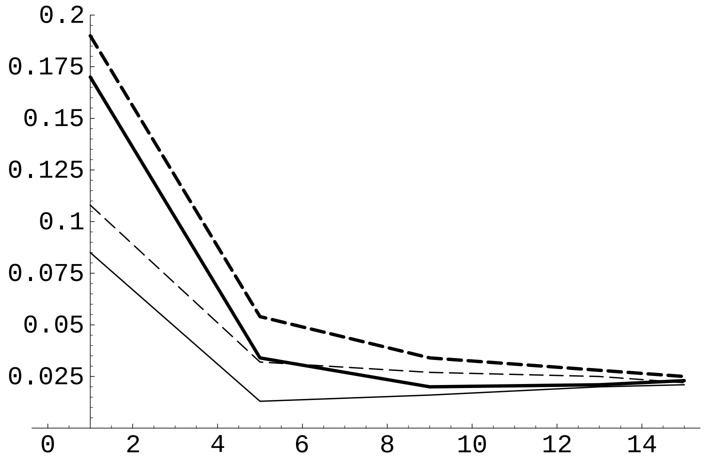
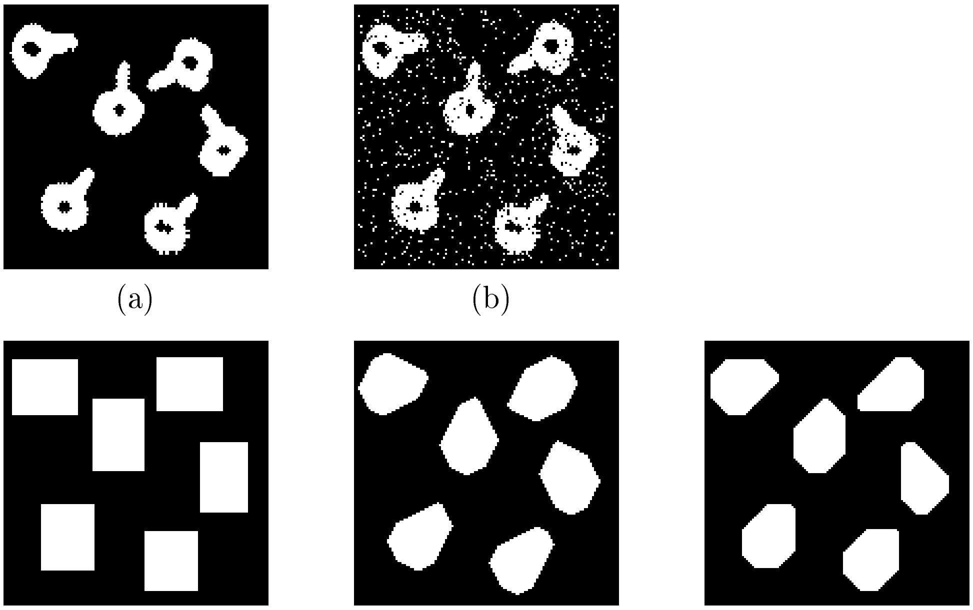


Fig. 2. For different values of r_2 (x-axis), dashed lines are proportional to $|(XoS)\ \Delta\ R_S^{|S|+1-r_2,r_2}(N(X))|$, full lines with $|X \Delta R_S^{|S|+1-r_2,r_2}(N(X))|$, where $N(X)$ denotes the original image of Fig 1(a) corrupted by 5% and 10%, respectively, of pepper and salt noise (thin and thick lines).



(a) (b)

(c) (d) (e)

Fig. 3. Fixed points of repeatedly applying $R^{3,4}_{S_1}(N(X))$ (Fig. 3(c)), $R^{7,4}_{S_2}(N(X))$ (Fig. 3(d)), $R^{6,5}_{S_2}(N(X))$ (Fig. 3(e)) where X is shown in Fig. 3(a), $N(X)$ is X corrupted with 5% pepper and salt noise (Fig. 3(b)), S_1 is a disk of radius 1 and S_2 a 3×3 square.

3 Extensions

3.1 Robust operators for grey scale images

One way to extend results for binary images to grey scale images is by making use of *slices* (thresholds) of an image defined by:

Definition 4 *We define for a grey scale image X and $u \in \mathbb{R}$*

$$T_u(X) = \{x : X_x \geq u\}.$$

Using the above, one generalization of (2) is:

Definition 5 *If X is a grey scale image and S a subset of pixels, we define a robust opening $R^{r_1,r_2}_S(X)$ as the image Z:*

$$Z_x = \sup\{u : x \in \rho_{r_2,S}(\rho_{r_1,S}(T_u(X)))\}. \tag{8}$$

The above definition allows a nice geometric interpretation: the value of the robust opening at x corresponds to the highest slice such that there are in that slice at least r_2 translations of S that have at least r_1 pixels in common with X. The next proposition offers another interpretation, relating the above operator with the well studied *rank filter*:

Proposition 6 *For a given structural element S,*

$$R_S^{r_1,r_2}(X) = Rank_S^{r_2}(Rank_S^{r_1}(X)),$$

where $Rank_S^r(X)$ denotes the $r-$rank filter with window S.

The proof is given at the end of this section.

For implementation purposes it is interesting to consider the grey scale image X as the set (umbra) $\{(x,u) : u \leq X_x)\}$. If we define C_v as $\{(x,v) : x \text{ arbitrary}\}$, it is not difficult to demonstrate that if $f()$ is an increasing set function (i.e. $A \subseteq B$ implies $f(A) \subseteq f(B)$) and Z an image:

$$\forall x : Z_x = \sup\{u : x \in f(T_u(X)\} \quad \text{iff} \quad Z = \cup_u(\mathbb{P}_1(f(X \cap C_u)), u), \quad (9)$$

where $\mathbb{P}_1$ is the projection operator: $\mathbb{P}_1((x,y)) = x$. In the sequel the righthand-side of (9) will be abbreviated as $\cup_u f(X \cap C_u)$.

Using (9), Definition (5) is equivalent to:

$$R_S^{r_1,r_2}(X) = \cup_u \rho_{r_2,S}(\rho_{r_1,S}(X \cap C_u)) \quad (10)$$

The above offers us an efficient parallel implementation: to each C_u one associates a processor that calculates twice $\rho_{r,S}(X \cap C_u)$. As mentioned previously, this corresponds to thresholding a convolution.

In Figure 4 the results are shown of the application of a robust opening to an image corrupted with 1% bit noise. The MSE and MAE between the robust opening and the opening of the original noise-free image is 52.7 resp. 702.9. This is to be compared with applying first a median filter (using as window a disk of radius 1) and afterwards a (classical) opening: the MSE and MEA are 57.1 resp. 790.7 .

Using the fact that for any $f()$ and $g()$,

$$\cup_v f((\cup_u g(X \cap C_u)) \cap C_v) = \cup_u f(g(X \cap C_u), \quad (11)$$

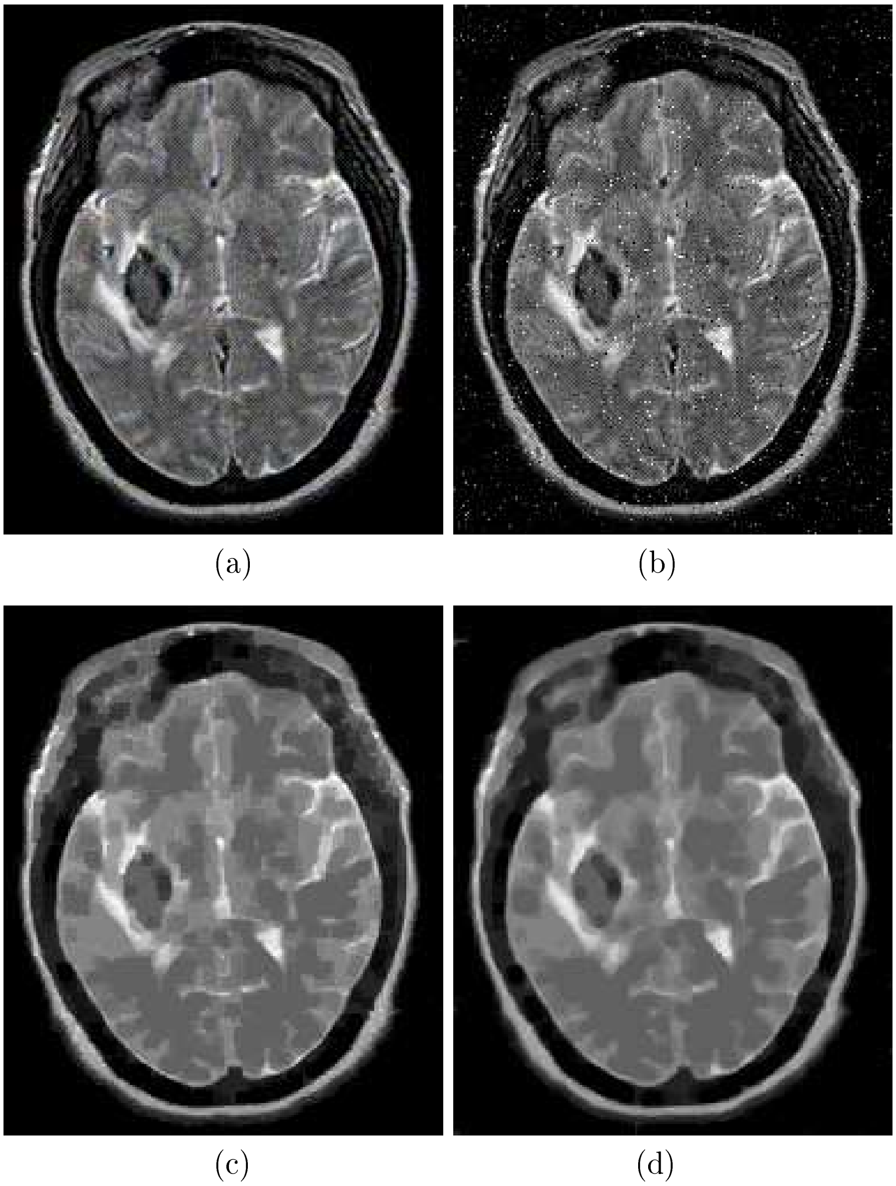


Fig. 4. Application of a robust opening: (a) original image X; (b) $N(X)$:X corrupted by 1% bit noise; (c) $X \circ S$ with S a 5×5 square; (d) $R_S^{22,3}(N(X))$.

if we define $f() = g() = R_S^{r_1,r_2}()$, repeated applications of $R_S^{r_1,r_2}(X)$ is equivalent to repeatedly applying this operator to each slice and taking the union (or equivalent the supremum) at the end. In this way we obtain in a straightforward way convergence to a fixed image using Property 3:

Proposition 7 *For a symmetric structural element, repeated applications of* $R_S^{r_1,r_2}(X)$ *will converge to a fixed image.*

Finally, to prove Proposition 6 we will make use of the following lemma:

Lemma 8 *The operator $R^r_S(X)$ defined as:*

$$R^r_S(X)_x = \sup\{u : x \in \rho_{r,S}(T_u(X))\} \tag{12}$$

coincides with the $r-$rank filter with window S.

Proof: Define $X^{S_{(x)}}(i)$ as the pixel value X_y, such that X_y is the ith largest value in $\{X_z : z \in S_{(x)}\}$. As we can constrain the values of u in (12) to $u \in \{X_y, y \in S_{(x)}\}$, there exists a k such that $R^r_S(X)_x = X^{S_{(x)}}(k)$. Consequently, it is sufficient to show that $X^{S_{(x)}}(k) = X^{S_{(x)}}(r)$.

(1) First of all, observe that $T_{X^{S_{(x)}}(r)}(X) \cap S_{(x)}$ has at least r elements. Hence $x \in \rho_{X^{S_{(x)}}(r),S}$ and

$$X^{S_{(x)}}(r) \leq R^r_S(X)_x \; (= X^{S_{(x)}}(k)). \tag{13}$$

(2) Suppose

$$X^{S_{(x)}}(r) < X^{S_{(x)}}(k). \tag{14}$$

This implies that the set $T_{X^{S_{(x)}}(k)}(X) \cap S_{(x)}$ has at most $r-1$ elements. Combining all this:

$$r \leq |T_{R^r_S(X)_x}(F) \cap S_{(x)}| = |T_{X^{S_{(x)}}(k)}(F) \cap S_{(x)}| \leq r-1 < r,$$

shows a contradiction. Consequently, because of the impossibility of (14), equation (13) implies that $X^{S_{(x)}}(k) = X^{S_{(x)}}(r)$.

Proof of Proposition 6: Combine (10) and (11) with $f() = \rho_{r_2,S}()$,$g() = \rho_{r_1,S}()$, and apply Lemma 8.

3.2 The general case

The above operators are constructed by counting elements in a particular set. If we make use of the counting measure, this is equivalent to the calculation of the *measure* of that set. In this way, we can extend (1) for an arbitrary measure $m()$ on the space S, to:

$$\rho^m_{r,S}(X) = \{x : m(S \cap X_{(-x)}) > r\}. \tag{15}$$

Observe that the change to a more general measure, implies the loss of the equivalence between having an empty intersection of two sets or not (hit-miss) on the one hand and having an intersection of measure zero or not on the other hand.

The above offers a methodology to extend the previous sections to continuous spaces. A natural choice for $m()$ is the Lebesgue measure, which roughly corresponds to calculating areas.

Proposition 9 *If $m()$ is the Lebesgue measure and S, X denote bounded, measurable sets, then the robust opening $R^{r_1,r_2}_{S,m}(X)$, defined as*

$$R^{r_1,r_2}_{S,m}(X) = \rho^m_{r_1,S}(\rho^m_{r_2,S}(X))$$

is always an open set.

The proof is given in the Appendix.

Another advantage of the formulation of (15), is the possibility to assign different weights (measure) to points depending on their distance to the origin. For example in the binary discrete case, given $A \subset$S taking $m(\{x\}) \sim r$ if $x \in A$ and $m(\{x\}) \sim 1$ otherwise, corresponds to the $r-$*soft morphology filter* [1].

4 Conclusions

In this paper we have presented a theoretical framework to study robust estimators for opening and closing following a point of view based on set–theory. The formulation by means of a cellular automata provided us insight into the limiting behaviour. On the fly, the established relation with alternating rank filters, offered us a valuable convergence result in this area, valid for any dimensional space.

Experiments confirm the quality of the introduced operators, at least for low noise levels. The statistical estimation of the optimal parameter seems to us an attractive point of departure for future research.

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Appendix

Proof of Proposition 9:
Observe that it is sufficient to show that $f(p) = m(S_{(p)} \cap X)$ is a continuous function in 0, or equivalently, for any ϵ, we can find a δ such that $||p|| < \delta$ implies that

$$|m(S_{(p)} \cap X) - m(S \cap X)| < \epsilon.$$

As

$$\begin{aligned} |m(S_{(p)} \cap X) - m(S \cap X)| &\leq m((S_{(p)} \triangle S) \cap X) \leq m(S_{(p)} \triangle S) \\ &\leq m(S_{(p)} \backslash S) + m(S \backslash S_{(p)}) \end{aligned} \tag{16}$$

it is sufficient to show that:

$$m(S_{(p)} \backslash S) + m(S \backslash S_{(p)}) \leq \epsilon.$$

As the Lebesgue measure is regular, there exists a compact measurable set K and an open measurable set U such that:

$$K \subseteq S : m(S) - m(K) < \epsilon/4, \tag{17}$$

$$S \subseteq U : m(U) - m(S) < \epsilon/4. \tag{18}$$

On the other hand, as $K \subset U$ and K compact and U open, there exists a δ such that $||p|| < \delta$ implies:

$$K_{(p)} \subseteq U. \tag{19}$$

and as $|| - p|| = ||p||$, $K_{(-p)} \subseteq U$ and:

$$K \subseteq U_{(p)}. \tag{20}$$

Combining (17) y (18) results in:

$$S \backslash S_{(p)} \subseteq U \backslash K_{(p)}$$

and

$$m(S \backslash S_{(p)}) \leq m(U \backslash K_{(p)}) \stackrel{(19)}{=} m(U) - m(K_{(p)}) = m(U) - m(K) =$$

$$= m(U) - m(S) + m(S) - m(K) < \epsilon/2$$

In the same way:

$$m(S_{(p)} \backslash S) \leq m(U_{(p)} \backslash K) \leq \epsilon/2$$

Combining the above, gives

$$||p|| < \delta \text{ implies } m(S \backslash S_{(p)}) + m(S_{(p)} \backslash S) < \epsilon.$$