QR, SV and Polar Decomposition and the Elliptically Contoured Distributions

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Abstract

Considering the QR factorization, $Y = H_1T$, the polar decomposition, $Y = H_1R$, the SVD, $Y = H_1DW_1$, of matrix Y, and the following decompositions: spectral, Cholesky's, and symmetric non-negative definite square root, of matrix $S = Y'Y$, the Jacobians associated to this transformations are found. Assuming that Y has a singular elliptically contoured distribution, the distributions of matrices T, R, (D, W_1) and D are determined, for central and non-central cases, as well as their relationship to the Wishart and Pseudo-Wishart generalised distributions. These results are applied to two subfamilies of elliptical distributions, the matrix variate normal distribution and the matrix variate symmetric Pearson type VII distribution.

Key words and phrases: Singular matrix distribution, generalised Wishart and Pseudo-Wishart distributions, noncentral distribution, elliptical distribution, SVD, QR and polar decomposition.

1. INTRODUCTION

Under different circumstances, given a random matrix Y, there is need to consider some form of decomposition, e.g., $Y = QR$, in order to find the density of matrices Q, R, or

*This article was written by the authors while at the Dept. of Mathematics of the Instituto Tecnológico de Estudios Superiores de Monterrey, México; the first author as a Visiting Professor and the second as an Associate Professor both. Thus if, for instance, Y has a matrix variate normal distribution, it may be written as $Y = H_1T$, the QR decomposition (see Lemma 2.1). The interest lies in finding the distribution of matrix T, since the distributions of $|Y'Y|$ or of $\operatorname{tr} Y'Y$ can be found as a function of it, see Dahel and Giri (1994). In the context of shape theory, the distribution of T is called size- and-shape distribution, also known in the literature as the rectangular coordinates distribution, see Goodall and Mardia (1993), and Rao (1973), p. 597. In the same setting of shape theory, when considering the SV ($Y = H_1DW_1$) or polar ($Y = H_1R$) decompositions (see Lemmas 2.3 and 2.4), the matrices (D, W_1) and R may both be thought of as an alternative coordinates system, in such a way that the corresponding distributions play the role of size-and-shape distributions, see Goodall (1991), and Le and Kendall (1993). Similarly, matrix D is considered as yet another coordinate system, and its corresponding distribution is called size-and-shape cone distribution, see Goodall and Mardia (1993), Díaz-García et al. (1998a, 1998b). Some of these results were extended to the case in which Y has a singular Gaussian and elliptically contoured distribution, see Díaz- García et al. (1998a, 1998b).

Now, let $Y \in \mathbb{R}^{N\times m}$ be a random matrix with $r(Y) = q \leq \min(N, m)$ and density function given by

$$\frac{1}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)\left(\prod_{j=1}^{k}\delta_j^{r/2}\right)} h\left(\operatorname{tr}\Sigma^{-}(Y-\mu)'\Theta^{-}(Y-\mu)\right) \tag{1}$$

$$\left.\begin{array}{rcl} H_1'(Y-\mu)M_2' & = & 0 \\ H_2'(Y-\mu)M_1' & = & 0 \\ H_2'(Y-\mu)M_2' & = & 0 \end{array}\right\} \text{ a. s.} \tag{2}$$

where A^{-} is a symmetric generalised inverse of A, λ_i and δ_j are the nonzero eigenvalues of Σ and Θ, respectively, and $H_1 \in \mathcal{V}_{k,N}$ (the Stiefel manifold, see section 2), $H_2 \in \mathcal{V}_{N-k,N}$, $M_1' \in \mathcal{V}_{r,m}$ and $M_2' \in \mathcal{V}_{m-r,m}$. This is called Singular Elliptically Contoured Distribution and is denoted as;

$$Y \sim \mathcal{E}_{N\times m}^{k,r}(\mu, \Sigma, \Theta, h)$$

where $\Sigma : m \times m$, $r(\Sigma) = r \leq m$ and/or $\Theta : N \times N$, $r(\Theta) = k \leq N$, see Díaz-García et al. (1998b).

Alternatively, this density may be expressed as

$$\frac{1}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)\left(\prod_{j=1}^{k}\delta_j^{r/2}\right)} h\left(\operatorname{tr}\Sigma^{-}(Y-\mu)'\Theta^{-}(Y-\mu)\right)\nu(dY), \tag{3}$$

where $\nu()$ is the Hausdorff measure, which coincides with that of Lebesgue when this is defined on the subspace $\mathcal{M}$ given by the hyperplane (2), see Díaz-García et al. (1998b), Cramér (1945), p. 297 and Billingsley (1979), p. 209.

The main objective of this paper is to find the densities of the matrices associated to the QR, SV and polar decompositions of matrix Y. The first step in that direction is finding the Jacobians associated to such transformations. When matrix Y has full rank (by rows or columns), some Jacobians have been studied in the literature, see James (1954), Herz (1955), Olkin and Rubin (1964), Henderson and Searle (1979), Srivastava and Khatri (1979), Muirhead

(1982), Eaton (1983), Farrell (1985), Goodall and Mardia (1992, 1993), Uhlig (1994), Cadet (1996), among others. Díaz-García et al. (1997) study Jacobians associated to the SVD in the case of a non-full rank matrix.

The fact that matrix Y is of full rank implies that Y has a density function with respect to the Lebesgue measure $\nu(dY)$, in $\mathbb{R}^{N\times m}$. Thus, given two spaces $\mathcal{K}$ and $\mathcal{N}$, when decomposing matrix Y as a matrix product, say $Y = KN$, with $K \in \mathcal{K}$ and $N \in \mathcal{N}$, the problem becomes formally that of factorising ν as $\nu = \upsilon_1 \times \upsilon_2$ ($\nu(dY) = \upsilon_1(dK) \times \upsilon_2(dN)$), where υ_1 is a measure on $\mathcal{K}$ and υ_2 is a measure on $\mathcal{N}$. For the decompositions of interest in this paper, the problem has been treated by various authors in the case where Y has a distribution with respect to Lebesgue measure, see Eaton (1983) and Farrell (1985). We are now interested in extending these results to the case where matrix Y has non-full rank, a case in which Y has a density function with respect to the Hausdorff measure.

The results on the Jacobians, have been applied by several authors to find the densities of various matrices associated to several decompositions of matrix Y, in the case when Y has a Gaussian distribution and full rank, see James (1954), Olkin and Rubin (1964), Srivastava and Khatri (1979), Muirhead (1982), Eaton (1983), Goodall and Mardia (1992, 1993), and when it has a Gaussian distribution and non-full rank, see Díaz-García et al. (1997). These results have been extended by different authors, assuming that Y has an elliptically contoured non-singular central distribution, see Fang and Zang (1990), and assuming it has an elliptically contoured singular non-central distribution, Díaz-García et al. (1998a, 1998b).

This paper studies Jacobians associated to the QR, SV and Polar decompositions, as well as some other decompositions closely related to these, namely: the spectral, Cholesky's, and symmetric positive square root of a matrix decompositions, and some of their modifications (see section 2). In section 3 the densities of matrices associated to the mentioned decompositions are found with respect to the Hausdorff measure, both for the non-central (see Theorem 3.1) and central cases (see Corollary 3.2). Finally, these results are applied to two subfamilies of elliptically contoured distributions, the matrix variate normal distribution and the matrix variate symmetric Pearson type VII distribution.

2. FACTORIZATION AND JACOBIANS

Let $\mathcal{L}_{m,N}(q)$ be the linear space of all $N \times m$ real matrices of rank $q \le \min(N, m)$. The set of matrices $H_1 \in \mathcal{L}_{m,N}$ such that $H_1' H_1 = I_m$ is a manifold denoted $\mathcal{V}_{m,N}$, called Stiefel manifold. In particular, $\mathcal{V}_{m,m}$ is the group of orthogonal matrices $\mathcal{O}(m)$. Denote $\mathcal{S}_m$, the homogeneous space of $m \times m$ positive definite symmetric matrices; $\mathcal{S}_m^+(q)$, the $(mq - q(q-1)/2)$-dimensional manifold of rank q positive semidefinite $m \times m$ symmetric matrices with q distinct positive eigenvalues; $\mathcal{T}_{m,N}$ the set of $N \times m$ upper quasi-triangular matrices; $\mathcal{D}(m) \subset \mathcal{T}_m^+$ the diagonal matrices. In particular $\mathcal{T}_{m,m}$ denote the group of $m \times m$ upper triangular matrices $\mathcal{T}_m$ and $\mathcal{T}_m^+$ is the group of $m \times m$ upper triangular matrices with positive diagonal elements.

Lemma 2.1. [QR decomposition].*Let $X \in \mathcal{L}_{m,N}(q)$, then there exist $H_1 \in \mathcal{V}_{q,N}$ and $T \in \mathcal{T}_{m,q}$ with $t_{ii} \ge 0$, $i = 1, 2, \ldots, \min(q, N-1)$ such that $X = H_1 T$, see Section 5.4 in Golub and Van Loan (1996) and Goodall and Mardia (1993).*

Lemma 2.2. [Modified Cholesky decomposition]. *Let* $S \in \mathcal{S}_m^+(q)$, *then* $S = T'T$, *where* $T \in \mathcal{T}_{m,q}$, *see Golub and Van Loan (1996), p. 148.*

Lemma 2.3. [Singular value decomposition, SVD]. *Let* $X \in \mathcal{L}_{m,N}(q)$, *then there exist* $H_1 \in \mathcal{V}_{q,N}$, $W_1 \in \mathcal{V}_{q,m}$ *and* $D \in \mathcal{D}(q)$, *such that* $X = H_1 D W_1'$, *it is called non-singular part of the SVD, see Rao (1973), p. 42 and Eaton (1983), p. 58.*

Remark 2.1. In Lemma 2.3, observe that when $X = X'$ then $W_1 = H_1$, thus obtaining the non- singular part of the spectral decomposition of X.

Corollary 2.1. [Spectral decomposition]. *Let* $S \in \mathcal{S}_m^+(q)$, *then* $S = W_1 L W_1'$, *where* $W_1 \in \mathcal{V}_{q,m}$ *and* $L \in \mathcal{D}(q)$, *it is called non-singular part of the spectral decomposition, see Díaz-García et al. (1997).*

Lemma 2.4. [Polar decomposition.] *Let* $X \in \mathcal{L}_{m,N}(q)$, $N \geq m$, *then there exist* $H_1 \in \mathcal{V}_{m,N}$, *and* $R \in \mathcal{S}_m^+(q)$, *such that* $X = H_1 R$, *see Herz (1955), Cadet (1996) and Golub and Loan (1996), p. 149.*

Observe that when $q = N \leq m$, in the QR and SVD decomposition, and $q = N = m$ in the Polar decomposition, two cases may be distinguished for their respective $H_1 \in \mathcal{O}(N)$:

1. H_1 includes reflection, $H_1 \in \mathcal{O}(N)$, $|H_1| = \pm 1$, denoting T, (D, W_1) and R by T^R $(D, W_1)^R$ and R^R, respectively. In addition, for matrices T and D, $T_{N,N} \geq 0$ and $D_{NN} > 0$.
2. H_1 excludes reflection, $H_1 \in \mathcal{SO}(N)$, $|H_1| = 1$, $T_{N,N}$ is not restricted, and in the case of SVD if $q = N = m$ $\operatorname{sign}(D_{NN}) = \operatorname{sign}(|X|)$. Matrices T, (D, W_1) and R are denoted as T^{NR}, $(D, W_1)^{NR}$ and R^{NR}, respectively, see Section 4 in Goodall (1991), Goodall and Mardia (1993) and Section 4 in Le and Kendall (1993).

Lemma 2.5.[Symmetric non-negative definite square root.] *If* $S \in \mathcal{S}_m^+(q)$ *then there exists* $R \in \mathcal{S}_m^+(q)$, *such that* $S = R^2$, *see Srivastava and Khatri (1979), p. 38, Muirhead (1983), p. 588, and Golub and Loan (1996), p. 148.*

Under the previous decompositions, we have he following Jacobians:

Theorem 2.1. *Under the assumptions of Lemma 2.1 we have*

$$(dX) = \prod_{i=1}^{q} t_{ii}^{N-i} (H_1' dH_1)(dT) \tag{4}$$

where $(H_1' dH_1)$ *is the Haar measure on* $\mathcal{V}_{q,N}$, *see James (1954) and Farrell (1985).*

Remark 2.2. When $N \geq m = q$, this result is given in Srivastava and Khatri (1979), p. 38, where in addition an explicit form for the measure $(H_1' dH_1)$ is given. On the same context, Muirhead (1982), pp. 63-66, gives the demonstration under the same guidelines as the one given in James (1954), Section 8, for the SVD case. Finally, Goodall and Mardia (1993) establish, without proof, that the result is true when $q = \min(N, m)$.

Proof. (THEOREM 2.1.) Observe that T may be written as $T = (T_1 \vdots T_2)$, where $T_1 \in \mathcal{T}_q$ and $T_2 \in \mathcal{L}_{m-q,q}$. Thus the demonstration reduces to the one given in Muirhead (1982), pp. 64-66, observing that $H_1' dH_1 T = [H_1' dH_1 T_1 \vdots H_1' dH_1 T_2]$, and that on computing the exterior product, column by column, $[H_1' dH_1 T_2]$ does not contribute anything to the exterior product, since the elements they consist of appear in previous columns. ■

THEOREM 2.2. *With the assumptions of Lemma 2.2 and Theorem 2.1,*

1. $(dS) = 2^q \prod_{i=1}^{q} t_{ii}^{m-i+1} (dT)$

2. $(dX) = 2^{-q} \prod_{i=1}^{q} \theta(S)_{ii}^{(N-m-1)/2} (dS)(H_1' dH_1)$

where $(dT) = \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} dt_{ij}$, $\theta(S)_{ii} = t_{ii}^2$ *and* $(dS) = \bigwedge_{i=1}^{q} \bigwedge_{j=i}^{m} ds_{ij}$.

Proof.

1. The proof is a copy of the one given in Muirhead (1982), p. 60], noting only that, on computing the exterior product (dS), only the $mq - q(q-1)/2$ mathematically independent elements are considered, ds_{ij}, $i = 1, \dots, q$; $j = i, \dots m$. Alternatively, the Jacobian may be computed via patterned matrices, see Henderson and Searle (1979) .

2. Observe that the t_{ii} element of matrix T may be expressed as a function of the s_{ij} elements of matrix S. Furthermore, $t_{ii}^2 = \theta(S)_{ii}$, where, in particular for $q = 2$, we have $t_{11}^2 = s_{11}$, $t_{22}^2 = (s_{11}s_{22} - s_{12}^2)/s_{11}$, see Graybill (1979), p. 232 and Khatri (1959). Now, simplifying (dT) in (i) and substituting it into (4), we obtain the desired result. ■

THEOREM 2.3. *With the hypothesis of Lemma 2.3*

$$(dX) = 2^{-q} |D|^{N+m-2q} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)(dD)(H_1' dH_1)(W_1' dW_1)$$

where $D = \operatorname{diag}(D_{11}, \dots, D_{qq})$ and $(dD) = \bigwedge_{i=1}^{q} dD_{ii}$.

For proof see Díaz- García et al. (1997).

Remark 2.3. When $N \geq m = q$, the Jacobian given in Theorem 2.3 has been studied by James (1954), Section 8.1; Le and Kendall (1993) Section 4 and by Uhlig (1994), Theorem 5.

THEOREM 2.4. *Under the assumptions of Corollary 2.1 and Theorem 2.3,* *1.* $(dS) = 2^{-q}|L|^{m-q}\prod_{i=1}^{q}(L_{ii}-L_{jj})(dL)(W_1'dW_1)$

2. $(dX) = 2^{-q}|L|^{(N-m-1)/2}(dS)(H_1'dH_1)$

where $L = \mathrm{diag}(L_{ii},\dots,L_{qq})$ *and* $(dL) = \bigwedge_{i=1}^{q} dL_{ii}$.

Remark 2.4. Observe that the Jacobian in Theorem 2.4 (i) is a particular case of Theorem 2.3, considering the symmetry of S. This Jacobian was demonstrated by Uhlig (1994). When $m = q$, the Jacobian has been studied by James (1954), Section 8.2, James (1964), eq. (93) (when S is Hermitian), Srivastava and Khatri (1979), p. 31 and by Muirhead (1982), pp. 104-105. Proof for Theorem 2.4 (ii) is given in Díaz-García et al. (1997).

THEOREM 2.5. *With the assumption of Lemma 2.5,*

$$(dS) = 2^q|D|^{m-q+1}\prod_{i<j}^{q}(D_{ii}+D_{jj})(dR) = |D|^{m-q}\prod_{i\le j}^{q}(D_{ii}+D_{jj})(dR) \tag{5}$$

where $R = P_1DP_1'$ *is the spectral decomposition of* R*,* $P_1 \in \mathcal{V}_{q,m}$ *and* $D = \mathrm{diag}(D_{11},\dots,D_{qq})$.

Proof. From Corollary 2.1, $R = P_1DP_1'$ with $P_1 \in \mathcal{V}_{q,m}$ and $D = \mathrm{diag}(D_{11},\dots,D_{qq})$. Applying Theorem 2.4

$$(dR) = 2^{-q}|D|^{m-q}\prod_{i<j}^{q}(D_{ii}-D_{jj})(dD)(P_1'dP_1). \tag{6}$$

Now let $S = R^2 = RR = P_1DP_1'P_1DP_1' = P_1D^2P_1'$, applying Corollary 2.1 once again, we have

$$(dS) = 2^{-q}|D^2|^{m-q}\prod_{i<j}^{q}(D_{ii}^2-D_{jj}^2)(dD^2)(P_1'dP_1).$$

Observing that $(dD^2) = \prod_{i=1}^{q} 2D_{ii}(dD) = 2^q|D|(dD)$, $(D_{ii}^2 - D_{jj}^2) = (D_{ii}+D_{jj})(D_{ii}-D_{jj})$, and from (6),

$$\begin{aligned}(dS) &= 2^q|D|^{m-q+1}\prod_{i<j}^{q}(D_{ii}+D_{jj})\left[2^{-q}|D|^{m-q}\prod_{i<j}^{q}(D_{ii}-D_{jj})(P_1'dP_1)(dD)\right]\\ &= 2^q|D|^{m-q+1}\prod_{i<j}^{q}(D_{ii}+D_{jj})(dR).\end{aligned}$$

The second expression for (dS) is found observing that

$$\prod_{i\le j}^{q}(D_{ii}+D_{jj}) = \prod_{i=1}^{q}2D_{ii}\prod_{i<j}^{q}(D_{ii}+D_{jj}).\blacksquare$$

Remark 2.5. The Jacobian for the case where $S \in \mathcal{S}_m$, i.e., $q = m$, was studied by Olkin and Rubin (19649, Hendreson and Searle (1979) and Cadet (1996).

Theorem 2.6 *For the assumptions of Lemma 2.4 and Theorem 2.5.*

1. $(dX) = \dfrac{|D|^{N-q}}{\operatorname{Vol}(\mathcal{V}_{m-q,N-q})}\displaystyle\prod_{i<j}^{q}(D_{ii}-D_{jj})(dR)(H_1'dH_1)$

2. $(dX) = \dfrac{2^{-q}}{\operatorname{Vol}(\mathcal{V}_{m-q,N-q})}|L|^{(N-m-1)/2}(dS)(H_1'dH_1)$

where $L = D^2$ *and* $\operatorname{Vol}(\mathcal{V}_{m-q,N-q}) = \displaystyle\int_{\mathcal{V}_{m-q,N-q}}(K_1'dK_1) = \frac{2^{(m-q)}\pi^{(m-q)(N-q)/2}}{\Gamma_{m-q}[\frac{1}{2}(N-q)]}$.

Proof.

1. From Díaz, Gutiérrez and Mardia (1997) we have that, nondegenerate density of $S = X'X = Y'\Theta^{-}Y$ (central case) is

$$\frac{\pi^{qk/2}|L|^{(K-m-1)/2}}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}S)(dS).$$

Let $S = R^2$, with $(dS) = 2^q|D|^{m-q+1}\prod_{i<j}^{q}(D_{ii}+D_{jj})(dR)$ and $L = D^2$ (see Theorem 2.5). Then

$$\frac{\pi^{qk/2}|L|^{(K-m-1)/2}}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}S)(dS) = \frac{2^q\pi^{qk/2}|D|^{(K-q)}\displaystyle\prod_{i<j}^{q}(D_{ii}+D_{jj})}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^{-}R^2)(dR)$$

denote this function as $f_R(R)$.

Now, the nondegenerate density of X ($\mu_x = 0$) is

$$\frac{1}{\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}}h(\operatorname{tr}\Sigma^{-}X'X)(dX).$$

Let $X = H_1R$ with Jacobian, $(dX) = \alpha(dR)(H_1dH_1)$, where α is independent of H_1. Then the nondegenerate joint density of R, H_1 is

$$\frac{\alpha}{\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}}h(\operatorname{tr}\Sigma^{-}R^2)(dR)(H_1dH_1).$$

Integrating with respect to $H_1 \in \mathcal{V}_{m,k}$ we have that

$$\frac{\alpha 2^m\pi^{km/2}}{\Gamma_m[\frac{1}{2}k]\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}}h(\operatorname{tr}\Sigma^{-}R^2)(dR).$$

denote this function as $g_R(R)$. Thus considering the quotient $f_R(R)/g_R(R) = 1$ and from the fact that $\mathcal{V}_{m,k}/\mathcal{V}_{m-q,k-q} = \mathcal{V}_{m,k}$, the result follows.

2. The result is obtained substituting (dR), from (5), in Theorem 2.6 (i). $\blacksquare$

Remark 2.6. The Jacobian in Theorem 2.6 (i) was studied by Cadet (1996) when $q = m$, computing Grams determinant on Riemannian manifold. In Cadet's notation, ds denotes the Riemannian measure on $\mathcal{V}_{q,m}$ (the Haar Measure on $\mathcal{V}_{q,m}$), which has the normalizing constant

$$\int_{\mathcal{V}_{q,m}} ds = \frac{2^{p(p+3)/4}\pi^{qm/2}}{\Gamma_q[\frac{1}{2}m]}.$$

Which differs from the normalizing constant proposed by James (1954), for $(H_1'dH_1)$, see also Srivastava and Khatri (1979), p. 75 and Muirhead (1982), p. 70. But it is known that the invariant measure on $\mathcal{V}_{q,m}$ is unique, in the sense that if there are two invariant measures on $\mathcal{V}_{q,m}$, one is a finite multiple of the other, see James (1954) and Farrell (1985), p. 43. In particular

$$ds = 2^{p(p-1)/4}(H_1'dH_1). \tag{7}$$

From expression (7) the Jacobian in Theorem 2.6 (ii) is found, when $q = m$, with respect to the measure $(H_1'dH_1)$, or any of the Jacobians here studied may be expressed as a function of the ds measure proposed by Cadet, considering the different normalizing constants, see Remark (4) in Cadet (1996). The result given in Theorem 2.6 (i), also under the assumption of $q = m$, was proposed (without proof) by Herz (1955).

3. DENSITY FUNCTIONS

Let $Y \sim \mathcal{E}^{k,r}_{N\times m}(\mu, \Sigma, \Theta, h)$, and define the generalised Wishart ($N \geq m$) or Pseudo-Wishart ($N < m$) matrix as $S = Y'\Theta^{-}Y$. Let $Q \in \mathcal{L}_{N,k}$, such that $\Theta = Q'Q$, and define $X = (Q^{-})'Y$. Then

$$X \sim \mathcal{E}^{k,r}_{k\times m}(\mu_x, \Sigma, I_k, h)$$

with $\mu_x = (Q^{-})'\mu$ in such a way that

$$S = Y'\Theta^{-}Y = ((Q^{-})'Y)'(Q^{-})'Y = X'X.$$

In this section, assuming that $X \sim \mathcal{E}^{k,r}_{k\times m}(\mu_x, \Sigma, I_k, h)$ and that h is expanding in power series, the densities of matrices T, R, (D, W_1) and D associated with the QR, SV, and Polar decompositions of matrix X are found.

THEOREM 3.1

1. *For $k \geq m$ or $k < m$, with $q = \min(k, r)$, the (reflection) density de T is given by*

$$\frac{2^q \pi^{qk/2} \displaystyle\prod_{i=1}^{q} t_{ii}^{k-i}}{\Gamma_q\left[\frac{1}{2}k\right]\left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2}\right)} \sum_{t=0}^{\infty}\sum_{\kappa} \frac{h^{(2t)}(\mathrm{tr}(\Sigma^{-}T'T + \Omega))}{t!} \frac{C_\kappa(\Omega\Sigma^{-}T'T)}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(T - T_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x}T_{\mu_x}$ *is the QR decomposition of* μ_x.

2. *Assuming that* $k \geq m$, *with* $q = \min(k, r)$, *the (reflection) density of* R *is,*

$$\frac{2^q \pi^{qk/2}|D|^{k-q}\prod_{i<j}^{q}(D_{ii} + D_{jj})}{\Gamma_q\left[\frac{1}{2}k\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)} \sum_{t=0}^{\infty}\sum_{\kappa} \frac{h^{(2t)}(\operatorname{tr}(\Sigma^- R^2 + \Omega))}{t!} \frac{C_\kappa(\Omega\Sigma^- R^2)}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(R - R_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x}R_{\mu_x}$ *is the polar decomposition of* μ_x.

3. *The joint (reflection) density of* D *and* W_1 *is*

$$\frac{2^{-q}\pi^{q(k-m)/2}\Gamma_q\left[\frac{1}{2}m\right]|D|^{k+m-2q}\prod_{i<j}^{q}(D_{ii}^2 - D_{jj}^2)}{\Gamma_q\left[\frac{1}{2}k\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}$$

$$\sum_{t=0}^{\infty}\sum_{\kappa}\frac{h^{(2t)}(\operatorname{tr}(\Sigma^- W_1 D^2 W_1' + \Omega))}{t!}\frac{C_\kappa(\Omega\Sigma^- W_1 D^2 W_1')}{\left(\frac{1}{2}k\right)_\kappa}$$

$$(DW_1' - D_{\mu_x}W_{1\mu_x}')M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x}D_{\mu_x}W_{1\mu_x}$ *is the SVD of* μ_x, $(dD) = \bigwedge_{i=1}^{q} dD_{ii}$ *and* $(dW_1) = \dfrac{(W_1'dW_1)}{\operatorname{Vol}(\mathcal{V}_{q,m})}$.

4. *The density of* D *is given by*

$$\frac{2^q\pi^{q(k+m)/2}\prod_{i=1}^{q}D_{ii}^{k+m-2q}\prod_{i<j}^{q}(D_{ii}^2 - D_{jj}^2)}{\Gamma_q\left[\frac{1}{2}k\right]\Gamma_q\left[\frac{1}{2}m\right]\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{\theta,\kappa}^{\infty}\sum_{\phi\in\theta,\kappa}\frac{h^{(2t+l)}(\operatorname{tr}\Omega)}{t!l!}\frac{\Delta_\phi^{\theta,\kappa}C_\phi(D^2)C_\phi^{\theta,\kappa}(\Sigma^-, \Omega\Sigma^-)}{\left(\frac{1}{2}k\right)_\kappa C_\phi(I_m)}$$

$$(D - D_{\mu_x})M_2' = 0 \qquad a.s.$$

where $\mu_x = H_{1\mu_x}D_{\mu_x}W_{1\mu_x}$ *is the SVD of* μ_x.

with $\Omega = \Sigma^-\mu'\Theta^-\mu$, $\left(\frac{1}{2}k\right)_\kappa$ *is the generalised hypergeometric coefficient and* $C_\kappa(.)$ *is the zonal polynomial, see James (1964), Farrell (1985) and Muirhead (1982). The multiple addition operators multiples,* $\Delta_\phi^{\theta,\kappa}$ *and* $C_\phi^{\theta,\kappa}$ *are given in Davis (1980), see also Chikuse (1980).*

Proof.

1. Considering the non-degenerated part of the density of X we have

$$\frac{1}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^-(X-\mu_x)'(X-\mu_x))(dX)$$

developing the argument

$$\frac{1}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}\Sigma^-(X'X+\mu_x'\mu_x)-2\operatorname{tr}\Sigma^-X'\mu_x)(dX).$$

Factoring, $X = H_1T$, from Theorem 2.2 (i) we have the joint density (non-degenerated part) of H_1 and T is given by

$$\frac{\left(\prod_{i=1}^{q}t_{ii}^{k-i}\right)}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}h(\operatorname{tr}(\Sigma^-T'T+\Omega)-2\operatorname{tr}\Sigma^-T'H_1'\mu_x)(H_1'dH_1)(dT)$$

where $\Omega = \Sigma^-\mu_x'\mu_x = \Sigma^-\mu'\Theta^-\mu$. Assuming that $h(\cdot)$ can be expanded in power series, see Fan (1990a), i.e.,

$$h(v) = \sum_{t=0}^{\infty}a_tv^t$$

and developing the binomial, we have

$$\frac{\left(\prod_{i=1}^{q}t_{ii}^{k-i}\right)}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}a_t\sum_{\eta=0}^{t}\binom{t}{\eta}(\operatorname{tr}(\Sigma^-T'T+\Omega))^{t-\eta}(\operatorname{tr}(-2\mu_x\Sigma^-T'H_1'))^{\eta}(H_1'dH_1)(dT).$$

Integrating on $H_1 \in \mathcal{V}_{q,k}$, noting that this integral equals zero when η is odd, see James (1964), eqs.(34)-(36), the marginal (non-degenerated) density of T may be expressed as

$$\frac{\left(\prod_{i=1}^{q}t_{ii}^{k-i}\right)}{\left(\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}a_t\sum_{\eta=0}^{[\frac{t}{2}]}\binom{t}{\eta}(\operatorname{tr}(\Sigma^-T'T+\Omega))^{t-\eta}\int_{H_1\in\mathcal{V}_{q,k}}(\operatorname{tr}(-2\mu_x\Sigma^-T'H_1')^{2\eta}(H_1'dH_1)(dT).$$

Integrating, see Muirhead (1982), Lemma 9.5.3, p. 397 and James (1964), eq. 22, we have

$$\begin{aligned}\int_{H_1\in\mathcal{V}_{q,k}}(\operatorname{tr}(-2\mu_x\Sigma^-T'H_1'))^{2\eta}(H_1'dH_1) &= \frac{2^q\pi^{qk/2}}{\Gamma_q[\frac{1}{2}k]}\sum_{\kappa}\frac{\left(\frac{1}{2}\right)_{\eta}C_\kappa(4\mu_x\Sigma^-T'T\Sigma^-\mu_x')}{\left(\frac{1}{2}\right)_{\kappa}}\\ &= \frac{2^q\pi^{qk/2}}{\Gamma_q[\frac{1}{2}k]}\sum_{\kappa}\frac{\left(\frac{1}{2}\right)_{\eta}4^{\eta}}{\left(\frac{1}{2}\right)_{\kappa}}C_\kappa(\Omega\Sigma^-T'T).\end{aligned}$$

Observing that $4^\eta\left(\frac{1}{2}\right)_\eta = \dfrac{(2\eta)!}{\eta!} = 2^\eta(2\eta-1)!!$, the non-degenerated part is obtained, see Teng, Fang and Deng (1989).

The degenerated part is still considered simply the QR decomposition of $\mu_x = H_{1_x}T_{\mu_x}$.

2. Proof is similar to the one given in (i), considering in this case the Jacobian given in Theorem 2.6 (i).
3. For proof of (iii) and (iv) see Díaz-García et al. (1998b). ∎

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Remark 3.1. Observe that, if function $h(\cdot)$ is not easily expandable in power series, an integral expression for the densities of T, R, (D, W_1) and D may be found in a form analogous to the one given by Fan (1990b), for the generalised Wishart matrix case.

¿From the Wishart matrix (or generalised Wishart matrix), the $S = T'T$ factorization is known in the literature as Bartlett decomposition (central or non-central). The density of T has been studied by different authors for the central non-singular case ($q = m \leq N$), as a function of both the density of S and the density of X ($S = X'X$) (see Srivastava and Khatri (1979), p. 74], Muirhead (1982), p. 99, Eaton (1983), p. 314, Fang and Zhang (1990), p. 119, among others. In the normal, non-central, non-singular case, Goodall and Mardia (1992, 1993) study the density of T when $q = \min(k, m)$, with $k \geq m$ and $k < m$, in the shape theory setting. Later Dahel and Giri (1994), also under normal theory, find the density of T for the case when $r(\mu_x) = 1$.

Also, Olkin and Rubin (1964) study the density of R under a non-singular central normal distribution, expressing the eigenvalues of R as a function of the elements of S, for the case when $q = m = 2$. Díaz-García et al. (1997), under normal theory, find the non-central density of D^2, when $q = \min(k, m)$. This result is extended to the case of a singular non-central elliptical model by Díaz-García et al. (1998b). Among other results, Díaz-García et al. (1998a) show that the density of $D/||D||$ in the central case, is invariant under all the elliptically contoured distributions.

Corollary 3.1 *Corollary 3.1. When $q = k \leq m$ and $r(\mu) < k$, the densities of T^{NR}, R^{NR} and $(D, W_1)^{NR}$ are the same as those given in Theorem 3.1 (i), (ii), and (iii), respectively, divided by 2.In particular, for the density of T^{NR}, $t_{ii} \geq 0$, for $i = 1, \dots, (k-1)$ and t_{kk} non-restricted, similarly for the density of (D, W_1), if $q = k = m$, $\operatorname{sign}(d_{kk}) = \operatorname{sign}(|X|)$. When $k > m$, t_{kk} is not present, see Srivastava and Khatri (1979), Goodall and Mardia (1993) and Le and Kendall (1993). For the case of the distribution of D, the densities, including and excluding reflection, are equal, see Goodall and Mardia (1993), Section 7.*

Proof. Expanding the exponential in Goodall and Mardia (1993), eq. 2.10 in power series, and integrating term by term, it is established for $r(Z) < k$ that

$$\int_{\mathcal{SO}(k)} (\operatorname{tr} ZH)^{2t}(H'dH) = \frac{1}{2}\int_{\mathcal{O}(k)} (\operatorname{tr} ZH)^{2t}(H'dH),$$

from which the result is obtained. ■

Next, the densities of T, R, (D, W_1) and D are presented for the central case, $\mu_x = 0$.

Corollary 3.2.

1. *The (refection) central density of T is*

$$\frac{2^q \pi^{kq/2} \displaystyle\prod_{i=1}^{q} t_{ii}^{k-i}}{\Gamma_q[\frac{1}{2}k] \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2} \right)} h(\operatorname{tr} \Sigma^- T'T)$$

$$TM_2' = 0 \quad a.s.$$

2. *The (refection) central density of R is*

$$\frac{2^q \pi^{kq/2} |D|^{k-q} \displaystyle\prod_{i<j}^{q} (D_{ii} + D_{jj})}{\Gamma_q[\frac{1}{2}k] \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2} \right)} h(\operatorname{tr} \Sigma^- R^2)$$

$$RM_2' = 0 \quad a.s.$$

3. *The (refection) central density of (D, W_1) is*

$$\frac{2^{-q} \pi^{q(k-m)/2} \Gamma_q[\frac{1}{2}m] |D|^{k+m-2q} \displaystyle\prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}k] \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2} \right)} h(\operatorname{tr} \Sigma^- W_1 D^2 W_1')$$

$$DW_1' M_2' = 0 \quad a.s.$$

4. *The central density of D is*

$$\frac{2^q \pi^{q(k+m)/2} \displaystyle\prod_{i=1}^{q} D_{ii}^{k+m-2q} \prod_{i<j}^{q} (D_{ii}^2 - D_{jj}^2)}{\Gamma_q[\frac{1}{2}k] \Gamma_q[\frac{1}{2}m] \left(\displaystyle\prod_{i=1}^{r} \lambda_i^{k/2} \right)} \sum_{t=0}^{\infty} \sum_{\kappa} \frac{h^t(0) C_\kappa(\Sigma^-) C_\kappa(D^2)}{t! C_\kappa(I_m)}$$

$$DM_2' = 0 \quad a.s.$$

Now two particular cases of elliptically contoured distributions are considered, the matrix variate normal distribution, and the class of matrix variate symmetric Pearson type VII distributions, Gupta and Varga (1993), pp. 75-76, for which the density of R is found. The densities of T, (D, W_1) and D are obtained in a similar form.

Corollary 3.3*Let $X \sim \mathcal{E}_{k \times m}^{k,r}(\mu_x, \Sigma, I_k, h)$, with h expanding in series of powers. Then,*

1. *if X has a matrix variate normal distribution, the (reflection) density of R is*

$$\frac{2^{(2q-kr)/2}\pi^{k(q-r)/2}|D|^{k-q}\displaystyle\prod_{i<j}^{q}(D_{ii}+D_{jj})}{\Gamma_q[\frac{1}{2}k]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\operatorname{etr}(-\tfrac{1}{2}(\Sigma^- R^2+\Omega))_0F_1(\tfrac{1}{2}k;\tfrac{1}{4}\Omega\Sigma^- R^2)$$

$$(R-R_{\mu_x})M_2'=0 \quad a.s.$$

2. *if X has a matrix variate symmetric Pearson type VII distribution, the (reflection) density of R is*

$$\frac{2^{q}\pi^{k(q-r)/2}\Gamma[b]|D|^{k-q}\displaystyle\prod_{i<j}^{q}(D_{ii}+D_{jj})}{\Gamma_q[\frac{1}{2}k]a^{kr/2}\Gamma[\frac{1}{2}(2b-kr)]\left(\displaystyle\prod_{i=1}^{r}\lambda_i^{k/2}\right)}\sum_{t=0}^{\infty}\sum_{\kappa}\frac{(b)_{2t}\left(1+\frac{\operatorname{tr}(\Sigma^- R^2+\Omega)}{a}\right)^{-(b+2t)}}{t!}\frac{C_\kappa\left(\frac{1}{a^2}\Omega\Sigma^- R^2\right)}{(\frac{1}{2}k)_\kappa}$$

$$(R-R_{\mu_x})M_2'=0 \quad a.s.$$

where ${}_0F_1()$ is a hypergeometric function of matrix argument, James (1964) and Muirhead (1982), p. 258.

Proof. The proof follows from Theorem 3.1 (*ii*), observing in addition that:

1. For the normal case
$$h(v)=\frac{1}{(2\pi)^{kr/2}}\exp(-\tfrac{1}{2}v),$$
therefore
$$h^{(2t)}(v)=\frac{1}{2^{2t+kr/2}\pi^{kr/2}}\exp(-\tfrac{1}{2}v).$$

2. For the Pearson type VII case
$$h(v)=\frac{\Gamma[b]}{(\pi\ a)^{kr/2}\Gamma[b-kr/2]}(1+v/a)^{-b},$$
then
$$h^{(2t)}(v)=\frac{\Gamma[b]}{(\pi\ a)^{kr/2}\Gamma[b-kr/2]}\frac{(b)_{2t}}{a^{2t}}(1+v/a)^{-(b+2t)}.$$

¿From which the results are obtained. ■

Finally, observe that from the densities of T, R and (D, W_1), the density of $S=Y'\Theta^- Y$ may be found, with the help of theorems 2.2 (*i*) (or 2.7 (*ii*)), 2.5 and 2.4 (*i*), respectively. Or alternatively, from the density of X, $S=X'X$, with the help of theorems 2.2 (*ii*) (or 2.7 (*iii*)), 2.6 (*ii*) and 2.4 (*ii*), respectively, for the following cases:

1. *QR decomposition.* In this case the density of $S = T'T$ may be found in all cases, i.e., $N \geq m$, $N < m$ and $q = \min(k, r)$, observing that under Theorem 2.2, $f_S(S)$ is the joint density of s_{ij}, $i = 1, 2, \ldots, q$; $j = i, i+1, \ldots, m$, whose volume is given by $(dS) = \bigwedge_{i=1}^{q}\bigwedge_{j=i}^{m}(ds_{ij})$ (see Theorem 2.2), while under Theorem 2.7, the volume (dS) is given by Theorem 2.4 (i).

2. *Polar decomposition.* Given the definition of the polar decomposition, the density of $S = R^2$ may be found when $N \geq m$ and $q = \min(k, r)$.

3. *Singular value decomposition.* Here $S = W_1 L W_1'$, $D^2 = L$, and its density exists for any relationship between N, m and k, r, i.e., for $N \geq m$, $N < m$ and $q = \min(k, r)$, and is studied in detail by Díaz-García et al. (1998b).

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