ON PICARD BUNDLES OVER PRYM VARIETIES

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Abstract. Let P be the Prym variety associated with a covering $\pi : Y \to X$ between non-singular irreducible projective curves. If $\widetilde{P}$ is a principally polarized Prym-Tyurin variety associated with P, we prove that the induced Abel-Prym morphism $\widetilde{\rho}\colon Y \to \widetilde{P}$ is birational onto its image for genus $g_X > 2$ and $\deg \pi \neq 2$. We use such result to prove that the Picard bundle over the Prym variety is simple and moreover is stable when $\widetilde{\rho}$ is not birational. As a consequence we obtain that the Picard bundle on the moduli space of stable vector bundles with fixed determinant is simple for $g_X \geq 2$.

Introduction

Let X be a non-singular irreducible projective curve of genus $g_X \geq 2$. We denote by $\mathcal{M}(n,d)$ the moduli space of stable vector bundles over X of rank n and degree d. If n and d are coprime there exists a universal family $\mathcal{U}$ parametrized by $\mathcal{M}(n,d)$, called the Poincaré bundle. The higher direct images of the Poincaré bundle on $\mathcal{M}(n,d)$ are called Picard sheaves. If $d > 2n(g_X-1)$, the direct image $\mathcal{W}$ of $\mathcal{U}$ is locally free and it is called the Picard bundle.

For $n=1$, $\mathcal{M}(1,d)$ is the Jacobian $J^d(X)$. Ein and Lazarsfeld in [?] proved the stability of the Picard bundle $\mathcal{W}_J$ when $d > 2g_X-1$ and Kempf (see [?]) for $d = 2g_X-1$. For $n \geq 2$, Li proved the stability of $\mathcal{W}$ over $\mathcal{M}(n,d)$ when $d > 2g_X n$. Moreover, he proves that the restriction of $\mathcal{W}$ to an embedding of $J(X)$ is also stable (see [?]). Denote by $\mathcal{W}_\xi$ the restriction of $\mathcal{W}$ to the subvariety $\mathcal{M}_\xi \subset \mathcal{M}(n,d)$ determined by stable bundles with fixed determinant $\xi \in J^d(X)$. Balaji and Vishwanath proved in [?] that for rank 2, $\mathcal{W}_\xi$ is simple. Actually, they compute the deformations of $\mathcal{W}_\xi$. In

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this paper we prove (see Theorem 3.1) that for any rank

$$\mathcal{W}_\xi \quad is \quad simple$$

when $g_X \geq 2$ and $d > n(n+1)(g_X - 1) + 4$.

The result follows from studying the restriction of the Picard bundle over the Jacobian of a curve which is a covering space to some natural subvarieties, namely the Prym variety and the Jacobian of the base space. That is, let $\pi : Y \to X$ be a covering between non singular irreducible projective curves. Let $(J(Y), \Theta_Y)$ be the principally polarized Jacobian of degree zero of Y and $\mathcal{P}_Y$ the Poincaré bundle over $Y \times J(Y)$. Fixing a line bundle L_0 on Y of degree $d > 2g_Y - 2$, we identify $J(Y)$ with $J^d(Y)$ and consider the Picard bundle $\mathcal{W}_J = p_{2*}(p_1^* L_0 \otimes \mathcal{P}_Y)$ on $J(Y)$. We denote the restriction of $\mathcal{W}_J$ to the subvarieties P and $\pi^*(J(X))$ of $J(Y)$ by $\mathcal{W}_P$ and $\mathcal{W}_\pi$ respectively.

We prove that if $\pi_*(L_0)$ is stable, then $(\pi^*)^*(\mathcal{W}_J)$ is Θ_X-stable on $J(X)$ where $\pi^*\colon J(X) \to J(Y)$ is the pull-back morphism (see Theorem **??** and Remark **??**). We deduce that

$$\mathcal{W}_\pi \quad is \quad stable$$

with respect to the polarization $\Theta_Y|_{\pi^*(J(X))}$.

The restriction Θ_P of Θ_Y to P need not be a multiple of a principal polarization. However, it is possible to construct (see [**?**] and [**?**]) a principally polarized abelian variety $(\widetilde{P}, \Xi)$ and an isogeny $f\colon \widetilde{P} \to P$ with $f^{-1}(\Theta_P) \equiv n\Xi$ such that there exists a map $\widetilde{\rho}\colon Y \to \widetilde{P}$ with $\widetilde{\rho}(Y)$ and Ξ^{m-1} numerically equivalent where $m = \dim P = \dim \widetilde{P}$.

We prove that if $n \neq 2$ and $g_X > 2$, the morphism $\widetilde{\rho}\colon Y \to \widetilde{P}$ is birational onto its image (Theorem **??**) and we use some properties of the Fourier-Mukai transform to prove that in this case $\mathcal{W}_P$ is simple.

Moreover, we give explicitly the cases where $\widetilde{\rho}$ may not be birational. In this case, we prove that the Prym variety is the image in $J(Y)$ of the Jacobian of the normalization of the curve $\widetilde{\rho}(Y)$. As a consequence of stability of $\mathcal{W}_\pi$, we have that the Picard bundle $\mathcal{W}_P$ is stable with respect to $\Theta_Y|_P$ when $\widetilde{\rho}$ is not birational. In particular, for $g_X \geq 2$ and $d > 2g_Y + 4$ (Theorem **??**),

$$\mathcal{W}_P \quad is \quad simple$$

In the first section we recall some of the principal properties of the Prym variety associated with a covering and the related properties of principally polarized Prym-Tyurin varieties $\widetilde{P}$ associated with P. We prove that the induced Abel-Prym map $\widetilde{\rho}\colon Y \to \widetilde{P}$ is birational onto its image when $n \neq 2$ and $g_X > 2$.

In section §2, we study the restrictions $\mathcal{W}_\pi$ and $\mathcal{W}_P$ of the Picard bundle $\mathcal{W}_J$ and in the last section, we prove that $\mathcal{W}_\xi$ on $\mathcal{M}_\xi$ is simple.

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1. Prym varieties

In this section we recall some properties of Prym varieties which we shall use. We shall denote by $J(X)$ the Jacobian of degree zero of a non-singular projective irreducible curve X and by Θ_X the natural polarization on $J(X)$ given by the Riemann theta divisor.

Let $\pi\colon Y \to X$ be a covering of degree n between non-singular irreducible projective curves of genus g_Y and g_X respectively. The norm map and the pull back map are defined by

$$\begin{aligned} N_\pi\colon J(Y) &\to J(X) & \qquad \pi^*\colon J(X) &\to J(Y) \\ N_\pi(\mathcal{O}_Y(D)) &= \mathcal{O}_X(\pi(D)) & \qquad \pi^*(\mathcal{O}_X(D')) &= \mathcal{O}_Y(\pi^{-1}(D')). \end{aligned}$$

Note the restriction of the theta divisor of $J(Y)$ to $\pi^*(J(X))$ in general is not a multiple of a principal polarization. However, it is known that the map $h\colon J(X) \to \pi^*(J(X))$ induced by π^* is an isogeny such that $h^{-1}\left(\Theta_Y|_{\pi^*(J(X))}\right) \equiv n\Theta_X$. We will denote by Θ_π the restriction of the theta divisor of $J(Y)$ to $\pi^*(J(X))$.

The Prym variety P associated with the covering is by definition the abelian subvariety of $J(Y)$ defined by

$$P = \operatorname{Im}(n \operatorname{Id}_{J(Y)} - \pi^* \circ N_\pi).$$

We shall denote by μ the morphism $(n \operatorname{Id}_{J(Y)} - \pi^* \circ N_\pi)\colon J(Y) \longrightarrow P$.

On P there exists a natural polarization Θ_P given by the restriction of the theta divisor Θ_Y to P. In general Θ_P is not a multiple of principal polarization (see [**?**], Theorem 3.3, pp. 376). The following theorem gathers several standard results about Prym-Tyurin varieties for which we refer to [?], [?] and [?].

Theorem 1.1. *Let $\pi\colon Y \to X$ be a covering of degree n. There exists a principally polarized abelian variety $(\widetilde{P}, \Xi)$, an isogeny $f : \widetilde{P} \to P$ and a morphism $\widetilde{\mu}\colon J(Y) \to \widetilde{P}$ such that:*

1. $f^{-1}(\Theta_Y|_P) \equiv n\Xi$.
2. *The composition $f \circ \widetilde{\mu}\colon J(Y) \to P$ coincides with the morphism μ defined as above.*
3. *The restriction $\widetilde{\mu}|_P$ is surjective and the composition map $f \circ \widetilde{\mu}|_P$ coincides with the multiplication by n on P.*

Moreover, the following relation holds

$$N_\pi^{-1}(\Theta_X) + \widetilde{\mu}^{-1}(\Xi) \equiv n\Theta_Y,$$

where $\equiv$ denotes algebraic equivalence between divisors.

¿From now on we shall consider a fixed isogeny $f\colon \widetilde{P} \to P$ enjoying the properties listed in Theorem 1.1. Let y_0 be a fixed point on Y. Let $\widetilde{\rho}_{y_0}\colon Y \to \widetilde{P}$ and $\rho_{y_0}\colon Y \to P$ be the morphisms

$$\widetilde{\rho}_{y_0} = \widetilde{\mu} \circ \alpha_{y_0} \qquad \text{and} \qquad \rho_{y_0} = f \circ \widetilde{\rho}_{y_0} = \mu \circ \alpha_{y_0}$$

where $\alpha_{y_0}\colon Y \to J(Y)$ is the Abel-Jacobi map defined by $y \mapsto \mathcal{O}(y - y_0)$. We shall refer to ρ_{y_0} and $\widetilde{\rho}_{y_0}$ as *the Abel-Prym maps* of P and $\widetilde{P}$ respectively. When there is no confusion on the point y_0 we shall omit it in the notations.

By the theory of Prym-Tyurin varieties (see [?] and [?]) the morphism $\widetilde{\rho}$ has the following property:

$$\widetilde{\rho}_*[Y] = \frac{n}{(m-1)!} \bigwedge^{m-1} [\Xi],$$

where $m = \dim P = \dim \widetilde{P}$. This last property is usually expressed by saying that the curve Y (or rather the morphism $\widetilde{\rho}$) is of class n in $(\widetilde{P}, \Xi)$.

In general the Abel-Prym map $\rho\colon Y \to P$ need not be birational. However, we have the following theorem.

Theorem 1.2. *Let $\pi\colon Y \to X$ be a covering of degree n. If $g_X > 2$ and $n \neq 2$ then Abel-Prym morphism $\widetilde{\rho}\colon Y \to \widetilde{P}$ is birational onto its image.*

Before the proof of Theorem **??** we prove a lemma for any principal polarized abelian variety.

Let (A, Θ_A) be a principally polarized abelian variety of dimension d and C a non-singular projective irreducible curve. Let $\nu\colon C \to A$ be a morphism and let $u\colon J(C) \to A$ be the unique morphism (up to a translation) obtained by the universal property of the Jacobian such that $\nu = u \circ \alpha$. Using the polarizations, we define the transposed morphism $u^t\colon A \to J(C)$ by

$$u^t = \phi_{\Theta_A} \circ \hat{u} \circ \phi_{\Theta_C}^{-1},$$

where $\hat{u}\colon \hat{A} \to \widehat{J(C)}$ is the dual morphism of u and $\phi_{\Theta_A}\colon A \to \hat{A}$, $\phi_{\Theta_C}\colon J(C) \to \widehat{J(C)}$ are the isomorphisms induced by the principal polarizations Θ_A and Θ_C respectively.

Recall that a curve C is of class n in (A, Θ_A), that is $\nu_*[C] = \frac{n}{(d-1)!} \bigwedge^{d-1}[\Theta_A]$, if and only if $u \circ u^t = n \operatorname{Id}_A$ (see [**?**]).

Lemma 1.3. *Let (A, Θ_A) be a principally polarized abelian variety and $\nu\colon C \to A$ a curve of class n in A. Let Z be the normalization of the image $\nu(C)$ and let $p\colon C \to Z$ and $\widetilde{\nu}\colon Z \to A$ be the induced maps. If r is the degree of p, then r divides n and the curve Z is of class $\frac{n}{r}$ in A.*

Proof. Let $N_p\colon J(C) \to J(Z)$ be the norm map of p and $u\colon J(C) \to A$ the morphism corresponding to ν obtained by the universal property of the Jacobian. Similarly, $\widetilde{u}\colon J(Z) \to A$ corresponds to $\widetilde{\nu}$. Since $\nu\colon C \to A$ is of class n and $N_p \circ p^* = N_p \circ N_p^t = r \operatorname{Id}_{J(Z)}$, we have:

$$n \operatorname{Id}_A = u \circ u^t = \widetilde{u} \circ N_p \circ N_p^t \circ \widetilde{u}^t = r \cdot \left(\widetilde{u} \circ \widetilde{u}^t\right).$$

Thus, for every $x \in A_r$ in the r-torsion of A, $r \cdot (\widetilde{u} \circ \widetilde{u}^t)(x) = 0$ and therefore $nx = 0$. Hence, $A_r \subset A_n$ and r divides n. Since A is connected and A_r is discrete, the morphism $A \to A_r$ given by $x \mapsto (\widetilde{u} \circ \widetilde{u}^t)(x) - \frac{n}{r}x$ is identically zero. Hence $\widetilde{u} \circ \widetilde{u}^t = \frac{n}{r} \operatorname{Id}_A$. ☐

Proof of Theorem **??**. Suppose that $\widetilde{\rho}$ is not birational onto its image and $n \neq 2$. Let Z be the normalization of the curve $\widetilde{\rho}(Y)$ and let $\widetilde{\pi}\colon Y \to Z$ be the morphism induced by $\widetilde{\rho}$. Let $\tilde{n}$ be the degree of $\tilde{\pi}$. That is,

$$\begin{tikzcd} & Y \arrow[dl, "n:1"', "\pi"] \arrow[dr, "\tilde{n}:1", "\tilde{\pi}"'] & \\ X & & Z \end{tikzcd}$$

Since the morphism $\tilde{\mu}$ is surjective, Z generates $\widetilde{P}$. Therefore the map $\tilde{u}\colon J(Z) \to \widetilde{P}$ defined by the universal property of the Jacobian is surjective. Hence $g_Y - g_X = \dim \widetilde{P} \leq g_Z$, where g_Z is the genus of Z.

By Riemann-Hurwitz formula for the coverings π and $\tilde{\pi}$, we have the following inequality

$$g_Y \leq \frac{g_Y - 1 - (\deg R_\pi)/2}{n} + \frac{g_Y - 1 - (\deg R_{\tilde{\pi}})/2}{\tilde{n}} + 2 \tag{1}$$

where R_π and $R_{\tilde{\pi}}$ are the ramification divisors.

¿From Lemma **??**, it follows that $\tilde{n}$ divides n; let $q = n/\tilde{n}$. Hence, inequality (**??**) implies that

$$(n-q-1)g_Y \leq 2n - q - 1 - ((\deg R_\pi)/2 + q(\deg R_{\tilde{\pi}})/2)\,.$$

Since $((\deg R_\pi)/2 + q(\deg R_{\tilde{\pi}})/2) \geq 0$ and $n \neq q+1$ because $n \neq 2$, we have

$$g_Y \leq 2 + \left[\frac{q+1}{n-q-1}\right]$$

where $[-]$ denotes the integral part of a rational number. Since $\tilde{n} > 1$, it is easy to check that $\left[\frac{q+1}{n-q-1}\right] \geq 1$ if and only if $\tilde{n} = 2, 3, 4$. Studying these cases we obtain that if $\widetilde{\rho}$ is not birational and $n \neq 2$, then

$$n = 3, \quad \tilde{n} = 3 \text{ and } g_Y \leq 4; \qquad n = 4, \begin{cases} \text{if } \tilde{n} = 4 & \text{then } g_Y \leq 3 \\ \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 5 \end{cases};$$

$$n = 6, \begin{cases} \text{if } \tilde{n} = 6 & \text{then } g_Y \leq 2 \\ \text{if } \tilde{n} = 3 & \text{then } g_Y \leq 3 \\ \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 4 \end{cases}; \qquad n \neq 3, 4, 6, \begin{cases} \text{if } \tilde{n} = 2 & \text{then } g_Y \leq 3 \\ \text{if } \tilde{n} \neq 2 & \text{then } g_Y \leq 2 \end{cases}.$$

In all cases, we have that if $\widetilde{\rho}$ in not birational onto its image and $n \neq 2$, then $g_Y \leq 5$.

Now if we also suppose that $g_X \geq 2$, by the Riemann-Hurwitz formula for the covering π, we deduce that if $\deg \pi \neq 2$ and $g_X \geq 2$, then $\widetilde{\rho}$ is not birational onto its image if the following cases occur:

$$\begin{aligned} &n = 3, \quad \tilde{n} = 3, \quad g_Y = 4, \quad g_X = 2, \quad g_Z = 2, \quad R_\pi = R_{\tilde{\pi}} = 0 \\ &n = 4, \quad \tilde{n} = 2, \quad g_Y = 5, \quad g_X = 2, \quad g_Z = 3, \quad R_\pi = R_{\tilde{\pi}} = 0 \end{aligned} \tag{2}$$

Therefore, if we assume that $g_X > 2$ and $n \neq 2$, the theorem is proved. □

Remark 1.4. From the proof of Theorem **??** we deduce that if $g_X \geq 2$, then the Abel-Prym map $\widetilde{\rho}$ may not be birational onto its image possibly in the cases given in (**??**) and when $n = 2$. For $n = 2$, it is well-known that if Y is not an hyperelliptic curve then ρ is birational onto its image and therefore $\widetilde{\rho}$ is birational onto its image.

We finish this section with a lemma that we will use later.

Lemma 1.5. *For any $\tilde{\xi} \in \widetilde{P}$, we have*

$$\widetilde{\rho}^*\left(\tau^*_{\tilde{\xi}}\mathcal{O}_{\widetilde{P}}(\Xi)\right) \cong \widetilde{\rho}^*\mathcal{O}_{\widetilde{P}}(\Xi) \otimes f(\widetilde{\xi})^{-1},$$

where $\tau_{\tilde{\xi}}$ is the translation by $\tilde{\xi}$.

Proof. By Theorem **??**, there exists $\xi \in P$ such that $\tilde{\mu}(\xi) = \tilde{\xi}$. Using the relation between the polarizations given in this theorem, we obtain

$$\begin{aligned}\widetilde{\rho}^*\left(\tau^*_{\tilde{\xi}}\mathcal{O}_{\widetilde{P}}(\Xi)\right) &\cong \alpha^*\left(\widetilde{\mu}^*\left(\tau^*_{\tilde{\xi}}\mathcal{O}_{\widetilde{P}}(\Xi)\right)\right) \cong \alpha^*\left(\tau^*_{\xi}\left(\widetilde{\mu}^*\mathcal{O}_{\widetilde{P}}(\Xi)\right)\right)\\ &\cong \alpha^*\left(\tau^*_{\xi}\left(\mathcal{O}_{J(Y)}(n\Theta_Y) \otimes N^*_{\pi}\mathcal{O}_{J(X)}(\Theta_X)^{\vee} \otimes \mathcal{N}\right)\right),\end{aligned}$$

where $\mathcal{N} \in \mathrm{Pic}^0(J(Y))$. In particular, $\mathcal{N}$ is invariant under translations. Moreover, we have $\tau^*_{\xi}\left(N^*_{\pi}\mathcal{O}_{J(X)}(\Theta_X)^{\vee}\right) \cong N^*_{\pi}\left(\mathcal{O}_{J(X)}(\Theta_X)^{\vee}\right)$ because $\xi \in P$ and the isomorphism

$$\alpha^*\left(\tau^*_{\xi}\mathcal{O}_{J(Y)}(n\Theta_Y)\right) \cong \alpha^*\left(\mathcal{O}_{J(Y)}(n\Theta_Y)\right) \otimes \xi^{-n}.$$

Therefore, it follows that $\widetilde{\rho}^*\left(\tau^*_{\tilde{\xi}}\mathcal{O}_{\widetilde{P}}(\Xi)\right) \cong \widetilde{\rho}^*\mathcal{O}_{\widetilde{P}}(\Xi) \otimes \xi^{-n}$. Since $\xi^{-n} = f(\widetilde{\mu}|_P(\xi^{-1})) = f(\widetilde{\xi})^{-1}$, the lemma is proved. □

2. Stability and restrictions of Picard bundles

Let (M, H) be a m-dimensional polarized non singular algebraic variety, where H is an ample divisor on M. A vector bundle E is said to be $H-$ *stable* if for every proper subsheaf F one has

$$(c_1(F) \cdot H^{m-1})/rk(F) < (c_1(E) \cdot H^{m-1})/rk(E).$$

Let Y be a non singular irreducible projective curve of genus $g_Y \geq 2$. The Jacobian $J(Y)$ carries a natural bundles namely *the Picard bundles*. That is, fix a line bundle L_0 on Y of degree d and a point $y_0 \in Y$. Let $\mathcal{P}_Y$ be the Poincaré bundle over $Y \times J(Y)$ normalized with respect to the point y_0, i.e. $\mathcal{P}_Y|_{\{y_0\}\times J(Y)} \cong \mathcal{O}_{J(Y)}$. The Picard sheaves (relative to L_0) on $J(Y)$ are defined as the higher direct images $\mathcal{R}^i(L_0) := R^ip_{2*}(p_1^*L_0 \otimes \mathcal{P}_Y)$ where p_i are the canonical projections of $Y \times J(Y)$ in the $i = 1, 2$ factor. If $d > 2g_Y - 2$, then $\mathcal{R}^1(L_0)$ is zero and $\mathcal{R}^0(L_0)$ is a vector bundle which we will denote as

$$\mathcal{W}_J := p_{2*}(p_1^*L_0 \otimes \mathcal{P}_Y)$$

and call it *the Picard bundle.*

The Picard sheaves have been studied from different points of view (see [?], [?], etc.). In particular, Ein and Lazarsfeld ([?]) for $d > 2g_Y - 1$ and Kempf ([?]) for $d = 2g_Y - 1$ proved the stability. We shall consider the restriction of the Picard bundle $\mathcal{W}_J$ to some subvarieties when the curve is a covering space.

Let $\pi\colon Y \to X$ be a covering of degree n between non-singular projective irreducible curves of genus g_Y and g_X respectively and let $\pi^*\colon J(X) \to J(Y)$ be the pull-back morphism. As in §1 consider the Prym variety (P, Θ_P) and $(\pi^*(J(X)), \Theta_\pi)$ in $J(Y)$.

¿From now on we fix a line bundle L_0 on Y of degree $d > 2g_Y - 2$. Denote by $\mathcal{W}_P$ and $\mathcal{W}_\pi$ the restrictions of the Picard bundle $\mathcal{W}_J = p_{2*}(p_1^* L_0 \otimes \mathcal{P}_Y)$ (relative to L_0) to P and $\pi^*(J(X))$ respectively.

Remark 2.1. We can take a line bundle $L_0 \in J^d(Y)$ such that $\pi_*(L_0)$ is stable and $\deg \pi_*(L_0) > 2ng_X$ without loss of generality. Indeed: for a generic line bundle L on Y, A. Beauville in [?] has proved that $\pi_* L$ is stable on X if $|\chi(L)| \leq g_X + \frac{g_X^2}{n}$ or $\deg \pi < \max\{g_X(1+\sqrt{3}) - 1, 2g_X + 2\}$. Therefore, if $L' \in J^{d'}(Y)$ with

$$g_Y - g_X - 1 - \frac{g_X^2}{n} \leq d' \leq g_Y + g_X - 1 + \frac{g_X^2}{n}, \tag{3}$$

then $\pi_* L'$ is stable on X and $\pi_*(L' \otimes \pi^* M)$ is stable for any line bundle M on X. Thus, for every line bundle $L \in J^d(Y)$ with d such that:

(i) $d > \max\{2g_Y - 2 + 2n - (\deg R_\pi)/2\ ,\ 2g_Y - 2\}$,

(ii) d is equivalent (modulo n) to a number d' such that d' fulfills the condition (??),

we have that $\pi_*(L)$ is stable and $\deg \pi_*(L) = \deg L - (\deg R_\pi)/2 > 2ng_X$. In the case $\deg \pi < \max\{g_X(1+\sqrt{3}) - 1, 2g_X + 2\}$, it is sufficient that $\deg L$ fulfills the condition (i).

Theorem 2.2. *If $\pi_*(L_0) = E_0$ is stable and $\deg(E_0) > 2ng_X$, then $(\pi^*)^*(\mathcal{W}_J)$ is Θ_X-stable on $J(X)$.*

Proof. For convenience of writing, we set up the following commutative diagrams:

$$\begin{tikzcd}
X & Y \arrow[l, "\pi"'] & \\
X \times J(X) \arrow[u, "p_X"] \arrow[dr, "p_{J(X)}"'] & Y \times J(X) \arrow[l, "\pi \times id"'] \arrow[u, "q_1"'] \arrow[r, "id \times \pi^*"] \arrow[d, "q_2"] & Y \times J(Y) \arrow[ul, "p_1"'] \arrow[d, "p_2"'] \\
& J(X) \arrow[r, "\pi^*"] & J(Y)
\end{tikzcd} \tag{4}$$

where the vertical morphisms are the natural projections.

Fix the Poincaré bundle $\mathcal{P}_X$ on $X \times J(X)$ normalized with respect to the point $\pi(y_0) \in X$. Then, the vector bundle $(id \times \pi^*)^*\mathcal{P}_Y$ on $Y \times J(X)$ is isomorphic to the bundle $(\pi \times id)^*\mathcal{P}_X$. ¿From the second square of diagram (**??**) and the base change formula, we have

$$\begin{aligned}(\pi^*)^*(\mathcal{W}_J) &= (\pi^*)^* \left(p_{2*}(p_1^* L_0 \otimes \mathcal{P}_Y)\right) \\ &\cong q_{2*}\left(q_1^* L_0 \otimes (id \times \pi^*)^*\mathcal{P}_Y\right) \\ &\cong q_{2*}\left(q_1^* L_0 \otimes (\pi \times id)^*\mathcal{P}_X\right).\end{aligned}$$

But from the first square of diagram (**??**), we obtain

$$\begin{aligned}q_{2*}\left(q_1^* L_0 \otimes (\pi \times id)^*\mathcal{P}_X\right) &\cong p_{J(X)*}\left((\pi \times id)_* \left(q_1^* L_0 \otimes (\pi \times id)^*\mathcal{P}_X\right)\right) \\ &\cong p_{J(X)*}\left((\pi \times id)_*(q_1^* L_0) \otimes \mathcal{P}_X\right) \\ &\cong p_{J(X)*}(p_X^*(\pi_*(L_0) \otimes \mathcal{P}_X)).\end{aligned}$$

Hence, $(\pi^*)^*(\mathcal{W}_J) \cong p_{J(X)*}(p_X^*(E_0) \otimes \mathcal{P}_X)$.

Since E_0 is stable of degree $d > 2ng_X$, $p_X^*(E_0) \otimes \mathcal{P}_X$ is a family of stable bundles parametrized by $J(X)$. Such family corresponds to an embedding of the Jacobian in the moduli space $\mathcal{M}(n,d)$ of stable vector bundles of rank n and degree d. As in the proof of Theorem 2.5 in [**?**], it follows that $p_{J(X)*}(p_X^*(E_0) \otimes \mathcal{P}_X)$ is Θ_X-stable. □

With the hypothesis of the previous theorem, we obtain

Corollary 2.3. *The restriction $\mathcal{W}_\pi$ of $\mathcal{W}_J$ to $\pi^*(J(X))$ is Θ_π-stable.*

Proof. The map $h : J(X) \to \pi^*(J(X))$ is an isogeny such that $h^{-1}(\Theta_\pi) \equiv n\Theta_X$. Since $h^*(\mathcal{W}_\pi) \cong (\pi^*)^*(\mathcal{W}_J)$, we have from Lemma 2.1 in [**?**] that $\mathcal{W}_\pi$ is Θ_π-stable. □

To study the restriction $\mathcal{W}_P$ of $\mathcal{W}_J$ to the Prym variety P we consider two cases, namely when the Abel-Prym map $\tilde{\rho}$ is not birational and when it is. In both cases we can reduce our study to consider the corresponding vector bundle over $(\widetilde{P}, \Xi)$ where $f : \widetilde{P} \to P$ is a fixed isogeny as in Theorem 1.1. That is, if $j_p : P \hookrightarrow J(Y)$ is the natural inclusion, then we will denote by $\beta : \widetilde{P} \to J(Y)$ the composition map $f \circ j_P$ and by $\mathcal{W}_{\widetilde{P}}$ the vector bundle $\beta^*(\mathcal{W}_J)$. Actually, $f^*(\mathcal{W}_P) \cong \mathcal{W}_{\widetilde{P}}$.

Proposition 2.4. *If $\mathcal{W}_{\widetilde{P}}$ is Ξ-stable (resp. simple), then $\mathcal{W}_P$ is Θ_P-stable (resp. simple).*

Proof. The stability follows, as in Corollary **??**, from Lemma 2.1 in [**?**].

Suppose $\mathcal{W}_{\widetilde{P}}$ is simple. Since $\mathcal{O}_P \hookrightarrow f_*\mathcal{O}_{\widetilde{P}}$ is injective,

$$\mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P) \hookrightarrow \mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P \otimes f_*\mathcal{O}_{\widetilde{P}})$$

is injective. Moreover,

$$\begin{aligned}\mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P \otimes f_*\mathcal{O}_{\widetilde{P}}) &\cong \mathrm{Hom}(\mathcal{W}_P, f_*f^*\mathcal{W}_P)\\ &\cong \mathrm{Hom}(\mathcal{W}_{\widetilde{P}}, \mathcal{W}_{\widetilde{P}}) \cong \mathbb{C}\end{aligned}$$

Hence, $\mathrm{Hom}(\mathcal{W}_P, \mathcal{W}_P) \cong \mathbb{C}$ and $\mathcal{W}_P$ is simple. □

Suppose $\widetilde{\rho}$ is not birational. As in section §1, let Z be the normalization of $\widetilde{\rho}(Y)$ and $\widetilde{\pi} : Y \to Z$ the morphism induced by $\widetilde{\rho}$ of degree $\tilde{n}$.

We shall relate the Jacobian variety $J(Z)$ with $\widetilde{P}$ and its images on $J(Y)$.

Proposition 2.5. *If the Abel-Prym map* $\widetilde{\rho}\colon Y \to \widetilde{P}$ *is not birational onto its image and* $\deg L_0 > 2g_Y + 4$*, then the Picard bundle* $\mathcal{W}_P$ *is* Θ_P*-stable.*

Proof. We consider separately each one of the cases where $\widetilde{\rho}$ is not birational (see Remark **??**).

1) Suppose $n = 2$.

By lemma **??**, $\deg \widetilde{\pi} = 2$ and $\widetilde{\rho}(Y)$ is of class one in $(\widetilde{P}, \varXi)$. Then, by Matsusaka's Criterion, $Z = \widetilde{\rho}(Y)$ and $(J(Z), \Theta_Z) \cong (\widetilde{P}, \varXi)$.

¿From the construction of the map $\widetilde{\rho}$, we have $\mathrm{Nm}\, \widetilde{\pi} = \widetilde{\mu}$. Moreover, $\widetilde{\mu}^t = j_P \circ f = \beta$ (see section §1 in [**?**]). Therefore the map $\widetilde{\pi}^* : J(Z) \to J(Y)$ coincides with the map $\beta : \widetilde{P} \to J(Y)$ via the isomorphism $J(Z) \cong \widetilde{P}$. By definition of $\mathcal{W}_{\widetilde{P}}$, we have

$$\mathcal{W}_{\widetilde{P}} \cong (\widetilde{\pi}^*)^*(\mathcal{W}_J).$$

In this case, $\deg \widetilde{\pi} = 2 < \max\{g_Z(1+\sqrt{3})-1, 2g_Z+2\}$ because $g_Z \geq 1$. Therefore, by Remark **??** and Theorem **??**, if $\deg L_0 = d > 2g_Y+2 \geq \max\{2g_Y+2-\deg(R_{\widetilde{\pi}})/2, 2g_Y-2\}$, then $(\widetilde{\pi}^*)^*(\mathcal{W}_J) \cong \mathcal{W}_{\widetilde{P}}$ is $\varXi$-stable and from Proposition **??**, $\mathcal{W}_P$ is Θ_P-stable.

2) Suppose $n = 3$, $\tilde{n} = 3$, $g_Y = 4$, $g_X = 2$, $g_Z = 2$, $R_\pi = R_{\widetilde{\pi}} = 0$.

Since $\widetilde{\rho}(Y)$ is of class one in $(\widetilde{P}, \varXi)$, $Z = \widetilde{\rho}(Y)$ and $(J(Z), \Theta_Z) \cong (\widetilde{P}, \varXi)$. In this case, $\deg \widetilde{\pi} = 3 < \max\{g_Z(1+\sqrt{3})-1, 2g_Z+2\}$. Hence, if $\deg L_0 = d > 2g_Y + 4 = 12$, then $\mathcal{W}_{\widetilde{P}}$ is $\varXi$-stable and $\mathcal{W}_P$ is Θ_P-stable.

3) Suppose $n = 4$, $\tilde{n} = 2$, $g_Y = 5$, $g_X = 2$, $g_Z = 3$, $R_\pi = R_{\widetilde{\pi}} = 0$.

We have the following commutative diagrams:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{\pi}"] \arrow[d, hook, "\alpha_{y_0}"'] \arrow[rr, bend left, "\widetilde{\rho}"] & Z \arrow[r] \arrow[d, hook, "\alpha_{\tilde{\pi}(y_0)}"'] & \widetilde{P} \\ J(Y) \arrow[r, "\mathrm{Nm}\,\widetilde{\pi}"] \arrow[rru, bend right, "\tilde{\mu}"'] & J(Z) \arrow[ru, "u"] & \end{tikzcd}$$

where u is the map induced by $Z \to \widetilde{P}$. Therefore,

$$\beta = j_P \circ f = \tilde{\mu}^t = \tilde{\pi}^* \circ u^t \colon \widetilde{P} \longrightarrow J(Y)$$

and $\operatorname{Im} \tilde{\mu}^t = P \subset \operatorname{Im} \tilde{\pi}^*$. Since $\dim P = 3 = \dim J(Z) = \dim \tilde{\pi}^*(J(Z))$, it follows that $\tilde{\pi}^*(J(Z)) = P \subset J(Y)$.

Since $\deg \tilde{\pi} = 2 < \max\{g_Z(1+\sqrt{3}) - 1, 2g_Z + 2\}$, if $\deg L_0 = d > 2g_Y + 2 = 12$ then, from Corollary **??**, the restriction of $\mathcal{W}_J$ to $\tilde{\pi}^*(J(Z))$ is $\Theta_Y|_{\tilde{\pi}^*(J(Z))}$-stable, i.e. W_P is Θ_P-stable. □

Before we consider the case when the map $\tilde{\rho}$ is birational we describe the bundle $\mathcal{W}_{\widetilde{P}}$ in different way.

The abelian variety $(\widetilde{P}, \Xi)$ is principally polarized, so we can identify $\widetilde{P}$ with its dual abelian variety. The Poincaré bundle on $\widetilde{P} \times \widetilde{P}$ is given by

$$\mathcal{Q} \cong m^*\mathcal{O}_{\widetilde{P}}(\Xi) \otimes q_1^*\mathcal{O}_{\widetilde{P}}(\Xi)^\vee \otimes q_2^*\mathcal{O}_{\widetilde{P}}(\Xi)^\vee,$$

where m is the multiplication law on $\widetilde{P}$ and q_i the canonical projections of $\widetilde{P} \times \widetilde{P}$ in the $i = 1, 2$ factor.

If $\mathcal{P}_Y$ is the normalized Poincaré bundle over $Y \times J(Y)$ then denote by $\mathcal{P}_P$ the line bundle $(id \times j_P)^*(\mathcal{P}_Y)$ on $Y \times P$ and by $\mathcal{P}_{\widetilde{P}}$ the line bundle $(id \times f)^*(\mathcal{P}_P)$ on $Y \times \widetilde{P}$, where $j_p : P \to J(X)$ is the natural inclusion and $f : \widetilde{P} \to P$ is a fixed isogeny as in Theorem 1.1. Thus we have the maps:

$$Y \times P \xleftarrow{id \times f} Y \times \widetilde{P} \xrightarrow{\widetilde{\rho} \times id} \widetilde{P} \times \widetilde{P}$$

Proposition 2.6. *The line bundles $(\tilde{\rho} \times id)^*\mathcal{Q}^\vee$ and $\mathcal{P}_{\widetilde{P}}$ are isomorphic.*

Proof. From the normalization of $\mathcal{P}_Y$, the restrictions of both line bundles to $\{y_0\} \times \widetilde{P}$ are trivial. Therefore it is enough to check that, for every $\tilde{\xi} \in \widetilde{P}$, the restrictions of the bundles $\mathcal{P}_{\widetilde{P}}$ and $(\tilde{\rho} \times id)^*\mathcal{Q}^\vee$ to $Y \times \{\tilde{\xi}\}$ are isomorphic.

¿From Lemma **??** we obtain

$$\begin{aligned}\left((\tilde{\rho}\times id)^{*}\mathcal{Q}^{\vee}\right)|_{Y\times\{\tilde{\xi}\}} &\cong \left((\tilde{\rho}\times id)^{*}\left(m^{*}\mathcal{O}_{\widetilde{P}}(\Xi)^{\vee}\otimes q_1^{*}\mathcal{O}_{\widetilde{P}}(\Xi)\right)\right)|_{Y\times\{\tilde{\xi}\}}\\ &\cong \widetilde{\rho}^{*}\left(\tau_{\tilde{\xi}}^{*}\mathcal{O}_{\widetilde{P}}(\Xi)^{\vee}\right)\otimes\widetilde{\rho}^{*}\mathcal{O}_{\widetilde{P}}(\Xi)\cong f(\tilde{\xi}).\end{aligned}$$

By the definition of $\mathcal{P}_{\widetilde{P}}$ it follows that $\mathcal{P}_{\widetilde{P}}|_{Y\times\{\tilde{\xi}\}}\cong\left((id\times f)^{*}\mathcal{P}_{P}\right)|_{Y\times\{\tilde{\xi}\}}\cong f(\tilde{\xi})$. Therefore, the proposition is proved. □

Given a sheaf M on $\tilde{P}$, we can consider the higher direct images

$$R^{i}q_{2*}(q_1^{*}M\otimes\mathcal{Q}^{\vee}).$$

When there is only one non-null higher direct image, this sheaf is called the Fourier-Mukai transform of M. For the general definition of the Fourier-Mukai transform as a functor between derived categories and its properties we refer to [**?**].

Remark 2.7. ¿From the invertibility of the Fourier-Mukai transform, it follows that the transform of a simple sheaf is simple (see [**?**], Corollary 2.5).

The following proposition allows us to express the vector bundle $\mathcal{W}_{\widetilde{P}}$ as a Fourier-Mukai transform.

Proposition 2.8. *The vector bundle $\mathcal{W}_{\widetilde{P}}$ is the Fourier-Mukai transform of the sheaf $\widetilde{\rho}_{*}(L_0)$. That is, $\mathcal{W}_{\widetilde{P}}\cong q_{2*}\left(q_1^{*}\left(\widetilde{\rho}_{*}(L_0)\right)\otimes\mathcal{Q}^{\vee}\right)$, where q_i are the projection of $\widetilde{P}\times\widetilde{P}$ in the $i=1,2$ factor.*

Proof. Consider the commutative diagrams:

$$\begin{tikzcd} Y \arrow[r, "\widetilde{\rho}"] & \widetilde{P} \\ Y\times\widetilde{P} \arrow[u, "p_1"] \arrow[r, "\widetilde{\rho}\times id"] \arrow[dr, "p_2"'] & \widetilde{P}\times\widetilde{P} \arrow[u, "q_1"'] \arrow[d, "q_2"] \\ & \widetilde{P} \end{tikzcd}$$

From the definition of $\mathcal{W}_{\widetilde{P}}$ and Proposition **??** we obtain:

$$\mathcal{W}_{\widetilde{P}}\cong p_{2*}\left(p_1^{*}L_0\otimes\mathcal{P}_{\widetilde{P}}\right)\cong q_{2*}\left((\tilde{\rho}\times id)_{*}\left(p_1^{*}L_0\otimes(\tilde{\rho}\times id)^{*}\mathcal{Q}^{\vee}\right)\right)$$

and using the projection formula and the base-change formula, we have

$$\mathcal{W}_{\widetilde{P}}\cong q_{2*}\left((\tilde{\rho}\times id)_{*}(p_1^{*}L_0)\otimes\mathcal{Q}^{\vee}\right)\cong q_{2*}\left(q_1^{*}\left(\widetilde{\rho}_{*}(L_0)\right)\otimes\mathcal{Q}^{\vee}\right).$$

□

Proposition 2.9. *If the Abel-Prym map $\widetilde{\rho}$ is birational onto its image then the restriction of the Picard bundle $\mathcal{W}_J$ to the Prym variety is simple.*

Proof. From Proposition **??** it is enough to prove that $\mathcal{W}_{\widetilde{P}}$ is simple.

The bundle $\mathcal{W}_{\widetilde{P}}$ is the Fourier-Mukai transform of the sheaf $\widetilde{\rho}_*(L_0)$ by Proposition **??**. Since $\deg(L_0) > 2g_Y - 2$, the sheaf $\widetilde{\rho}_*(L_0) \otimes \mathcal{Q}^\vee$ has only one non-null direct image. Then the Fourier-Mukai transform is a complex concentrated in degree zero and, from Remark **??**, $\mathcal{W}_{\widetilde{P}}$ is simple if $\widetilde{\rho}_*(L_0)$ is simple.

To prove that $\widetilde{\rho}_*(L_0)$ is simple write $\widetilde{\rho}$ as a composite $j \circ \tilde{\nu}$ where $\tilde{\nu}\colon Y \to \widetilde{\rho}(Y)$ is a birational morphism between curves and $j\colon \widetilde{\rho}(Y) \to \widetilde{P}$ is a closed embedding. Since $\widetilde{\rho}$ is birational onto its image, $\tilde{\nu}_*(L_0)$ is of rank 1. Therefore the sheaf K defined by

$$0 \to K \to \tilde{\nu}^* \tilde{\nu}_* L_0 \to L_0 \to 0$$

is a torsion sheaf. From the exact sequence

$$0 \to \operatorname{Hom}(L_0, L_0) \to \operatorname{Hom}(\tilde{\nu}^* \tilde{\nu}_* L_0, L_0) \to \operatorname{Hom}(K, L_0).$$

and the fact that $\operatorname{Hom}(K, L_0) = 0$, we obtain

$$\operatorname{Hom}(\tilde{\nu}^* \tilde{\nu}_* L_0, L_0) \cong \operatorname{Hom}(L_0, L_0) \cong \mathbb{C}.$$

Hence, by adjunction, we have that

$$\operatorname{Hom}(\tilde{\nu}^* \tilde{\nu}_* L_0, L_0) \cong \operatorname{Hom}(\tilde{\nu}_* L_0, \tilde{\nu}_* L_0) \cong \mathbb{C}.$$

Since j is a closed embedding, $j^* j_* E \cong E$ for any sheaf E and applying the adjunction formula again, we obtain that $j_* \tilde{\nu}_*(L_0) \cong \widetilde{\rho}_* L_0$ is simple. □

From Proposition **??** and Proposition **??** we have that:

Theorem 2.10. *The restriction of the Picard bundle $\mathcal{W}_J$ to the Prym variety is simple if $g_X \geq 2$ and $\deg L_0 > 2g_Y + 4$.*

3. Picard bundles over the moduli spaces

We shall recall the construction and properties of the spectral coverings given by Beauville, Narasimhan and Ramanan in [**?**].

Let X be a non-singular projective irreducible curve of genus $g_X \geq 2$. Denote by $\mathcal{M}(n, d)$ the moduli space of stable vector bundles over X of rank n and degree d. If n and d are coprime there exists a universal family $\mathcal{U}$ parametrized by $\mathcal{M}(n, d)$ called the

Poincaré bundle. If $d > 2n(g_X - 1)$ the Picard bundle $\mathcal{W}$ is the direct image of $\mathcal{U}$ and it is locally free. Denote by $\mathcal{W}_\xi$ the restriction of $\mathcal{W}$ to the subvariety $\mathcal{M}_\xi \subset \mathcal{M}(n,d)$ determined by stable bundles with fixed determinant $\xi \in J^d(X)$.

Let K be the canonical bundle over X and $W = \bigoplus_{i=1}^n H^0(X, K^i)$. For every element $s = (s_1, \dots, s_n) \in W$, denote by Y_s the associated spectral curve (see [?]). For a general $s \in W$, Y_s is non-singular of genus $g_{Y_s} = n^2(g_X - 1) + 1$ and the morphism

$$\pi_s \colon Y_s \longrightarrow X$$

is of degree n.

In [?] it is proved that if $\delta = d + n(n-1)(g_X - 1)$ then there is an open subvariety T_δ of $J^\delta(Y_s)$ such that the morphism $T_\delta \to \mathcal{M}(n,d)$ defined by $L \mapsto \pi_{s*}(L)$ is dominant. Moreover, the direct image induces a dominant rational map

$$f \colon P' \dashrightarrow \mathcal{M}_\xi$$

defined on an open subvariety $T' \subseteq P'$, where P' is a translate of the Prym variety P_s of π_s (see [?], Proposition 5.7). The complement of the open subvariety $T' \subseteq P'$ is of codimension at least 2. Actually, $f \colon T' \longrightarrow \mathcal{M}_\xi$ is generically finite.

Consider the following commutative diagram:

$$\begin{tikzcd}
Y_s \times T' \arrow[r, "\pi_s \times id"] \arrow[dr, "q_2'"'] & X \times T' \arrow[r, "id \times f"] \arrow[d, "q_2"] & X \times \mathcal{M}_\xi \arrow[d, "p_2"] \\
& T' \arrow[r, "f"] & \mathcal{M}_\xi
\end{tikzcd}$$

where q_2', q_2, p_2 are the projections to the second factor.

If $\mathcal{P}_{T'}$ is the restriction of the Poincaré bundle over $Y_s \times P'$ to $Y_s \times T'$ and $\mathcal{U}_\xi$ is the restriction of the universal bundle $\mathcal{U}$ to $X \times \mathcal{M}_\xi$, then, by the definition of f,

$$(id \times f)^*(\mathcal{U}_\xi) \cong (\pi_s \times id)_*(\mathcal{P}_{T'}) \otimes q_2^*(M)$$

for some line bundle M over T', which depends on the choice of $\mathcal{U}$. Therefore,

$$\begin{aligned}
f^*(\mathcal{W}_\xi) &\cong (q_2)_*(id \times f)^*(\mathcal{U}_\xi) \\
&\cong (q_2)_*((\pi_s \times id)_*(\mathcal{P}_{T'})) \otimes M \\
&\cong (q_2')_*(\mathcal{P}_{T'}) \otimes M.
\end{aligned}$$

Actually, $(q_2')_*(\mathcal{P}_{T'})$ is just the restriction of the Picard bundle $\mathcal{W}_{P'}$ over P' to T'.

Theorem 3.1. *The Picard bundle $\mathcal{W}_\xi$ on $\mathcal{M}_\xi$ is simple for $d > n(n+1)(g_X - 1) + 4$ and $g_X \geq 2$.*

Proof. Since $\operatorname{codim}(T') \geq 2$, we have

$$\operatorname{End}(\mathcal{W}_{P'}) \cong H^0(P', \mathcal{E}nd(\mathcal{W}_{P'})) \cong H^0(T', \mathcal{E}nd(\mathcal{W}_{P'}|_{T'})) \cong \operatorname{End}(f^*\mathcal{W}_\xi).$$

The Abel-Prym map for the spectral cover is birational onto its image if $n \neq 2$ or $g_X > 2$. In this case, from Proposition **??**, the bundle $\mathcal{W}_{P'}$ is simple if $\delta > 2g_{Y_s} - 2$. Since the map $f\colon T' \to \mathcal{M}_\xi$ is dominant and generically finite, as in the proof of Proposition **??**, we deduce that the Picard bundle $\mathcal{W}_\xi$ on $\mathcal{M}_\xi$ is simple for degree $d > n(n+1)(g_X - 1)$.

For $n = 2$ and $g_X = 2$, if the map $\widetilde{\rho}$ is not birational onto its image, then by the proof of Proposition **??**, $\mathcal{W}_\xi$ is simple for $d > n(n+1)(g_X - 1) + 4$. If $\widetilde{\rho}$ is birational then, as above, $\mathcal{W}_\xi$ is simple for $d > n(n+1)(g_X - 1)$. □

Remark 3.2. i) Denote by Θ_ξ a generalized theta divisor on $\mathcal{M}_\xi$. By [**?**], Theorem 4.3, we have that

$$f^*(\mathcal{O}(\Theta_\xi)) \cong \mathcal{O}(\Theta_{P'})|_{T'}$$

where $\Theta_{P'}$ is the restriction of the theta divisor of $J^\delta(Y_s)$ to P'. Since $\mathcal{W}_{P'}|_{T'} \cong f^*\mathcal{W}_\xi$, by Lemma 2.1 in [**?**], it follows that if $\mathcal{W}_{P'}$ is $\Theta_{P'}$-stable, then $\mathcal{W}_\xi$ is Θ_ξ-stable. Moreover, from Theorem 2.2 in [**?**], $\mathcal{W}_{P'}$ is stable if $\widetilde{\rho}_L^*(\mathcal{W}_{P'})$ is stable on Y for a general line bundle L.

ii) Let $n = 2$ and $g_X = 2$. If the spectral curve can be choosen to be hyperelliptic then, as in Proposition **??**, $\mathcal{W}_{P'}$ is stable. Therefore, $\mathcal{W}_\xi$ will be stable and will answer the question arised by Balaji and Vishwanath in [**?**], for this case.

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