Large Deviations for Double Itô Equations

Víctor Pérez-Abreu
Centro de Investigación en Matemáticas, A.C.
CIMAT, Apdo. Postal 402,
Guanajuato, 36000 Gto. Mexico

Constantin Tudor
University of Bucharest and CIMAT
Department of Mathematics
70109 Bucharest, Romania

Abstract

A large deviations principle is proved for double integro-differential Itô equations. The proof is based on an approximation for multiple stochastic integral with random integrand by multiple Wiener-Itô integral with deterministic integrand.

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1. Introduction

The subject of large deviations for stochastic differential equations driven by semimartingales has been developed for the last two decades since the publication of the book by Freidlin and Wentzell [5] and followed by several papers like those by Azencot [1], Baldi and Chaleyat-Maurel [2], Cutland [4], Dozzi [6], Liptser and Pukhalskii [9], Nualart and Rovira [12], Priouret [14], Rovira and Sanz-Solé [16].

On the other hand, in a pioneering work, Meyer [11] introduced multiple stochastic integrals with respect to a particular class of martingales and for random integrands. These results were extended by Ruiz de Chavez [17] for a class of semimartingales.
Recently, stochastic differential equations driven by multiple stochastic integrals are considered in [18]. The set up is that general that it includes the classical Itô stochastic equations, as well as integro-differential and Volterra stochastic equations (Berger and Mizel [3], Protter [15]).
In the present paper we prove large deviations for double Itô equations, i.e., stochastic differential equations driven by double stochastic integrals. Our method of proof is based on a result by Xiong [20] (see also Kallianpur and Xiong [7]) who obtained a large deviations criterium for Banach space valued random variables which come from random transformations of the Wiener process. It is worthwhile to mention that since the transformation is not pointwise, the classical contraction principle for large deviations does not apply.
In particular, Xiong 's criterium allows to prove large deviations for stochastic equations by replacing the random integrands by deterministic ones, and using large deviations for the latter (see [19]). Therefore, it is natural to ask whether or not the same technique could be used to prove large deviations for multiple Itô equations, since large deviations for multiple Wiener-Itô integrals (multiple integrals with deterministic integrands) have been obtained by Ledoux [8] and Mayer-Wolf, Nualart and Pérez-Abreu [10].
In this paper we use such an alternative for double Itô equations, together with an exponential inequality for a type of stochastic integrals that are not martingales which is due to Nualart and Rovira (see [12]).
In order to make the presentation self-contained, Section 2 collects several results that are needed in the sequel, while Section 3 contains the main result and its proof, as well an extension to large deviations principle for double Volterra stochastic equations.

2. Preliminaries

Let (S, d) be a Polish space, $\{\mu_\varepsilon\}_{\varepsilon>0}$ a sequence of probability measures on the Borel σ-algebra of S and $\Lambda : S \to [0, +\infty]$ be a function.
Recall that the sequence $\{\mu_\varepsilon\}_{\varepsilon>0}$ is said to satisfy the large deviations principle with the rate function Λ if :

(a) Λ is lower semicontinuous.
(b) (Lower estimate). For every nonempty open set $G \subset S$

$$-\Lambda(G) \leq \underline{\lim}_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(G). \tag{2.1}$$

(c) (Upper estimate). For every closed set $F \subset S$

$$\overline{\lim}_{\varepsilon \to 0} \varepsilon \log \mu_\varepsilon(F) \leq -\Lambda(F), \tag{2.2}$$

where, for a set $A \subset S$, $\Lambda(A) = \inf_{x \in A} \Lambda(x)$.
The rate function Λ is said to be *good* if the level set $\Psi_\Lambda(r) = \{x \in S : \Lambda(x) \leq r\}$ is compact for each $r \geq 0$.
A useful criterion for proving large deviations is due to Xiong [20] (see also Kallianpur and Xiong [7]). We present part of it in a slightly different form which is more appropriate for our purposes.

Theorem 2.1 *Let B be a real separable Banach space, $\mathcal{A}_B$ be a linear space of B-valued random variables defined on a probability space $(\Omega, \mathcal{F}, P)$.*
Let $\Lambda : B \to [0, \infty]$ be a good rate function with the skeleton $K_\Lambda = \bigcup_{r>0} \Psi_\Lambda(r)$.
Let $H : B \to B$, $H_i : \mathcal{A}_B \to \mathcal{A}_B, i = 1, ..., n$, $F : B \times K_\Lambda \to B$ and $\{X^\varepsilon\}_{\varepsilon>0}$ be a family of B-valued random variables. Assume that:
(a) $B \subset \mathcal{A}_B$.
(b) $F(x, .) \mid_{\Psi_\Lambda(r)}$ is continuous for all $x \in B, r > 0$ and H, F are Lipschitz continuous, i.e., there are positive constants K_1, $\{K(r)\}_{r>0}$ such that

$$\|H(x) - H(y)\| \leq K_1 \|x - y\|, \ \ \forall x, y \in B,$$

$$\|F(x, \varphi) - F(y, \varphi)\| \leq K(r) \|x - y\|, \ \ \forall x, y \in B, \varphi \in \Psi_\Lambda(r).$$

(c) $X^\varepsilon \in \mathcal{A}_B$ and

$$X^\varepsilon = H(X^\varepsilon) + \sum_{i=1}^{n} \varepsilon^{\frac{i}{2}} H_i(X^\varepsilon), \ a.s. \ in \ B.$$

(d) H_i are stochastically exponentially continuous in the following sense: for any $M > 0$ there exist $\delta > 0$ such that for any $\{X_1^\varepsilon\}, \{X_2^\varepsilon\} \subset \mathcal{A}_B$ we have

$$\max_{1 \leq i \leq n} \overline{\lim}_{\varepsilon \to 0} \varepsilon \log P(\varepsilon^{\frac{i}{2}} \|H_i(X_1^\varepsilon) - H_i(X_2^\varepsilon)\| > \sqrt{\delta}, \|X_1^\varepsilon - X_2^\varepsilon\| < \delta) \leq -M.$$

(e) $\{X^\varepsilon\}_{\varepsilon>0}$ *is exponentially tight, i.e., for any* $M > 0$ *there exist a compact* $C_M \subset B$ *such that*

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log\ P(X^\varepsilon \notin C_M) \leq -M.$$

(f) For any $x \in B$, *the family* B*-valued random variables*

$$X^{\varepsilon,x} = H(x) + \sum_{i=1}^{n} \varepsilon^{\frac{i}{2}} H_i(X^{\varepsilon,x}),\ \varepsilon > 0,$$

satisfies the LDP in B *with the good rate function*

$$I^x(y) = \inf\left\{\Lambda(\varphi) : \varphi \in K_\Lambda, F(x,\varphi) = y\right\},\ y \in B. \tag{2.3}$$

Then the function $I : B \to [0,\infty]$ *defined by*

$$I(x) = \inf\left\{\Lambda(\varphi) : \varphi \in K_\Lambda, F(x,\varphi) = x\right\},\ x \in B, \tag{2.4}$$

is a rate function (not necessarily good!) and the upper large deviations bound holds for the sequence $\mu_\varepsilon = P \circ (X^\varepsilon)^{-1}$.

Proof. The upper bound follows as in Kallianpur and Xiong [7] (Theorem 10.1.5 and 10.1.6). The fact that I is lower semicontinuous is easily seen. ■

For the case of multiple Wiener-Itô integrals we have the following LDP criterium (Ledoux [8] and Mayer-Wolf, Nualart and Pérez-Abreu [10], see also [13]).

Theorem 2.2 *For* $T > 0$ *let* $C'([0,T])$ *be the Cameron-Martin space and*

$$C_n(T) = \left\{(t_1,...,t_n) \in [0,T]^n : 0 < t_1 < ... < t_n \leq T\right\}.$$

For each $t \in [0,T], n \geq 1$ *and* $h_i^t \in L^2(C_i(T)), i = 1,...,n,$ *let*

$$I_i(\tilde{h}_i^t) = i! \int_{C_i(T)} h_i^t(t_1,...,t_i) dW_{t_1}...dW_{t_i},\ t \in [0,T] \tag{2.5}$$

be the ith-multiple Wiener-Itô integral (the right side in (2.5) is the multiple Itô integral), where $\tilde{h}$ is the symmetrization of h.
Assume that the processes $\left\{ I_i(\tilde{h}_i^t)\right\}_{t\in[0,T]}$ *have continuous versions.*
Then the family of multiple Wiener-Itô integrals processes

$$\left\{V_{n,\varepsilon}^t\right\}_{t\in[0,T]} = \left\{\sum_{i=1}^{n} \varepsilon^{\frac{i}{2}} I_i(\tilde{h}_i^t)\right\}_{t\in[0,T]}, \varepsilon > 0,$$

satisfies the LDP in $C([0,T])$ *with the good rate function*

$$I_n(f) = \inf\left\{\frac{1}{2}\|\dot{\varphi}\|^2_{L^2([0,T])} : \varphi \in C'([0,T]),\right. \tag{2.6}$$

$$\left. f(t) = \sum_{i=1}^{n} \int_{[0,T]^i} \tilde{h}_i^t(t_1, ..., t_i)\dot{\varphi}_{t_1}...\dot{\varphi}_{t_i} dt_1...dt_i, t \in [0,T]\right\}.$$

A key feature in the proof of our main result is the following exponential inequality for stochastic Volterra integrals due to Nualart and Rovira (see [12]).

Theorem 2.3 *Let* $Z : [0,T]^2 \times \Omega \to R$ *be a* $\mathcal{B}_{[0,T]^2} \otimes \mathcal{F}$*-measurable process with the properties:*

(i) $Z(t,s) = 0$ *if* $s > t$.

(ii) $Z(t,s)$ *is* $\mathcal{F}_s$*-adapted.*

(iii) There exists a positive random variable ξ *and* $\alpha \in (0,2]$ *such that*

$$\int_0^{\min(r,t)} |Z(t,s) - Z(r,s)|^2 ds \le \xi |t-r|^\alpha, \ \forall r, t \in [0,T]. \tag{2.7}$$

Then, for every $0 < \beta \le \min(1, \alpha)$, *there exist positive constants* K_1 *(depending only on* β*),* K_2 *and* K_3 *such that*

$$P\left(\sup_{0\le t\le T}\left|\int_0^t Z(t,s)dW_s\right| > L, \|Z\|_\infty \le K_Z, \xi \le C_Z\right) \le$$

$$\exp\left\{-\frac{L^2}{(TK_Z^2 + T^\alpha C_Z)}K_3\right\}, \tag{2.8}$$

for any $L, C_Z \ge 0$ *and* $K_Z \ge 0$ *satisfying*

$$\frac{L}{(T^{\alpha-\beta}C_Z + T^{1-\beta}K_Z^2)} \ge \max\left(K_1, K_2(1+T)T^{\frac{\beta}{2}}\right).$$

3. Main Result

Consider the double Itô equation

$$X_t^\varepsilon = \xi + \int_{C_2(t)} h_1(t_1, t_2, X_{t_1}^\varepsilon)dt_1dt_2 +$$

$$\varepsilon^{\frac{1}{2}} \int_{C_2(t)} h_2(t_1, t_2, X_{t_1}^{\varepsilon}) dt_1 dW_{t_2} + \varepsilon^{\frac{1}{2}} \int_{C_2(t)} h_3(t_1, t_2, X_{t_1}^{\varepsilon}) dW_{t_1} dt_2$$
$$+\varepsilon \int_{C_2(t)} h_4(t_1, t_2, X_{t_1}^{\varepsilon}) dW_{t_1} dW_{t_2}, \tag{3.1}$$

where $\xi \in R^d$, $\{W_t\}_{t\in[0,T]}$ is a real Brownian Motion and $h_i(t_1, t_2, x) : C_2(T) \times R^d \to R^d, i = 1, .., 4$, satisfy the following conditions:
(a) $h_k(t_1, t_2, x)$ is measurable in (t_1, t_2, x) and bounded,
(b) $h_k(t_1, t_2, x)$ satisfies the Lipschitz condition in x uniformly in t_1, t_2
(c) There exist $K > 0,\ 0 < \alpha \leq 1$ such that

$$\|h_4(s, t_1, x) - h_4(s, t_2, x)\| \leq K \left|t_1 - t_2\right|^{\alpha},\ \forall t_1, t_2 \geq s, x \in R.$$

The stochastic integral in (3.1) is the double stochastic integral defined in [11] and [17].

Remark 3.1. By Theorem 2.2 of [18] (see also Berger and Mizel [3]) we have that (3.1) has a pathwise unique continuous solution $\{X_t^{\varepsilon}\}_{t\in[0,T]}$ such that

$$E\left[\sup_{0\leq t\leq T} \|X_t^{\varepsilon}\|^2\right] < \infty.$$

The following is the main result of this paper.
Theorem 3.2 *The family* $\{X^{\varepsilon}\}_{\varepsilon>0}$ *satisfies a LDP in* $C\left([0,T]\right)$ *with the good rate function*

$$I(f) = \inf\left\{\frac{1}{2}\left\|\dot{\varphi}\right\|^2_{L^2([0,T])} : \varphi \in C'\left([0,T]\right), f = F(f,\varphi)\right\}, \tag{3.2}$$

where

$$F(f,\varphi) : C\left([0,T], R^d\right) \times C'\left([0,T], R^d\right) \to C\left([0,T], R^d\right)$$

is defined by

$$F(f,\varphi)_t = \xi + \int_{C_2(t)} \tilde{h}_1(t_1, t_2, f_{t_1}) dt_1 dt_2 +$$

$$\int_{C_2(t)} \tilde{h}_2(t_1, t_2, f_{t_1}) \dot{\varphi}_{t_2} dt_1 dt_2 + \int_{C_2(t)} \tilde{h}_3(t_1, t_2, f_{t_1}) \dot{\varphi}_{t_1} dt_1 dt_2$$

$$+\int_{C_2(t)} \tilde{h}_4(t_1,t_2,f_{t_1})\dot{\varphi}_{t_1}\dot{\varphi}_{t_2}dt_1dt_2. \tag{3.3}$$

For the proof of Theorem 3.2 we need the following lemma. For a process $\{U_t\}$ and $\alpha > 0$ we define

$$[U]_\alpha = \sup_{0\le s<t\le T} \frac{|U_t - U_s|}{|t-s|^\alpha},$$

and if $\{V_{t_1,t_2}\}_{t_1<t_2\le T}$ is a predictable process such that

$$E\left[\int_{C_2(T)} V^2_{t_1,t_2}dt_1dt_2\right] < \infty,$$

we consider the double stochastic integral

$$I_2^t(V) = \int_{C_2(t)} V_{t_1,t_2}dW_{t_1}dW_{t_2},\ t\in[0,T].$$

The following result is similar to Lemma 10.2.2 in Kallianpur-Xiong [7], but we obtain a more precise estimate of the upper bound in (3.4).

Lemma 3.3. *Let $\{f(t)\}_{t\in[0,T]}$ be a nonanticipating real process such that*

$$\int_0^T |f(t)|^4\, dt \le K_1^2,\ K_1 > 0,$$

and let $0<\alpha<\frac{1}{4}$. Then there exists a positive constant $K(\alpha)$ such that for any $a>0$,

$$P(\frac{1}{K(\alpha)}[I_1(f)]_\alpha > a) \le \sqrt{2}T^2\exp\left\{\left(\frac{1}{6\tilde{K}_1}+\frac{\tilde{K}_1}{16}\right)-\left(\frac{a}{2\sqrt{3\tilde{K}_1}}-1\right)^2\right\}, \tag{3.4}$$

where $\tilde{K}_1 = K_1T^{\frac{1-4\alpha}{2}}$.

Moreover if $h(t_1,t_2)$ is a bounded predictable real process which satisfies (2.7) with ξ constant, then

$$\lim_{x\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\ P([I_2(h)]_\alpha > \frac{x}{\varepsilon}) = -\infty. \tag{3.5}$$

Proof. We proceed as in the proof of Lemma 10.2.2 in Kallianpur-Xiong [7]. Fix $0\le s<t\le T$ and define

$$g(r) = f(r)1_{[s,t]}(r)(t-s)^{-\alpha},\ 0\le r\le T,$$

$$\gamma(u) = \int_0^u g(r) dW_r, \quad 0 \le u \le T.$$

By Itô's formula we have

$$d\left(1+|\gamma(u)|^2\right)^{\frac{1}{2}} = \frac{1}{2}\left(1+|\gamma(u)|^2\right)^{-\frac{1}{2}} |g(u)|^2 \, du -$$

$$\frac{1}{2}\left(1+|\gamma(u)|^2\right)^{-\frac{3}{2}} |\gamma(u) g(u)|^2 \, du + \gamma(u) g(u) \left(1+|\gamma(u)|^2\right)^{-\frac{1}{2}} dW_u. \tag{3.6}$$

By the time change in stochastic integral we choose a Brownian Motion $\left\{\tilde{W}_u\right\}_{u\in[0,T]}$ such that

$$\int_0^u \gamma(r) g(r) \left(1+|\gamma(r)|^2\right)^{-\frac{1}{2}} dW_r = \tilde{W}_{\tau(u)},$$

where

$$\tau(u) = \int_0^u |\gamma(r) g(r)|^2 \left(1+|\gamma(r)|^2\right)^{-1} dr, \ u \in [0,T].$$

Next we have

$$\tau(u) = (t-s)^{-2\alpha} \int_s^t |\gamma(r) f(r)|^2 \left(1+|\gamma(r)|^2\right)^{-1} dr \le$$

$$(t-s)^{-2\alpha} \int_s^t |f(r)|^2 \, dr \le (t-s)^{\frac{1-4\alpha}{2}} \left[\int_0^T |f(r)|^4 \, dr\right]^{\frac{1}{2}} \le K_1 T^{\frac{1-4\alpha}{2}} = \tilde{K}_1.$$

Then, from (3.6) we obtain

$$\left(1+|\gamma(u)|^2\right)^{\frac{1}{2}} \le 1 + \sup_{u\le \tilde{K}_1} \left|\tilde{W}_u\right| + \frac{\tilde{K}_1}{2},$$

and thus

$$|\gamma(T)|^2 \le \left(1 + \sup_{u\le \tilde{K}_1} \left|\tilde{W}_u\right| + \frac{\tilde{K}_1}{2}\right)^2 - 1,$$

$$|\gamma(T)|^2 \le 2 + 3 \sup_{u\le \tilde{K}_1} \left|\tilde{W}_u\right|^2 + \frac{3\tilde{K}_1^2}{4}. \tag{3.7}$$

Since

$$E\left[\exp\left\{\frac{1}{4\tilde{K}_1}\sup_{u\leq\tilde{K}_1}\left|\tilde{W}_u\right|^2\right\}\right]=\sqrt{2},$$

we obtain that

$$E\left[\exp\left\{\frac{|\gamma(T)|^2}{12\tilde{K}_1}\right\}\right]\leq\sqrt{2}\exp\left\{\frac{1}{6\tilde{K}_1}+\frac{\tilde{K}_1}{16}\right\}. \tag{3.8}$$

From Kallianpur-Xiong [7, pp 313-314], it follows that

$$\left|I_1^t(f)-I_1^s(f)\right|\leq 2K(\alpha)\sqrt{3\tilde{K}_1}\left(\sqrt{|\log\eta|}+1\right)(t-s)^{\alpha}, \tag{3.9}$$

where $K(\alpha)$ is a positive constant depending only on α and

$$\eta=\int_0^T\int_0^T\exp\left\{\frac{1}{12\tilde{K}_1}\frac{|I_1^t(f)-I_1^s(f)|^2}{(t-s)^{2\alpha}}\right\}dsdt.$$

From (3.8) it follows

$$E(\eta)\leq\sqrt{2}T^2\exp\left\{\frac{1}{6\tilde{K}_1}+\frac{\tilde{K}_1}{16}\right\}. \tag{3.10}$$

Next (3.9) and (3.10) imply

$$P\left(\frac{1}{K(\alpha)}\left[I_1(f)\right]_{\alpha}>a\right)\leq P\left[\eta>\exp\left\{\left(\frac{a}{2\sqrt{3\tilde{K}_1}}-1\right)^2\right\}\right]\leq$$

$$\exp\left\{-\left(\frac{a}{2\sqrt{3\tilde{K}_1}}-1\right)^2\right\}E(\eta)\leq\sqrt{2}T^2\exp\left\{\frac{1}{6\tilde{K}_1}+\frac{\tilde{K}_1}{16}-\left(\frac{a}{2\sqrt{3\tilde{K}_1}}-1\right)^2\right\},$$

i.e., the estimate (3.4).
Of course

$$I_2^t(h)=\int_0^t I_1^{t_1}(h_{t_1,\cdot})dW_{t_1},$$

and therefore

$$P\left(\frac{1}{K(\alpha)}\left[I_2(h)\right]_\alpha > \frac{x}{\varepsilon}\right) \leq P\left(\int_0^T \left|I_1^{t_1}(h_{t_1,\cdot})\right|^4 dt_1 > K_1^2\right) +$$

$$P\left(\int_0^T \left|I_1^{t_1}(h_{t_1,\cdot})\right|^4 dt_1 \leq K_1^2, \frac{1}{K(\alpha)}\left[I_2(h)\right]_\alpha > \frac{x}{\varepsilon}\right)$$

$$\leq P\left(\sup_{0\leq t\leq T}\left|I_1^t(h_{t,\cdot})\right| > \left(K_1 T^{-1}\right)^{\frac{1}{2}}\right) +$$

$$P\left(\frac{1}{K(\alpha)}\left[I_1\left(I_1^{\cdot}\left(h1_{\left\{\int_0^{\cdot}\left|I_1^{t_1}(h)\right|^4 dt_1 \leq K_1^2\right\}}\right)\right)\right]_\alpha > \frac{x}{\varepsilon}\right),$$

and using (3.4) and (2.8) we get

$$P\left(\frac{1}{K(\alpha)}\left[I_2(h)\right]_\alpha > \frac{x}{\varepsilon}\right) \leq \exp\left\{-MK_1\right\} +$$

$$\sqrt{2}T^2 \exp\left\{\frac{1}{6\tilde{K}_1} + \frac{\tilde{K}_1}{16} - \left(\frac{x}{2\varepsilon\sqrt{3\tilde{K}_1}} - 1\right)^2\right\}.$$

Now, taking

$$K_1 = \frac{x}{\varepsilon T^{\frac{1-4\alpha}{2}}}, \text{ so } \tilde{K}_1 = \frac{x}{\varepsilon},$$

we obtain

$$P\left(\frac{1}{K(\alpha)}\left[I_2(h)\right]_\alpha > \frac{x}{\varepsilon}\right) \leq \exp\left\{-\frac{M_1 x}{\varepsilon}\right\} +$$

$$\sqrt{2}T^2 \exp\left\{\frac{\varepsilon}{6x} + \frac{x}{16\varepsilon} - \left(\sqrt{\frac{x}{12\varepsilon}} - 1\right)^2\right\},$$

and finally

$$\overline{\lim_{\varepsilon\to 0}}\, \varepsilon \log\, P(\frac{1}{K(\alpha)}\left[I_2(h)\right]_\alpha > \frac{x}{\varepsilon}) \leq$$

$$\max\left\{-M_1 x, -\frac{1}{48}x\right\} \to -\infty \text{ as } x \to \infty. \quad \blacksquare$$

Proof of Theorem 3.2. Let $B = C([0,T])$, $\mathcal{A}_B$ be the family of R^d-valued continuous adapted processes $\{f_t\}_{t\in[0,T]}$,

$$\Lambda : B \to [0,\infty],\ \Lambda(\varphi) = \begin{cases} \frac{1}{2}\|\dot{\varphi}\|^2_{L^2([0,T])}, & \varphi \in C'([0,T]) \\ \infty, & \varphi \notin C'([0,T]), \end{cases}$$

$$H : B \to B,\ H(f)_t = \xi + \int_{C_2(t)} h_1(t_1,t_2,f_{t_1})dt_1dt_2,$$

$$H_1(X)_t = H_1'(X)_t + H_1''(X)_t,$$

$$H_1'(X)_t = \int_{C_2(t)} h_2(t_1,t_2,X_{t_1})dt_1dW_{t_2} = \int_0^t \left(\int_0^{t_2} h_2(t_1,t_2,X_{t_1})dt_1\right)dW_{t_2},$$

$$H_1''(X)_t = \int_{C_2(t)} h_3(t_1,t_2,X_{t_1})dW_{t_1}dt_2 = \int_0^t \left(\int_{t_1}^{t} h_3(t_1,t_2,X_{t_1})dt_2\right)dW_{t_1},$$

$$H_2(X)_t = \int_{C_2(t)} h_4(t_1,t_2,X_{t_1})dW_{t_1}dW_{t_2},$$

and $F : B \times C'([0,T]) \to B$ defined by (3.3).
Now we verify that the assumptions of Theorem 2.1 are satisfied in the above setting.
The conditions (a), (b) are easily seen. In order to see (d), we fix $0 < \alpha < \frac{1}{4}$ and define for $t \in [0,T]$, $(t_1,t_2) \in C_2(T)$,

$$f(t) = \delta^{-1}\int_0^t \left[h_2(t_1,t,X_\varepsilon^1(t_1)) - h_2(t_1,t,X_\varepsilon^2(t_1))\right]1_{\{|X_\varepsilon^1(t_1)-X_\varepsilon^2(t_1)|<\delta\}}dt_1,$$

$$h(t_1,t_2) = \delta^{-1}\left[h_4(t_1,t_2,X_\varepsilon^1(t_1)) - h_4(t_1,t_2,X_\varepsilon^2(t_1))\right]1_{\{|X_\varepsilon^1(t_1)-X_\varepsilon^2(t_1)|<\delta\}}.$$

We have by (3.4)

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon \log P\left(\varepsilon^{\frac{1}{2}}\sup_{0\le t\le T}\left|H_1'^1(X_\varepsilon^1)_t - H_1'(X_\varepsilon^2)_t\right| > \sqrt{\delta},\ \sup_{0\le t\le T}\left|X_\varepsilon^1(t) - X_\varepsilon^2(t)\right| < \delta\right)$$

$$\le \lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log P\left([I_1(f)]_\alpha > \sqrt{\frac{1}{\delta\varepsilon}}\right) = -\infty. \tag{3.11}$$

Also, using (b), (c) and (2.8) we obtain

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log P\left(\varepsilon^{\frac{1}{2}}\sup_{0\le t\le T}\left|H_1''(X_\varepsilon^1)_t - H_1''(X_\varepsilon^2)_t\right| > \sqrt{\delta},\ \sup_{0\le t\le T}\left|X_\varepsilon^1(t)-X_\varepsilon^2(t)\right|<\delta\right)\le$$

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log P\left(\varepsilon^{\frac{1}{2}}\sup_{0\le t\le T}\left|\int_{C_2(t)}\left[h_3(t_1,t_2,X_\varepsilon^1(t_1)) - h_3(t_1,t_2,X_\varepsilon^2(t_1))\right]dW_{t_1}dt_2\right| > \sqrt{\delta},\right.$$

$$\left.\sup_{0\le t\le T}\left|X_\varepsilon^1(t)-X_\varepsilon^2(t)\right|<\delta\right)\le$$

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log P\left(\varepsilon^{\frac{1}{2}}\sup_{0\le t\le T}\left|\int_0^t\left[h_3(t_1,t,X_\varepsilon^1(t_1)) - h_3(t_1,t,X_\varepsilon^2(t_1))\right]dW_{t_1}\right| > T^{-1}\sqrt{\delta},\right.$$

$$\left.\sup_{0\le t\le T}\left|X_\varepsilon^1(t)-X_\varepsilon^2(t)\right|<\delta\right)=-\infty. \qquad (3.12)$$

Next, from (b) and (3.5) we have

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log P\left(\varepsilon\sup_{0\le t\le T}\left|H_2(X_\varepsilon^1)_t - H_2(X_\varepsilon^2)_t\right| > \sqrt{\delta},\ \sup_{0\le t\le T}\left|X_\varepsilon^1(t)-X_\varepsilon^2(t)\right|<\delta\right)\le$$

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log P\left(\sup_{0\le t\le T}\left|I_2^t(h)\right| > \frac{1}{\varepsilon\sqrt{\delta}}\right)\le$$

$$\lim_{\delta\to 0}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log P\left(\left[I_2(h)\right]_\alpha > \frac{1}{T^\alpha\sqrt{\delta}\varepsilon}\right)=-\infty. \qquad (3.13)$$

Now from (3.11)-(3.13) we verify (d) of Theorem 3.2.
In order to see that condition (e) is satisfied, define the compact set

$$C_M=\left\{x\in C\left([0,T]\right):\ [x]_\alpha\le M\right\},\ M>0,\ 0<\alpha<\frac{1}{4}.$$

Then we have

$$\lim_{M\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\ P(X^\varepsilon\notin C_M)\le$$

$$\lim_{M\to\infty}\max\left\{\overline{\lim_{\varepsilon\to 0}}\,\varepsilon\log\ P\left(\left[H_1'(X^\varepsilon)\right]_\alpha > \frac{M-(\|\xi\|+\|C\|_\infty T)}{3T^\alpha}\right),\right.$$

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P\left([H_1''(X^\varepsilon)]_\alpha > \frac{M-(\|\xi\|+\|C\|_\infty T)}{3T^\alpha}\right),$$

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P\left([I_2(H(X^\varepsilon))]_\alpha > \frac{M-(\|\xi\|+\|C\|_\infty T)}{3T^\alpha}\right)\right\} \leq$$

$$\lim_{M\to\infty} \max\left\{\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P\left(\left[I_1\left(\int_0^{\cdot} h_2(s,.,X_s)ds\right)\right]_\alpha > \frac{M-(\|\xi\|+\|C\|_\infty T)}{3T^\alpha}\right),\right.$$

$$\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P\left(\sup_{0\leq t\leq T}\left|\int_0^t h_3(t_1,t,X_{t_1})ddW_{t_1}\right| > \frac{M-(\|\xi\|+\|C\|_\infty T)}{3T^{1-2\alpha}}\right),$$

$$\left.\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P\left([I_2\left(h_4(t_1,t_2,X_{t_1})\right)]_\alpha > \frac{M-(\|\xi\|+\|C\|_\infty T)}{3T^\alpha}\right)\right\} = -\infty$$

by (2.8) and (3.5).
Now, defining the compact set

$$C_M' = \left\{x\in C\left([0,T]\right) :\ \sup_{0\leq t\leq T}|x(t)|\leq M,\ [x]_\alpha \leq M\right\},\ M>0,\ 0<\alpha<\frac{1}{4},$$

it follows that

$$\lim_{M\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P(X^\varepsilon \notin C_M') \leq$$

$$\lim_{M\to\infty} \max\left\{\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P(X^\varepsilon \notin C_M), \overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P(\sup_{0\leq t\leq T}|x(t)|>M)\right\} \leq$$

$$\lim_{M\to\infty}\overline{\lim_{\varepsilon\to 0}}\,\varepsilon \log\ P\left([X^\varepsilon]_\alpha\right) \geq \max\left(M, \frac{M-\|\xi\|}{T^\alpha}\right) = -\infty,$$

which is the condition (e).
Next, for $f\in C\left([0,T]\right)$,we have

$$H_1'(f)_t = I_1\left(\int_\cdot^t h_2(.,t_2,f_\cdot)dt_2\right)$$

and therefore, by Theorem 2.2, the family

$$X^{\varepsilon,f} = \xi + H(f) + \varepsilon^{\frac{1}{2}}H_1(f) + \varepsilon H_2(f),$$

satisfies the LDP in $C([0,T])$ with the good rate function

$$I^f(h) = \inf \left\{ \frac{1}{2} \|\dot{\varphi}\|^2_{L^2([0,T])} : \varphi \in C'([0,T]), h = F(f,\varphi) \right\},$$

verifying condition (f).
Therefore, by Theorem 2.1 the upper estimate of the large deviations principle holds.
For the lower estimate all we need to show is that for every $f \in C([0,T])$ with $I(f) < \infty$ and for all $\delta, \eta > 0$ we have

$$\varliminf_{\varepsilon \to 0} \varepsilon \log P(\sup_{0 \le t \le T} |X^\varepsilon_t - f_t| < \delta) \ge -I(f) - \eta.$$

Choose $\varphi \in C'([0,T])$ such that $f = F(f,\varphi)$ and $\frac{1}{2}\|\dot{\varphi}\|^2_{L^2([0,T])} \le I(f) + \eta$. Consider the process $Y^\varepsilon_t = X^\varepsilon_t - f_t$, $t \in [0,T]$. It is clear that Y^ε_t satisfies under P the stochastic equation

$$\begin{aligned}
Y^\varepsilon_t = &\int_{C_2(t)} \left[h_1(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) - h_1(t_1,t_2,f_{t_1}) \right] dt_1 dt_2 + \\
&\sqrt{\varepsilon} \int_{C_2(t)} h_2(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) dt_1 d\left(W_{t_2} - \frac{1}{\sqrt{\varepsilon}}\varphi_{t_2} \right) + \\
&\int_{C_2(t)} \left[h_2(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) - h_2(t_1,t_2,f_{t_1}) \right] \dot{\varphi}_{t_2} dt_1 dt_2 + \\
&\sqrt{\varepsilon} \int_{C_2(t)} h_3(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) d\left(W_{t_1} - \frac{1}{\sqrt{\varepsilon}}\varphi_{t_1} \right) dt_2 + \\
&\int_{C_2(t)} \left[h_3(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) - h_3(t_1,t_2,f_{t_1}) \right] \dot{\varphi}_{t_1} dt_1 dt_2 + \\
&\varepsilon \int_{C_2(t)} h_4(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) d\left(W_{t_1} - \frac{1}{\sqrt{\varepsilon}}\varphi_{t_2} \right) dW_{t_2} + \\
&\varepsilon \int_{C_2(t)} h_4(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) d\left(\frac{1}{\sqrt{\varepsilon}}\varphi_{t_1} \right) d\left(W_{t_2} - \frac{1}{\sqrt{\varepsilon}}\varphi_{t_2} \right) + \\
&\int_{C_2(t)} \left[h_4(t_1,t_2,Y^\varepsilon_{t_1} + f_{t_1}) - h_4(t_1,t_2,f_{t_1}) \right] \dot{\varphi}_{t_1}\dot{\varphi}_{t_2} dt_1 dt_2.
\end{aligned} \tag{3.14}$$

By Girsanov's theorem the process

$$\tilde{W}_t = W_t - \frac{1}{\sqrt{\varepsilon}}\varphi_t, t \in [0,T],$$

is a Brownian Motion with respect to the probability measure

$$\tilde{P} = \exp\left\{\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{\varphi}_t d_P W_t - \frac{1}{2\varepsilon}\int_0^T |\dot{\varphi}_t|^2\, dt\right\}.P,$$

(here $\int_0^T \dot{\varphi}_t d_P W_t$ is the stochastic integral with respect to P). Let

$$A^\varepsilon = \left\{\sup_{0\le t\le T} |Y_t^\varepsilon| < \delta\right\}.$$

Then, using the equality

$$\int_0^T \dot{\varphi}_t d_P W_t = \int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t + \frac{1}{\sqrt{\varepsilon}}\int_0^T |\dot{\varphi}_t|^2\, dt, \tag{3.15}$$

we obtain

$$P(A^\varepsilon) = \int_{A^\varepsilon} \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{\varphi}_t d_P W_t + \frac{1}{2\varepsilon}\int_0^T |\dot{\varphi}_t|^2\, dt\right\} d\tilde{P} =$$

$$\exp\left\{-\frac{1}{2\varepsilon}\int_0^T |\dot{\varphi}_t|^2\, dt\right\}\int_{A^\varepsilon} \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t\right\} d\tilde{P},$$

and from Hölder and Jensen's inequalities,

$$\varepsilon \log P(A^\varepsilon) = -\frac{1}{2}\int_0^T |\dot{\varphi}_t|^2\, dt + \varepsilon \log \int_{A^\varepsilon} \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t\right\} d\tilde{P} =$$

$$-\frac{1}{2}\int_0^T |\dot{\varphi}_t|^2\, dt + \varepsilon \log \tilde{P}(A^\varepsilon) + \varepsilon \log \int \exp\left\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t\right\} d\tilde{P}\ |_{A^\varepsilon} \ge$$

$$-I(f) - \eta + \varepsilon \log \tilde{P}(A^\varepsilon) - \sqrt{\varepsilon}\int \left(\int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t\right) d\tilde{P}\ |_{A^\varepsilon},$$

which shows that it is enough to prove that

$$\log \tilde{P}(A^\varepsilon) \text{ and } \int \left(\int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t\right) d\tilde{P}\ |_{A^\varepsilon},$$

are bounded for $\varepsilon \leq \varepsilon_0$.
Since

$$\left| \int \left(\int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t \right) d\tilde{P} \mid_{A^\varepsilon} \right| \leq \frac{1}{\tilde{P}(A^\varepsilon)} \left[\int \left| \int_0^T \dot{\varphi}_t d_{\tilde{P}} \tilde{W}_t \right|^2 d\tilde{P} \right]^{\frac{1}{2}} \tilde{P}(A^\varepsilon)^{\frac{1}{2}}$$

$$= \frac{\left[\int_0^T |\dot{\varphi}_t|^2 \, dt \right]^{\frac{1}{2}}}{\tilde{P}(A^\varepsilon)^{\frac{1}{2}}},$$

the assertion follows if we prove that $\tilde{P}(A^\varepsilon) \to 1$ as $\varepsilon \to 0$. In fact we shall prove that $\tilde{E}\left[\sup_{0 \leq t \leq T} |Y_t^\varepsilon|^2\right] \to 0$ as $\varepsilon \to 0$.
Indeed, from (3.14), (3.15) it follows that under $\tilde{P}$ the process Y_t^ε satisfies the double Itô equation

$$Y_t^\varepsilon = \int_{C_2(t)} \left[h_1(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) - h_1(t_1, t_2, f_{t_1}) \right] dt_1 dt_2 +$$

$$\sqrt{\varepsilon} \int_{C_2(t)} h_2(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) dt_1 d_{\tilde{P}} \tilde{W}_{t_2} +$$

$$\int_{C_2(t)} \left[h_2(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) - h_2(t_1, t_2, f_{t_1}) \right] \dot{\varphi}_{t_2} dt_1 dt_2 +$$

$$\sqrt{\varepsilon} \int_{C_2(t)} h_3(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) d_{\tilde{P}} \tilde{W}_{t_1} dt_2 +$$

$$\int_{C_2(t)} \left[h_3(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) - h_3(t_1, t_2, f_{t_1}) \right] \dot{\varphi}_{t_1} dt_1 dt_2 +$$

$$+\varepsilon \int_{C_2(t)} h_4(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) d_{\tilde{P}} \tilde{W}_{t_1} d_{\tilde{P}} \tilde{W}_{t_2} +$$

$$\sqrt{\varepsilon} \int_{C_2(t)} h_4(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) d_{\tilde{P}} \tilde{W}_{t_1} \dot{\varphi}_{t_2} dt_2 +$$

$$\sqrt{\varepsilon} \int_{C_2(t)} h_4(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) \dot{\varphi}_{t_1} dt_1 d_{\tilde{P}} \tilde{W}_{t_2} +$$

$$\int_{C_2(t)} \left[h_4(t_1, t_2, Y_{t_1}^\varepsilon + f_{t_1}) - h_4(t_1, t_2, f_{t_1}) \right] \dot{\varphi}_{t_1} \dot{\varphi}_{t_2} dt_1 dt_2,$$

and, by using a standard procedure, we have

$$\tilde{E}\left[\sup_{0\le t\le T}|Y_t^\varepsilon|^2\right]\le K_1\left\{C(\varepsilon)+\int_0^t\tilde{E}\left[\sup_{u\le s}|Y_u^\varepsilon|^2\right]ds\right\},$$

where $\lim_{\varepsilon\to 0}C(\varepsilon)=0$. From Gronwall's lemma we get

$$\tilde{E}\left[\sup_{0\le t\le T}|Y_t^\varepsilon|^2\right]\le KC(\varepsilon)e^{K_1T}\to 0 \text{ as } \varepsilon\to 0.$$

Finally, the exponential tightness and the lower estimate imply that the rate function is good. ■

We now prove that Theorem 3.2 extends easily to double Volterra stochastic equations of the form

$$X_t^\varepsilon=\xi+\int_{C_2(t)}h_1(t_1,t_2,t,X_{t_1}^\varepsilon)dt_1dt_2+$$

$$\varepsilon^{\frac{1}{2}}\int_{C_2(t)}h_2(t_1,t_2,t,X_{t_1}^\varepsilon)dt_1dW_{t_2}+\varepsilon^{\frac{1}{2}}\int_{C_2(t)}h_3(t_1,t_2,t,X_{t_1}^\varepsilon)dW_{t_1}dt_2$$

$$+\varepsilon\int_{C_2(t)}h_4(t_1,t_2,X_{t_1}^\varepsilon)h_5(t_2,t)dW_{t_1}dW_{t_2}, \qquad (3.16)$$

where $\xi\in R^d$, $\{W_t\}_{t\in[0,T]}$ is a real Brownian Motion and

$$h_i(t_1,t_2,t,x):C_3(T)\times R^d\to R^d, i=1,2,3,$$

$$h_4(t_1,t_2,x):C_2(T)\times R^d\to R^d,\ h_5(s,t):C_2(T)\to R,$$

satisfy the following conditions:
(a) h_k, $k=1,...,5$, are measurable and bounded.
(b) $h_k(t_1,t_2,t,x), k=1,...,4$, satisfies the Lipschitz condition in x uniformly in t_1,t_2.
(c) There exist $K>0,\ 0<\alpha\le 1$ such that

$$\|h_k(t_1,t_2,t,x)-h_k(t_1,t_2,r,x)\|\le K\left|t-r\right|^\alpha,\ \forall t,r\ge t_2\ge t_1,x\in R^d,\ k=2,3,$$

$$\|h_4(t_1,t,x)-h_4(t_1,r,x)\|\le K\left|t-r\right|^\alpha,\ \forall t,r\ge t_1,x\in R^d,$$

$$\|h_5(s,t) - h_5(s,r)\| \leq K\,|t-r|^{\alpha}\,,\ \forall t,r \geq s.$$

The following extension of Proposition 2.3 and Lemma 3.3 to stochastic integral processes of Volterra type are needed.

Proposition 3.4 *Let $h : C_2(T) \times \Omega \to R$ be a predictable process. Assume there exists a positive random variable ξ and $\alpha \in (0,2]$ such that*

$$\int_0^{\min(r,t)} |h(t,s) - h(r,s)|^2\,ds \leq \xi\,|t-r|^{\alpha}\,,\ \forall r,t \in [0,T]\,. \tag{3.17}$$

Define for $M_1, M_2 > 0$,

$$H(t) = \int_0^t h(s,t)dW_s,\ A = \{\|h\|_{\infty} \leq M_1, \xi \leq M_2\}\,, \tilde{K} = T^{1-2\beta}M_1^2 + T^{\alpha-2\beta}M_2.$$

Then, for every $\beta \leq \min\left(\frac{1}{2}, \frac{\alpha}{2}\right)$, there exist a positive constant $C = C(\alpha,\beta,T)$ and a positive random variable $\eta = \eta(\alpha,\beta,T)$ such that

$$E\left[1_A \exp\left(\eta^2\right)\right] < \infty, \tag{3.18}$$

$$|H(t) - H(r)| \leq C(\eta+1)\sqrt{3\tilde{K}}\,|t-r|^{\beta}\ \text{ on } A,\ \forall t,r \in [0,T]\,, \tag{3.19}$$

$$P\left([H]_{\beta} > x, A\right) \leq \sqrt{2}T^2 \exp\left\{-\left(\frac{x}{2\sqrt{3\tilde{K}}} - 1\right)^2 + \frac{1}{6\tilde{K}} + \frac{\tilde{K}}{6}\right\}. \tag{3.20}$$

In particular

$$P\left(\|H\|_{\infty} > x, A\right) \leq \sqrt{2}T^2 \exp\left\{-\left(\frac{x}{2T^{\beta}\sqrt{3\tilde{K}}} - 1\right)^2 + \frac{1}{6\tilde{K}} + \frac{\tilde{K}}{6}\right\}. \tag{3.21}$$

Proof. It follows as in Lemma 3.3 with

$$g(r) = 1_{\{s<r\leq t\}}h(r,t) + 1_{\{r\leq s\}}\left[h(r,t) - h(r,s)\right]. \quad \blacksquare$$

Corollary 3.5 *Let the assumptions of Proposition 3.5 be satisfied and suppose that h and ξ are bounded.*

Then

$$\lim_{x\to\infty}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log P\left([H]_\beta>\frac{x}{\varepsilon^{\frac{1}{2}}}\right)=-\infty. \tag{3.22}$$

In particular

$$\lim_{x\to\infty}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log P\left(\|H\|_\infty>\frac{x}{\varepsilon^{\frac{1}{2}}}\right)=-\infty. \tag{3.23}$$

Proposition 3.6 *Let $\{h_1(s,t)\}_{(s,t)\in C_2(T)}$ be a bounded, predictable process and $h_2(s,t):C_2(T)\to R$ be a bounded deterministic function. Assume that there exists $\alpha\in(0,1]$ such that h_i are α-Hölder in the second variable, that is*

$$|h_i(s,t)-h(s,r)|\le K\,|t-r|^\alpha\,,\ \forall r,t\ge s,$$

for some positive constant K.
Let

$$\tilde H(t)=\int_{C_2(t)}h_1(t_1,t_2)h_2(t_2,t)dW_{t_1}dW_{t_2},\ t\in[0,T]\,,$$

and let $\beta\le\min\left(\frac{1}{2},\alpha\right)$.
Then

$$\lim_{x\to\infty}\overline{\lim_{\varepsilon\to 0}}\ \varepsilon\log P\left(\left[\tilde H\right]_\beta>\frac{x}{\varepsilon}\right)=-\infty. \tag{3.24}$$

Proof. Define

$$H(t)=\int_0^t h_1(t_1,t)dW_{t_1},\ \tilde h(s,t)=H(s)h_2(s,t),$$

$$A=\left\{\left\|\tilde h\right\|_\infty\le M_1,K^2\|H\|_\infty^2\le M_2\right\}.$$

Then

$$\tilde H(t)=\int_0^t\tilde h(t_2,t)dW_{t_2},$$

and by (3.19) in Proposition 3.5 we have

$$|H(t)-H(r)|\le C_1(\eta_1+1)\,|t-r|^\beta\,,\ \forall t,r\ge 0. \tag{3.25}$$

Therefore

$$\left|\ \tilde h(t_2,t)-\ \tilde h(t_2,r)\right|\le K\,\|H\|_\infty\,|t-r|^\beta\,,\ \forall t,r\ge t_2,$$

$$\int_0^{\min(t,r)}\left|\ \tilde h(t_2,t)-\ \tilde h(t_2,r)\right|^2dt_2\le K^2T\,\|H\|_\infty^2\,|t-r|^{2\beta}\,,$$

and by (3.20) and (3.21) in Proposition 3.5 again we obtain

$$P\left(\left[\tilde{H}\right]_\beta > \frac{x}{\varepsilon}, A\right) \leq \sqrt{2}T^2 \exp\left\{-\left(\frac{x}{2\sqrt{3\tilde{K}\varepsilon}} - 1\right)^2 + \frac{1}{6\tilde{K}} + \frac{\tilde{K}}{6}\right\}. \tag{3.26}$$

Next

$$P\left(\left[\tilde{H}\right]_\beta > \frac{x}{\varepsilon}\right) \leq P\left(\left[\tilde{H}\right]_\beta > \frac{x}{\varepsilon}, A\right) +$$

$$P\left(\|H\|_\infty > \frac{M_1}{\|h_2\|_\infty}\right) + P\left(\|H\|_\infty > \frac{M_2^{\frac{1}{2}}}{KT^{\frac{1}{2}}}\right),$$

and

$$\overline{\lim_{\varepsilon\to 0}}\ \varepsilon \log P\left(\left[\tilde{H}\right]_\beta > \frac{x}{\varepsilon}\right) \leq$$

$$\max\left\{\overline{\lim_{\varepsilon\to 0}}\ \varepsilon \log P\left(\left[\tilde{H}\right]_\beta > \frac{x}{\varepsilon}, A\right), \overline{\lim_{\varepsilon\to 0}}\ \varepsilon \log P\left(\|H\|_\infty > \frac{M_1}{\|h_2\|_\infty}\right),\right.$$

$$\left.\overline{\lim_{\varepsilon\to 0}}\ \varepsilon \log P\left(\|H\|_\infty > \frac{M_2^{\frac{1}{2}}}{KT^{\frac{1}{2}}}\right)\right\}.$$

Finally, taking

$$M_1 = \sqrt{\frac{x}{2\varepsilon T^{1-2\beta}}},\ M_1 = \frac{x}{2T^{\alpha-2\beta}\varepsilon},\ \text{so } \tilde{K} = \frac{x}{\varepsilon},$$

and using (3.23) and (3.26) we get (3.24). ■

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