Andreas E. Kyprianou

Gerber–Shiu Risk Theory

Draft

May 15, 2013

Springer
Preface

These notes were developed whilst giving a graduate lecture course (Nachdiplomvorlesung) at the Forschungsinstitut für Mathematik, ETH Zürich. The title of these lecture notes may come as surprise to some readers as, to date, the term Gerber–Shiu Risk Theory is not widely used. One might be more tempted to simply use the title Ruin theory for Cramér-Lundberg models instead. However, my objective here is to focus on the recent interaction between a large body of research literature, spear-headed by Hans–Ulrich Gerber and Elias Shiu, concerning ever more sophisticated questions around the event of ruin for the classical Cramér-Lundberg surplus process, and the parallel evolution of the fluctuation theory of Lévy processes. The fusion of these two fields has provided economies to proofs of older results, as well as pushing much further and faster classical theory into what one might describe as exotic ruin theory. That is to say, the study of ruinous scenarios which involve perturbations to the surplus coming from dividend payments that have a historical path dependence. These notes keep to the Cramér–Lundberg setting. However, the text has been written in a form that appeals to straightforward and accessible proofs, which take advantage, as much as possible, of the fact that Cramér–Lundberg processes have stationary and independent increments and no upward jumps.

I would like to thank Paul Embrechts for the invitation to spend six months in at the FIM and the opportunity to present this material. I would also like to thank the attendees of the course in Zürich for their comments.

Zürich, Andreas E. Kyprianou
November, 2013
Contents

1 Introduction ................................................................. 1
  1.1 The Cramér–Lundberg process ........................................ 1
  1.2 The classical problem of ruin ........................................ 3
  1.3 Gerber–Shiu expected discounted penalty functions ............ 4
  1.4 Exotic Gerber–Shiu theory .......................................... 5
  1.5 Comments ............................................................. 7

2 The Esscher martingale and the maximum ......................... 9
  2.1 Laplace exponent ...................................................... 9
  2.2 First exponential martingale ....................................... 11
  2.3 Esscher transform .................................................... 12
  2.4 Distribution of the maximum ....................................... 14
  2.5 Comments ............................................................. 15

3 The Kella-Whitt martingale and the minimum ..................... 17
  3.1 The Cramér–Lundberg process reflected in its supremum ...... 17
  3.2 A useful Poisson integral .......................................... 18
  3.3 Second exponential martingale .................................... 21
  3.4 Duality ............................................................... 22
  3.5 Distribution of the minimum ...................................... 23
  3.6 The long term behaviour .......................................... 24
  3.7 Comments ............................................................. 25

4 Scale functions and ruin probabilities ............................. 27
  4.1 Scale functions and the probability of ruin ..................... 27
  4.2 Connection with the Pollaczek–Khintchine formula ........... 30
  4.3 Gambler’s ruin ....................................................... 33
  4.4 Comments ............................................................ 35
<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>The Gerber–Shiu measure</td>
<td>37-42</td>
</tr>
<tr>
<td>5.1</td>
<td>Decomposing paths at the minimum</td>
<td>37</td>
</tr>
<tr>
<td>5.2</td>
<td>Resolvent densities</td>
<td>38</td>
</tr>
<tr>
<td>5.3</td>
<td>More on Poisson integrals</td>
<td>40</td>
</tr>
<tr>
<td>5.4</td>
<td>Gerber–Shiu measure and gambler’s ruin</td>
<td>41</td>
</tr>
<tr>
<td>5.5</td>
<td>Comments</td>
<td>42</td>
</tr>
<tr>
<td>6</td>
<td>Reflection strategies</td>
<td>45-53</td>
</tr>
<tr>
<td>6.1</td>
<td>Perpetuities</td>
<td>46</td>
</tr>
<tr>
<td>6.2</td>
<td>Decomposing paths at the maximum</td>
<td>47</td>
</tr>
<tr>
<td>6.3</td>
<td>Derivative of the scale function</td>
<td>50</td>
</tr>
<tr>
<td>6.4</td>
<td>Net-present-value of dividends at ruin</td>
<td>52</td>
</tr>
<tr>
<td>6.5</td>
<td>Comments</td>
<td>53</td>
</tr>
<tr>
<td>7</td>
<td>Perturbation-at-maxima strategies</td>
<td>55-63</td>
</tr>
<tr>
<td>7.1</td>
<td>Re-hung excursions</td>
<td>55</td>
</tr>
<tr>
<td>7.2</td>
<td>Marked Poisson process revisited</td>
<td>57</td>
</tr>
<tr>
<td>7.3</td>
<td>Gambler’s ruin for the perturbed process</td>
<td>59</td>
</tr>
<tr>
<td>7.4</td>
<td>Continuous ruin with heavy tax</td>
<td>61</td>
</tr>
<tr>
<td>7.5</td>
<td>Net-present-value of tax</td>
<td>62</td>
</tr>
<tr>
<td>7.6</td>
<td>Comments</td>
<td>63</td>
</tr>
<tr>
<td>8</td>
<td>Refraction strategies</td>
<td>65-75</td>
</tr>
<tr>
<td>8.1</td>
<td>Pathwise existence and uniqueness</td>
<td>65</td>
</tr>
<tr>
<td>8.2</td>
<td>Gambler’s ruin and resolvent density</td>
<td>67</td>
</tr>
<tr>
<td>8.3</td>
<td>Resolvent density with ruin</td>
<td>73</td>
</tr>
<tr>
<td>8.4</td>
<td>Comments</td>
<td>75</td>
</tr>
<tr>
<td>9</td>
<td>Concluding discussion</td>
<td>77-88</td>
</tr>
<tr>
<td>9.1</td>
<td>Mixed-exponential jumps</td>
<td>77</td>
</tr>
<tr>
<td>9.2</td>
<td>Spectrally negative Lévy processes</td>
<td>79</td>
</tr>
<tr>
<td>9.3</td>
<td>Analytic properties of scale functions</td>
<td>82</td>
</tr>
<tr>
<td>9.4</td>
<td>Engineered scale functions</td>
<td>84</td>
</tr>
<tr>
<td>9.5</td>
<td>Comments</td>
<td>88</td>
</tr>
<tr>
<td>References</td>
<td></td>
<td>89</td>
</tr>
</tbody>
</table>
Chapter 1
Introduction

In this brief introductory chapter, we shall introduce the basic context of these lecture notes. In particular, we shall explain what we understand by so-called Gerber–Shiu theory and the role that it has played in classical ruin theory.

1.1 The Cramér–Lundberg process

The beginnings of ruin theory is based around a very basic model for the evolution of the wealth, or surplus, of an insurance company, known as the Cramér–Lundberg process. In the classical model, the insurance company is assumed to collect premiums at a constant rate $c > 0$, whereas claims arrive successively according to the times of a Poisson process, henceforth denoted by $N = \{N_t : t \geq 0\}$, with rate $\lambda > 0$. These claims, indexed in order of appearance $\{\xi_i : i = 1, 2, \cdots\}$, are independent and identically distributed (i.i.d.) with common distribution $F$, which is concentrated on $(0, \infty)$. The dynamics of the Cramér–Lundberg process are described by

$$X_t = ct - \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0. \quad (1.1)$$

Here, we use standard notation in that a sum of the form $\sum_{i=1}^{0} \cdot$ is understood to be equal to zero. We assume that $X$ is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, where $\mathcal{F} := \{\mathcal{F}_t : t \geq 0\}$ the natural filtration generated by $X$. When the initial surplus of our insurance company is valued at $x > 0$, we may consider the evolution of the surplus to follow the dynamics of $x + X$ under $\mathbb{P}$.

The Cramér–Lundberg process, $(X, \mathbb{P})$, is nothing more than a compound Poisson process with negative jumps and positive drift. Accordingly, it is easy to verify that it conforms to the definition of a so-called Lévy process, given below.

---

1 Henceforth written i.i.d. for short.
Definition 1.1. A process $X = \{X_t : t \geq 0\}$ with law $\mathbb{P}$ is said to be a Lévy process if it possesses the following properties:

(i) The paths of $X$ are $\mathbb{P}$-almost surely right-continuous with left limits.
(ii) $\mathbb{P}(X_0 = 0) = 1$.
(iii) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to $X_{t-s}$.
(iv) For all $n \in \mathbb{N}$ and $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2 \leq \cdots \leq s_n \leq t_n < \infty$, the increments $X_{t_i} - X_{s_i}$, $i = 1, \cdots, n$, are independent.

Whilst our computations in this text will largely remain within the confines of the Cramér–Lundberg model, we shall, as much as possible, appeal to mathematical reasoning which is handed down from the general theory of Lévy processes. Specifically, our analysis will predominantly appeal to martingale theory as well as excursion theory. The latter of these two concerns the decomposition of the path of $X$ into a sequence of sojourns from its running maximum or indeed from its running minimum.

Many of the arguments we give will apply, either directly, or with minor modification, into the setting of general spectrally negative Lévy processes. These are Lévy processes which do not experience positive jumps. In the forthcoming chapters, we have deliberately stepped back from treating the case of general spectrally negative Lévy processes in order to keep the presentation as mathematically light as possible, without disguising the full strength of the arguments that lie underneath. Nonetheless, at the very end, in Chapter 9, we will spend a little time making the connection with the general spectrally negative setting.

As a Lévy process, it is well understood that $X$ is a strong Markov process and, henceforth, we shall prefer to work with the probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$, where, thanks to spatial homogeneity, for $x \in \mathbb{R}$, $(X, \mathbb{P}_x)$ is equal in law $x + X$ under $\mathbb{P}$. For convenience, we shall always prefer to write $\mathbb{P}$ instead of $\mathbb{P}_0$.

Recall that $\tau$ is a stopping time with respect to $\mathcal{F}$ if and only if, for all $t \geq 0$, $\{\tau \leq t\} \in \mathcal{F}$. Moreover, for each stopping time, $\tau$, we associate the sigma algebra

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t \text{ for all } t \geq 0\}.$$  

(Note, it is a simple exercise to verify that $\mathcal{F}_\tau$ is a sigma algebra.) The standard way of expressing the strong Markov property for a one-dimensional process such as $X$ is as follows. For any Borel set $B$, on $\{\tau < \infty\}$,

$$\mathbb{P}(X_{\tau+s} \in B | \mathcal{F}_\tau) = \mathbb{P}(X_{\tau+s} \in B | \sigma(X_\tau)) = h(X_\tau, s),$$

where $h(x, s) = \mathbb{P}_x(X_s \in B)$. On account of the fact that $X$ has stationary and independent increments, we may also state the strong Markov property in a slightly refined form.

---

2 In the definition of a spectrally negative Lévy processes, we exclude the uninteresting cases of Lévy processes with no positive jumps and monotone paths.

3 We assume that the reader is familiar with the basic theory of Markov processes. In particular, the use of the (strong) Markov properties.
1.2 The classical problem of ruin

Theorem 1.2. Suppose that \( \tau \) is a stopping time. Define on \( \{ \tau < \infty \} \) the process \( \tilde{X} = \{ \tilde{X}_t : t \geq 0 \} \), where

\[
\tilde{X}_t = X_{\tau + t} - X_\tau, \; t \geq 0.
\]

Then, on the event \( \{ \tau < \infty \} \), the process \( \tilde{X} \) is independent of \( \mathcal{F}_\tau \) and has the same law as \( X \).

1.2 The classical problem of ruin

Financial ruin in the Cramér–Lundberg model (or just ruin for short) will occur if the surplus of the insurance company drops below zero. Since this will happen with probability one if \( \mathbb{P}(\liminf_{t \to \infty} X_t = -\infty) = 1 \), it is usual to impose an additional assumption that

\[
\lim_{t \to \infty} X_t = \infty.
\]

(1.2)

Write \( \mu = \int_{(0, \infty)} xF(dx) > 0 \) for the common mean of the i.i.d. claim sizes \( \{ \xi_i : i = 1, 2, \ldots \} \). A sufficient condition to guarantee (1.2) is that

\[
c - \lambda \mu > 0,
\]

(1.3)

the so-called security loading condition. To see why, note that the Strong Law of Large Numbers for Poisson processes, which states that \( \lim_{t \to \infty} N_t / t = \lambda \) a.s., and the obvious fact that \( \lim_{t \to \infty} N_t = \infty \) a.s. imply that

\[
\lim_{t \to \infty} \frac{X_t}{t} = \lim_{t \to \infty} \left( \frac{x}{t} + c - \frac{N_t}{t} \sum_{i=1}^{N_t} \xi_i \right) = \mathbb{E}(X_1) = c - \lambda \mu > 0 \quad \text{a.s.,}
\]

(1.4)

from which (1.2) follows. We shall see later that (1.3) is also a necessary condition for (1.2). Note that (1.3) also implies that \( \mu < \infty \).

Under the security loading condition, it follows that ruin will occur only with probability less than one. The most basic question that one can therefore ask under such circumstances is: what is the probability of ruin when the initial surplus is equal to \( x > 0 \)? This involves giving an expression for \( \mathbb{P}_x(\tau_0^- < \infty) \), where

\[
\tau_0^- := \inf\{ t > 0 : X_t < 0 \}.
\]

The Pollaczek–Khintchine formula does just this.

Theorem 1.3 (Pollaczek–Khintchine formula). Suppose that \( \lambda \mu / c < 1 \). For all \( x \geq 0 \),

\[
1 - \mathbb{P}(\tau_0^- < \infty) = (1 - \rho) \sum_{k=0}^{\infty} \rho^k \eta^k(x),
\]

(1.5)

where

\[
\rho = \lambda \mu / c \quad \text{and} \quad \eta(x) = \frac{1}{\mu} \int_{0}^{x} [1 - F(y)] dy.
\]
It is not our intention to dwell on this formula at this point in time, although we shall re-derive it in due course later in this text. This classical result and its connection to renewal theory are the inspiration behind a whole body of research literature addressing more elaborate questions concerning the ruin problem. Our aim in this text is to give an overview of the state of the art in this respect. Amongst the large number of names active in this field, one may note, in particular, the many and varied contributions of Hans Gerber and Elias Shiu. In recognition of their foundational work, we accordingly refer to the collective results that we present here as Gerber-Shiu risk theory.

1.3 Gerber–Shiu expected discounted penalty functions

Following Theorem 1.3, an obvious direction in which to turn one’s attention is looking at the joint distribution of $\tau_0^-, -X_{\tau_0^-}$ and $X_{\tau_0^-}$. That is to say, the joint law of the time of ruin, the deficit at ruin and the wealth prior to ruin. In their well-cited paper of 1998, Gerber and Shiu introduce the so-called expected discounted penalty function as follows. Suppose that $f: (0, \infty)^2 \to [0, \infty)$ is any bounded, measurable function. Then the associated expected discounted penalty function with force of interest $q \geq 0$ when the initial surplus is equal to $x \geq 0$ in value is given by

$$\phi_f(x, q) = \mathbb{E}_x \left[ e^{-q \tau_0^-} f(-X_{\tau_0^-}, X_{\tau_0^-}) I(\tau_0^- < \infty) \right].$$

Ultimately, one is really interested in, what we call here, the Gerber–Shiu measure. That is, the exponentially discounted joint law of the pair $(-X_{\tau_0^-}, X_{\tau_0^-})$. Indeed, writing

$$K^{(q)}(x, dy, dz) = \mathbb{E}_x \left[ e^{-q \tau_0^-} ; -X_{\tau_0^-} \in dy, X_{\tau_0^-} \in dz, \tau_0^- < \infty \right] \quad (1.6)$$

for the Gerber–Shiu measure one notes the simple relation

$$\phi_f(x, q) = \int_{(0, \infty)} \int_{(0, \infty)} f(y, z) K^{(q)}(x, dy, dz).$$

The expected discounted penalty function is now a well-studied object and there are many different ways to develop the expression on the right hand side of (1.6). As a consequence of the Poissonian path decomposition, which will drive many of the computations that we are ultimately interested in, we shall show how the Gerber–Shiu measure can be written in terms of so-called scale functions. Scale functions turn out to be a natural family of functions with which one may develop many of the identities around the event of ruin that we are interested in. We shall spend quite some time discussing the recent theory of scale functions later on in this text.

---

4 See the comments at the end of this chapter.
1.4 Exotic Gerber–Shiu theory

Again inspired by foundational work of Gerber and Shiu, and indeed many others, we shall also look at variants of the classical ruin problem in the setting that the Cramér–Lundberg process undergoes perturbations in its trajectory on account of pay-outs, typically due to dividend payments or taxation. Three specific cases that will interest us are the following.

**Reflection strategies:** An adaptation of the classical ruin problem, introduced by Bruno de Finetti in 1957, is to consider the continuous payment of dividends out of the surplus process to hypothetical share holders. Naturally, for a given stream of dividend payments, this will continuously change the net value of the surplus process and the problem of ruin will look quite different. Indeed, the event of ruin will be highly dependent on the choice of dividend payments. We are interested in finding an optimal way paying out of dividends such as to optimise the expected net present value of the total income of the shareholders, under force of interest, from time zero until ruin. The optimisation is made over an appropriate class of dividend strategies. Mathematically speaking, de Finetti’s dividend problem amounts to solving a control problem which we reproduce here.

Let \( \xi = \{ \xi_t : t \geq 0 \} \) with \( \xi_0 = 0 \) be a dividend strategy, consisting of a left-continuous non-negative non-decreasing process adapted to the filtration, \( \{ \mathcal{F}_t : t \geq 0 \} \), of \( X \). The quantity \( \xi_t \) thus represents the cumulative dividends paid out up to time \( t \geq 0 \) by the insurance company whose risk-process is modelled by \( X \). The aggregate, or controlled, value of the risk process when taking account of dividend strategy \( \xi \) is thus \( U_\xi = \{ U_\xi_t : t \geq 0 \} \) where \( U_\xi_t = X_t - \xi_t \), \( t \geq 0 \). An additional constraint on \( \xi \) is that \( L^{\xi}_t - L^\xi_0 \leq \max\{ U_\xi_t, 0 \} \) for \( t \geq 0 \) (i.e. lump sum dividend payments are always smaller than the available reserves).

Let \( \Xi \) be the family of dividend strategies as outlined in the previous paragraph and, for each \( \xi \in \Xi \), write \( \sigma^\xi = \inf\{ t > 0 : U^\xi_t < 0 \} \) for the time at which ruin occurs for the controlled risk process. The expected net present value, with discounting at rate \( q \geq 0 \), associated to the dividend policy \( \xi \) when the risk process has initial capital \( x \geq 0 \) is given by

\[
\nu^\xi(x) = \mathbb{E}_x \left( \int_0^{\sigma^\xi} e^{-qt} \, d\xi_t \right).
\]

De Finetti’s dividend problem consists of solving the stochastic control problem

\[
\nu^*(x) := \sup_{\xi \in \Xi} \nu^\xi(x) \quad x \geq 0.
\]

That is to say, if it exists, to establish a strategy, \( \xi^* \in \Xi \), such that \( \nu^* = \nu^{\xi^*} \).

We shall refrain from giving a complete account of their findings, other than to say that under appropriate conditions on the jump distribution, \( F \), of the Cramér–Lundberg process, \( X \), the optimal strategy consists of a so-called reflection strategy.
Specifically, there exists an \( a \in [0, \infty) \) such that
\[
\xi^*_t = (a \lor X_t) - a \quad t \geq 0.
\]
In that case, the \( \xi^* \)-controlled risk process, say \( U^*_t = \{ U^*_t : t \geq 0 \} \), is identical to the process \( \{ a - Y_t : t \geq 0 \} \) under \( \mathbb{P}_x \) where
\[
Y_t = (a \lor X_t) - X_t, \quad t \geq 0,
\]
and \( \overline{X}_t = \sup_{s \leq t} X_s \) is the running supremum of the Lévy insurance risk process.

Refraction strategies: An adaptation of the optimal control problem deals with the case that optimality is sought in a subclass, say \( \Xi_\alpha \), of the admissible strategies \( \Xi \). Specifically, \( \Xi_\alpha \) denotes the set of dividend strategies \( \xi \in \Xi \) such that
\[
\xi_t = \int_0^t \ell_s \, ds, \quad t \geq 0,
\]
where \( \ell = \{ \ell_t : t \geq 0 \} \) is uniformly bounded by some constant, say \( \alpha > 0 \). That is to say, dividend strategies which are absolutely continuous with uniformly bounded density.

Again, we refrain from going into the details of their findings, other than to say that under appropriate conditions the optimal strategy, \( \xi^*_\alpha = \{ \xi^*_\alpha_t : t \geq 0 \} \) in \( \Xi_\alpha \) turns out to satisfy
\[
\xi^*_\alpha_t = \alpha \int_0^t 1_{(Z_s > b)} \, ds, \quad t \geq 0,
\]
for some \( b > 0 \), where \( Z = \{ Z_t : t \geq 0 \} \) is the controlled Lévy risk process \( X - \xi^* \). The pair \( (Z, \xi^* \alpha) \) cannot be expressed autonomously and we are forced to work within the confines of the stochastic differential equation
\[
Z_t = X_t - \alpha \int_0^t 1_{(Z_s > b)} \, ds, \quad t \geq 0. \tag{1.8}
\]
For reasons that we shall elaborate on later, the process in (8.1) is called a refracted Lévy process.

Perturbation-at-maxima strategies: Another way of perturbing the path of our Cramér–Lundberg is by forcing payments from the surplus at times that it attains new maxima. This may be be interpreted, for example, as tax payments. To this end, consider the process
\[
U_t = X_t - \int_{[0,t]} \gamma(\overline{X}_u) \, d\overline{X}_u, \quad t \geq 0, \tag{1.9}
\]
where \( \gamma : [0, \infty) \to [0, \infty) \) satisfying appropriate conditions. The presentation we shall give here follows the last two references.

We distinguish two regimes, light- and heavy-tax regimes. The first corresponds to the case that \( \gamma : [0, \infty) \to [0, 1) \) and the second to the case that \( \gamma : [0, \infty) \to (1, \infty) \).
The light tax regime has a similar flavour to paying dividends at a weaker rate than a reflection strategy. In contrast, the heavy-tax regime is equivalent to paying dividends at a much stronger rate than a reflection strategy. A little thought reveals that the dividing case that \( \gamma(x) = 1_{\{x \leq a\}} \) corresponds precisely to a reflection strategy.

For each of the three scenarios described above, questions concerning the way in which ruin occurs remain just as pertinent as for the case of the Cramér–Lundberg process. In addition, we are also interested in the distribution of payments made out of the surplus process until ruin. For example in the case of reflection strategies, with a force of interest equal to \( q \geq 0 \), this boils down to understanding the distribution of

\[
\int_0^{\sigma_a} e^{-qt} 1_{\{X_t \geq x\}} dX_t,
\]

where

\[
\sigma_a = \inf\{t > 0 : (x \vee X_t) - X_t > a\}.
\]

1.5 Comments

For more on the standard model for an insurance risk process as described in Sect. 1.1, see Lundberg (1903), Cramér (1994a) and Cramér (1994b). See also, for example, the books of Embrechts et al. (1997) and Dickson (2010) to name but a few standard texts on the classical theory.

Within the setting of the classical Cramér–Lundberg model, Gerber and Shiu (1998) introduced the expected discounted penalty function. See also Gerber and Shiu (1997). It has been widely studied since with too many references to list here. The special issue, in volume 46, of the journal Insurance: Mathematics and Economics contains a selection of papers focused on the Gerber–Shiu expected discounted penalty function, with many further references therein.

An adaptation of the classical ruin problem was introduced within the setting of a discrete time insurance risk process by de Finetti (1957) in which dividends are paid out to share holders up to the moment of ruin, resulting in a control problem (1.7). This control problem was considered in framework of Cramér–Lundberg processes by Gerber (1969) and then, after a large gap, by Azcue and Muler (2005). Thereafter a string of articles, each one successively improving on the previous one in rapid succession; see Avram et al. (2007), Loeffen (2008), Kyprianou et al. (2010) and Loeffen and Renaud (2010). The variant of (1.7) resulting in refraction strategies was studied by Jeanblanc and Shiryaev (1995) and Asmussen and Taksar (1997) in the diffusive setting and Gerber (2006b) and Kyprianou et al. (2012) in the Cramér–Lundberg setting.

In the setting of the classical Cramér–Lundberg risk insurance model, Albrecher and Hipp (2007) introduced the idea of tax payments as in (1.9) for the case that \( \gamma \)
is a constant in $\(0, 1\)$. This model was quickly generalised in more general settings by \cite{Albrecher2008}, \cite{KyprianouZhou2009} and \cite{KyprianouOtt2012}.
Chapter 2
The Esscher martingale and the maximum

In this chapter, we shall introduce the first of our two key martingales and consider two immediate applications. In the first application we will use the martingale to construct a change of measure with respect to $\mathbb{P}$ and thereby consider the dynamics of $X$ under the new law. In the second application, we shall use the martingale to study the law of the process $\bar{X} = \{\bar{X}_t : t \geq 0\}$, where

$$\bar{X}_t := \sup_{s \leq t} X_s, \quad t \geq 0. \tag{2.1}$$

In particular, we shall discover that the position of the trajectory of $\bar{X}$ when sampled at an independent and exponentially distributed time is again exponentially distributed.

2.1 Laplace exponent

A key quantity in the forthcoming analysis is the Laplace exponent of the Cramér-Lundberg process, whose definition falls out of the following lemma.

**Lemma 2.1.** For all $\theta \geq 0$ and $t \geq 0$,

$$\mathbb{E}(e^{\theta X_t}) = \exp\{-\psi(\theta)t\},$$

where

$$\psi(\theta) := c\theta - \lambda \int_{(0,\infty)} \left(1 - e^{-\theta x}\right) F(dx). \tag{2.2}$$

**Proof.** Given the definition (1.1) one easily sees that it suffices to prove that

$$\mathbb{E}\left(e^{-\theta \sum_{i=1}^{N_t} \xi_i}\right) = \exp\left\{-\lambda t \int_{(0,\infty)} (1 - e^{-\theta x}) F(dx)\right\}, \tag{2.3}$$
for \( \theta, t \geq 0 \). However the equality (2.3) is the result of conditioning the expectation on its left-hand-side on \( N_t \), which is independent of \( \{ \xi_i : i \geq 1 \} \) and Poisson distributed with rate \( \lambda t \), to get

\[
\mathbb{E} \left( e^{-\theta \sum_{i=1}^{N_t} \xi_i} \right) = \sum_{n=0}^{\infty} \mathbb{E} \left( e^{-\theta \sum_{i=1}^{n} \xi_i} \right) e^{-\lambda t} \frac{(\lambda t)^n}{n!}
\]

\[
= \sum_{n=0}^{\infty} \left[ \mathbb{E} \left( e^{-\theta \xi_1} \right) \right]^n e^{-\lambda t} \frac{(\lambda t)^n}{n!}
\]

\[
= \exp \left\{ -\lambda t \left( 1 - \mathbb{E} \left( e^{-\theta \xi_1} \right) \right) \right\}
\]

\[
= \exp \left\{ -\lambda t \int_{(0,\infty)} (1 - e^{-\theta x}) F(dx) \right\},
\]

for all \( \theta, t \geq 0 \). Note, in particular, the integral in the final equality is trivially finite on account of the fact that \( F \) is a probability distribution. □

The Laplace exponent (2.2) is an important way of identifying the characteristics of Cramèr-Lundberg processes. The following theorem, which we refrain from proving, makes this clear.

**Theorem 2.2.** Any Lévy process with no positive jumps whose Laplace exponent agrees with (2.2) on \([0, \infty)\) must be equal in law to the Cramèr-Lundberg process with premium rate \( c \), arrival rate of claims \( \lambda \) and claim distribution \( F \).

We are interested in the shape of this Laplace exponent. Straightforward differentiation, with the help of the Dominated Convergence Theorem, tells us that, for all \( \theta > 0 \),

\[
\psi''(\theta) = \lambda \int_{(0,\infty)} x^2 e^{-\theta x} F(dx) > 0,
\]

which in turn implies that \( \psi \) is strictly convex on \((0, \infty)\). Integration by parts allows us to write

\[
\psi(\theta) = c\theta - \lambda \theta \int_{(0,\infty)} e^{-\theta x} F(x) dx, \quad \theta \geq 0,
\]

(2.4)

where we recall that \( F(x) = 1 - F(x) \), \( x \geq 0 \). This representation allows us to deduce, moreover, that

\[
\lim_{\theta \to \infty} \frac{\psi(\theta)}{\theta} = c
\]

and

\[
\psi'(0+) = c - \int_{(0,\infty)} x F(dx),
\]

where the left-hand-side is the right derivative of \( \psi \) at the origin and we understand the right-hand-side to be equal to \(-\infty\) in the case that \( \int_{(0,\infty)} x F(dx) = \infty \). In particular we see that

\[
\mathbb{E}(X_1) = \lim_{\theta \to 0} \mathbb{E}(X_1 e^{\theta X_1}) = \psi'(0+) \in [-\infty, \infty),
\]
where, again, we have used the the Dominated Convergence Theorem to justify
the first equality above. The net profit condition discussed in Sect. 1.1 can thus
otherwise be expressed simply as $\psi'(0+) > 0$.

A quantity which will also repeatedly appear in our computations is the right
inverse of $\psi$. That is,

$$
\Phi(q) := \sup\{ \theta \geq 0 : \psi(\theta) = q \},
$$

(2.5)

for $q \geq 0$. Thanks to the strict convexity of $\psi$, we can say that there is at most
one solution to the equation $\psi(\theta) = q$, when $q > 0$, and at most two when $q = 0$.
The number of solutions in the latter of these two cases depends on the value of
$\psi'(0+)$. Indeed, when $\psi'(0+) \geq 0$, then $\theta = 0$ is the only solution to $\psi(\theta) = 0$.
When $\psi'(0+) < 0$, there are two solutions, one at $\theta = 0$ and a second solution, in
$(0, \infty)$, which, by definition, gives the value of $\Phi(0)$. See Fig. 2.1.

2.2 First exponential martingale

Define, for each $\beta > 0$, process $\mathcal{E}(\beta) = \{ \mathcal{E}_t(\beta) : t \geq 0 \}$, where

$$
\mathcal{E}_t(\beta) := e^{\beta X_t - \psi(\beta)t}, \quad t \geq 0.
$$

(2.6)

**Theorem 2.3.** For each $\beta > 0$, the process $\mathcal{E}(\beta)$ is a $\mathbb{F}$-martingale with respect to $\mathbb{F}$.
Proof. Note that, for each \( \beta \geq 0 \), the process \( \mathcal{E}(\beta) \) is \( \mathbb{F} \)-adapted. With this in hand, it suffices to check that, for all \( \beta, s, t \geq 0 \), \( \mathbb{E}[\mathcal{E}_{t+s}(\beta) | \mathcal{F}_t] = \mathcal{E}_t(\beta) \). Indeed, on account of positivity, this would immediately show that \( \mathbb{E}[|\mathcal{E}(\beta)|] < \infty \), for all \( t \geq 0 \).

However, thanks to stationary and independent increments, \( \mathcal{F} \)-adaptedness as well as Lemma 2.1, for all \( \beta, s, t \geq 0 \),

\[
\mathbb{E}[\mathcal{E}_{t+s}(\beta) | \mathcal{F}_t] = \mathcal{E}_t(\beta) \mathbb{E}[e^{\beta(X_t-X_s)-\psi(\beta)(t-s)} | \mathcal{F}_t] = \mathcal{E}_t(\beta) \mathbb{E}[e^{\beta X_s}] e^{-\psi(\beta)(t-s)} = \mathcal{E}_t(\beta)
\]

and the proof is complete. \( \square \)

We call this martingale the \textit{Esscher martingale} on account of its connection to the exponential change of measure, discussed in the next section, which originates from the work of Esscher. See Sect. 2.5 for further historical details.

### 2.3 Esscher transform

Fix \( \beta > 0 \) and \( x \in \mathbb{R} \). Since \( \mathcal{E}(\beta) \) is a mean-one martingale, it may be used to perform a change of measure on \( (X, \mathbb{P}_x) \) via

\[
\frac{d\mathbb{P}_x^\beta}{d\mathbb{P}_x} \bigg|_{\mathcal{F}_t} = \frac{\mathcal{E}_t(\beta)}{\mathcal{E}_0(\beta)} = e^{\beta(X_t-X_s)-\psi(\beta)s}, \quad t \geq 0. \tag{2.7}
\]

In the special case that \( x = 0 \) we shall write \( \mathbb{P}_0^\beta \) in place of \( \mathbb{P}_0^\beta \). Since the process \( X \) under \( \mathbb{P}_x \) may be written as \( x + X \) under \( \mathbb{P} \), it is not difficult to see that the change of measure on \( (X, \mathbb{P}_x) \) is equivalent to the change of measure on \( (X, \mathbb{P}) \). Also known as the \textit{Esscher transform}, (2.7) alters the law of \( X \) and it is important to understand the dynamics of \( X \) under \( \mathbb{P}^\beta \).

**Theorem 2.4.** Fix \( \beta > 0 \). Then the process \( (X, \mathbb{P}^\beta) \) is equal in law to a Cramér-Lundberg process with premium rate \( c \) and claims that arrive at rate \( \lambda m(\beta) \) and that are distributed according to the probability measure \( e^{-\beta x} F(dx) / m(\beta) \) on \((0,\infty)\), where \( m(\beta) = \int_{(0,\infty)} e^{-\beta x} F(dx) \). Said another way, the process \( (X, \mathbb{P}^\beta) \) is equal in law to \( X^\beta \), where \( X^\beta := \{X^\beta_t : t \geq 0 \} \) is a Cramér-Lundberg process with Laplace exponent

\[
\psi_\beta(\theta) := \psi(\theta + \beta) - \psi(\beta) \quad \theta \geq 0.
\]

**Proof.** For all \( 0 \leq s \leq t \leq u < \infty, \theta \geq 0 \) and \( A \in \mathcal{F}_s \), we have with the help of the stationary and independent increments of \( (X, \mathbb{P}) \),
where in the second equality we have conditioned on \( \mathcal{F}_t \) and in the third equality we have conditioned on \( \mathcal{F}_s \) and used the martingale property of \( \mathcal{E}(\beta) \).

Using a straightforward argument by induction, it follows from (2.8) that, for all \( n \in \mathbb{N}, 0 \leq s_1 \leq t_1 \leq \cdots \leq s_n \leq t_n < \infty \) and \( \theta_1, \cdots, \theta_n \geq 0 \),

\[
\mathbb{E}^\beta \left[ \prod_{j=1}^{n} e^{\theta_j (X_{s_j} - X_{s_{j-1}})} \right] = \prod_{j=1}^{n} e^{\mathcal{E}(\beta)(t_j - t_{j-1})}. \tag{2.9}
\]

Moreover, a brief computation shows that

\[
\mathcal{E}(\theta) = c\theta - \lambda m(\beta) \int_{(0,\infty)} (1 - e^{-\theta x}) \frac{e^{-\beta x}}{m(\beta)} F(dx), \quad \theta \geq 0.
\]

Coupled with (2.9), this shows that \((X, \mathbb{P}^\beta)\) has stationary and independent increments, which are equal in law to those of a Cramér-Lundberg process with premium rate \( c \), arrival rate of claims \( \lambda m(\beta) \) and distribution of claims \( e^{-\beta x} F(dx) / m(\beta) \).

Since the measures \( \mathbb{P}^\beta \) and \( \mathbb{P} \) are equivalent on \( \mathcal{F}_t \), for all \( t \geq 0 \), then the property that \( X \) has paths that are almost surely right-continuous with left limits and no positive jumps on \([0,t]\) carries over to the measure \( \mathbb{P}^\beta \). Finally, taking note of Theorem 2.2, it follows that \((X, \mathbb{P}^\beta)\), which we now know is a spectrally negative Lévy process, has the law of the aforementioned Cramér-Lundberg process. \( \square \)

The Esscher transform may also be formulated at stopping times.

**Corollary 2.5.** Under the conditions of Theorem 2.2, if \( \tau \) is an \( \mathbb{F} \)-stopping time then

\[
\frac{d\mathbb{P}^\beta}{d\mathbb{P}} \bigg|_{\mathcal{F}_\tau} = \mathcal{E}_{\tau}(\beta) \text{ on } \{ \tau < \infty \}.
\]

Said another way, for all \( A \in \mathcal{F}_\tau \), we have

\[
\mathbb{P}^\beta(A, \tau < \infty) = \mathbb{E}(1_{(A, \tau < \infty)} \mathcal{E}(\beta)).
\]

**Proof.** By definition if \( A \in \mathcal{F}_\tau \), then \( A \cap \{ \tau \leq t \} \in \mathcal{F}_t \). Hence

\[
\mathbb{P}^\beta(A \cap \{ \tau \leq t \}) = \mathbb{E} \mathcal{E}(\beta) 1_{(A, \tau \leq t)} = \mathbb{E}(1_{(A, \tau \leq t)} \mathcal{E}(\beta)|\mathcal{F}_t) = \mathbb{E} \mathcal{E}_{\tau}(\beta) 1_{(A, \tau \leq t)}.
\]
where in the third equality we have used the Strong Markov Property as well as the martingale property for $\mathcal{E}(\beta)$. Now taking limits as $t \uparrow \infty$, the result follows with the help of the Monotone Convergence Theorem. \hfill \Box

### 2.4 Distribution of the maximum

Our first result here is to use the Esscher transform to characterise the law of the first passage times

$$\tau^+_x := \inf\{t > 0 : X_t > x\},$$

for $x \geq 0$, and subsequently the law of the running maximum when sampled at an independent and exponentially distributed time.

**Theorem 2.6.** For $q \geq 0$,

$$\mathbb{E}(e^{-q\tau^+_x} 1_{(\tau^+_x < \infty)}) = e^{-\Phi(q)x},$$

where we recall that $\Phi(q)$ is given by (2.5).

**Proof.** Fix $q > 0$. Using the fact that $X$ has no positive jumps, it must follow that $x = X_{\tau^+_x}$ on $\{\tau^+_x < \infty\}$. Note, with the help of the Strong Markov Property that,

$$\mathbb{E}(e^{\Phi(q)(X_t-X_{\tau^+_x})-qt} \mid F_{\tau^+_x}) = 1_{(\tau^+_x < t)} e^{\Phi(q)x - q\tau^+_x} \mathbb{E}(e^{\Phi(q)(X_t-X_{\tau^+_x})-q(t-\tau^+_x)} \mid F_{\tau^+_x}),$$

where in the final equality we have used the fact that $\mathbb{E}(\mathcal{E}(\Phi(q))) = 1$ for all $t \geq 0$. Taking expectations again we have

$$\mathbb{E}(e^{\Phi(q)(X_t-X_{\tau^+_x})-q(t-\tau^+_x)}) = 1.$$

Noting that the expression in the latter expectation is bounded above by $e^{\Phi(q)x}$, an application of dominated convergence yields

$$\mathbb{E}(e^{\Phi(q)x-qt} 1_{(\tau^+_x < \infty)}) = 1$$

which is equivalent to the statement of the theorem.

We recover the promised distributional information about the maximum process (3.1) in the next corollary. In its statement, we understand an exponential random variable with rate 0 to be infinite in value with probability one.

**Corollary 2.7.** Suppose that $q \geq 0$ and let $e_q$ be an exponentially distributed random variable which is independent of $X$. Then $Xe_q$ is exponentially distributed with parameter $\Phi(q)$. 

2.5 Comments

Theorem 2.2 is a simplified form of a much more general theorem that says that all Lévy processes are uniquely identified by the distribution of their increments. The idea of tilting a distribution by exponentially weighting its probability density function was introduced by Esscher (1932). This idea lends itself well to changes of measure in the theory of stochastic processes, in particular for Lévy processes. The Esscher martingale and the associated change of measure presented above is analogous to the exponential martingale for Brownian motion and the role that it plays in the classical Cameron-Martin-Girsanov change of measure. Indeed, the theory presented here may be extended to the general class of spectrally negative Lévy processes, which includes both Cramér-Lundberg processes as well as Brownian motion. See for example Chapter 3 of Kyprianou (2012). The Esscher transform plays a prominent role in fundamental mathematical finance as well as insurance mathematics; see for example the discussion in the paper of Gerber and Shiu (1994) and references therein. They style of reasoning in the proof of Theorem 2.6 is inspired by the classical computations of Wald (1944) for random walks.
Chapter 3
The Kella-Whitt martingale and the minimum

We move now to the second of our two key martingales. In a similar spirit to the previous chapter, we shall use the martingale to study the law of the process $\tilde{X} = \{\tilde{X}_t : t \geq 0\}$, where

$$\tilde{X}_t := \inf_{s \leq t} X_s, \quad t \geq 0. \quad (3.1)$$

In particular, as with the case of $\tilde{X}$, we shall consider the law of $\tilde{X}$ when sampled at an independent and exponentially distributed time. Unlike the case of $\tilde{X}$ however, this will turn out not to be exponentially distributed. In order to reach this objective, we will need to pass through two sections of preparatory material.

3.1 The Cramér–Lundberg process reflected in its supremum

Fix $x \geq 0$. Define the process $Y^x = \{Y^x_t : t \geq 0\}$, where

$$Y^x_t := (x \lor \tilde{X}_t) - \tilde{X}_t, \quad t \geq 0.$$  

Suppose that

**Lemma 3.1.** For each $x \geq 0$, $Y^x$ is a Markov process.

**Proof.** To this end, define for each $y \geq 0$, $Y^y_t = (y \lor \tilde{X}_t) - \tilde{X}_t$ and let $\tilde{X}_u = X_{t+u} - X_t$ for any $u \geq 0$. Note that for $t, s \geq 0$,

$$(y \lor \tilde{X}_{t+s}) - \tilde{X}_{t+s} = \left( y \lor \tilde{X}_t \lor \sup_{u \in [t,t+s]} X_u \right) - \tilde{X}_t - \tilde{X}_s$$

$$= \left[ (y \lor \tilde{X}_t - \tilde{X}_s) \lor \left( \sup_{u \in [t, t+s]} X_u - X_t \right) \right] - \tilde{X}_s$$

$$= \left[ Y^y_t \lor \sup_{u \in [0, s]} \tilde{X}_u \right] - \tilde{X}_s.$$
From the right-hand side above, it is clear that the law of $Y^x_{t+s}$ depends only on $Y^x_t$ and $\{X_u : u \in [0,s]\}$, the latter being independent of $\mathcal{F}_t$. Hence $\{Y^x_t : t \geq 0\}$ is a Markov process. \qed

Remark 3.2. Note that the argument given in the proof above shows that, if $\tau$ is any stopping time with respect to $\mathcal{F}$, then, on $\{\tau < \infty\}$,

$$ (x \vee X_{\tau+s}) - X_{\tau+s} = \left[ Y^x_t \vee \sup_{u \in [0,s]} \bar{X}_u \right] - \bar{X}_s, $$

where now, $\bar{X}_s = X_{\tau+s} - X_\tau$. In other words,

$$ Y^x_{\tau+s} = \bar{Y}_s^z \text{ such that } z = Y^x_\tau, $$

where $\bar{Y}_s^z$ is independent of $\{Y^x_u : u \leq \tau\}$ and equal in law to $Y^z$.

Remark 3.3. It is also possible to argue in the style of the proof of Lemma 3.1 that, for each $x \geq 0$, the triple $(Y^x, X, N)$ is also Markovian. Specifically, for each $s, t \geq 0$,

$$ (Y^x_{t+s}, X_{t+s}, N_{t+s}) = (\bar{Y}_s^z, y + \bar{X}_s, n + \bar{N}_s) \text{ such that } z = Y^x_t, y = X_t \text{ and } n = N_t, $$

where $\{(\bar{Y}_s^z, \bar{X}_s, \bar{N}_s) : s \geq 0\}$ is independent of $\mathcal{F}_t$ and equal in law to $(Y^z, X, N)$ under $\mathbb{P}$. Again, one also easily replaces $t$ by an $\mathcal{F}$-stopping time in the above observation, as in the previous remark.

### 3.2 A useful Poisson integral

In the next section, we will come across some functionals of the driving Poisson process $N = \{N_t : t \geq 0\}$ that lies behind $X$, and hence $Y^x$, for each $x \geq 0$. Specifically we will be interested in expected sums of the form

$$ \mathbb{E} \left[ \sum_{i=1}^{N_t} f(Y^x_{T_i^-}, \xi_i) \right], \quad x, t \geq 0, $$

where $f : [0, \infty) \times (0, \infty) \rightarrow [0, \infty)$ is measurable, $\{T_i : i \geq 1\}$ are the arrival times in the process $N$ and recall that $\{\xi_i : i \geq 1\}$ are the i.i.d. subsequent claim sizes of $X$ with common distribution $F$. We will use the following result.

**Theorem 3.4 (Compensation formula).** For all non-negative, bounded, measurable $f, g$ and $x, t \geq 0$,

$$ \mathbb{E} \left[ \sum_{i=1}^{N_t} f(Y^x_{T_i^-}, \xi_i) \right] = \lambda \int_0^t \int_{(0,\infty)} \mathbb{E}(f(Y^x_{s^-}), u) F(du) ds. \quad (3.2) $$
Proof. First note that, with the help of Fubini’s Theorem, we can write
\[
\mathbb{E} \left[ \sum_{i=1}^{\infty} \mathbf{1}_{(T_i \leq t)} f(Y_{T_i-}^x, \xi_i) \right] = \sum_{i=1}^{\infty} \mathbb{E} \left[ \mathbf{1}_{(T_i \leq t)} f(Y_{T_i-}^x, \xi_i) \right]. \tag{3.3}
\]
Note that \(T_i = \inf\{t > 0 : N_t = i\}\) and hence each \(T_i\) is a stopping time. Note also that, for each \(i \geq 1\), the terms \(f(Y_{T_i-}^x)\) are each measurable in the sigma-algebra \(\mathcal{H}_i := \sigma(\{N_s : s \leq T_i\}, \{\xi_j : j = 1, \ldots, i-1\})\). Breaking each of the expectations in the sum on the right-hand side of (3.3), by first conditioning on \(\mathcal{H}_i\), it follows that
\[
\sum_{i=1}^{\infty} \int_{(0, \infty)} \mathbb{E} \left[ \mathbf{1}_{(T_i \leq t)} f(Y_{T_i-}^x, u) \right] F(du) = \int_{(0, \infty)} \mathbb{E} \left[ \sum_{i=1}^{N_t} f(Y_{T_i-}^x, u) \right] F(du).
\]
The proof is therefore complete as soon as we show that, for all \(x, t \geq 0\) and \(u > 0\),
\[
\mathbb{E} \left[ \sum_{i=1}^{N_t} f(Y_{T_i-}^x, u) \right] = \lambda \int_0^t \mathbb{E}[f(Y_s^x, u)] ds. \tag{3.4}
\]
To this end define, for \(x, t \geq 0\) and \(u > 0\), \(\eta_u(x, t) = \mathbb{E} \left[ \sum_{i=1}^{N_t} f(Y_{T_i-}^x, u) \right]\). With the help of the Markov property for \(Y^x\) and stationary independent increments of \(N\), we have
\[
\eta_u(x, t + s) - \eta_u(x, t) = \mathbb{E} \left[ \mathbb{E} \left[ \sum_{i=N_{t+1}}^{N_{t+s}} f(Y_{T_i-}^x, u) \bigg| \mathcal{F}_t \right] \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E} \left[ \sum_{i=1}^{N_t} f(\tilde{Y}_{T_i}^x, u) \bigg| \mathcal{F}_t \right] \right]
\]
\[
= \mathbb{E} \left[ \mathbb{E}[\eta_u(Y_{T_i}^x, s)] \right],
\]
where the process \(\{(\tilde{Y}_t^x, N_s) : s \geq 0\}\) is independent of \(\mathcal{F}_t\) and equal in law to \((Y^x, N)\) under \(\mathbb{P}\). Note here that \(\{T_i : i \geq 1\}\) are the arrival times of the process \(N\). Next, note that, for all \(x, s \geq 0\), we have that \(s^{-1} \eta_u(x, s)\) is bounded by \(s^{-1} C \mathbb{E}(N_s) = \lambda C\), where \(C = \sup_{y > 0} f(y) < \infty\). Hence, with the help of the Dominated Convergence Theorem, our objective now is to compute the right-derivative of \(\eta_u(x, t)\) by evaluating the limit
\[
\lim_{s \downarrow 0} \frac{\eta_u(x, t + s) - \eta_u(x, t)}{s} = \mathbb{E} \left[ \lim_{s \downarrow 0} \frac{1}{s} \eta_u(Y_{T_i}^x, s) \right]. \tag{3.5}
\]
Note that, for all \(x, s \geq 0\),
\[
\frac{1}{s} \eta_u(x, s) = \frac{1}{s} \mathbb{E}[f(Y_{T_i}^x, u) \mathbf{1}_{(N_i = 1)}] + \frac{1}{s} \mathbb{E} \left[ \sum_{i=1}^{N_t} f(Y_{T_i}^x, u) \mathbf{1}_{(N_i \geq 2)} \right]. \tag{3.6}
\]
Recall, moreover, that, for \(v \leq s\),
\[ P(T_1 \in d\nu, N_s = 1) = P(T_1 \in d\nu, T_2 > s) \\
= P(T_1 \in d\nu, T_2 - T_1 > s - v) \\
= \lambda e^{-\lambda v} d\nu \times e^{-\lambda(s-v)} \\
= \lambda e^{-\lambda s} d\nu. \]

Hence for the first term on the right-hand side of (3.6) we have
\[
\lim_{s \downarrow 0} \frac{1}{s} \mathbb{E}[f(Y_{T_1}^x, u) \mathbf{1}(N_s = 1)] = \lim_{s \downarrow 0} \frac{\lambda e^{-\lambda s}}{s} \int_0^s f((x \lor cv) - cv, u) d\nu = \lambda f(x).
\]

For the second term on the right-hand side of (3.6), we also have
\[
\lim_{s \downarrow 0} \frac{1}{s} \mathbb{E} \left[ \sum_{i \geq 1} f(Y_{T_i}^x, u) \mathbf{1}(N_s \geq 2) \right] \leq C \lim_{s \downarrow 0} \frac{1}{s} \mathbb{E}[N_s \mathbf{1}(N_s \geq 2)] = \lim_{s \downarrow 0} \frac{1}{s} [\lambda s(1 - e^{-\lambda s})] = 0.
\]

Returning to (3.6) it follows that, for all \( x \geq 0 \), \( \lim_{s \downarrow 0} \frac{1}{s} \eta_u(x, s) = \lambda f(x) \) and hence, from (3.5), we have that
\[
\frac{\partial}{\partial t} \eta_u(x, t) = \lambda \mathbb{E}[f(Y_t^x, u)].
\]

A similar argument, looking at the difference \( \eta_u(x, t) - \eta_u(x, t - s) \), for \( x \geq 0 \) and \( t > s > 0 \), also shows that the left derivative
\[
\frac{\partial}{\partial t} \eta_u(x, t-) = \lambda \mathbb{E}[f(Y_t^x, u)].
\]

It follows that \( \eta_u(x, t) \) is differentiable in \( t \) on \((0, \infty)\) and hence, since \( \eta_u(x, 0) = 0 \),
\[
\eta_u(x, t) = \lambda \int_0^t \mathbb{E}[f(Y_s^x, u)] ds,
\]
which establishes (3.4) and completes the proof. \( \square \)

Remark 3.5. It is a straightforward exercise to deduce from Theorem 3.4 that the compensated process
\[
\sum_{i=1}^{N_t} f(Y_{T_i}^x, \xi_i) - \lambda \int_0^t \int_{(0,\infty)} \mathbb{E}(f(Y_s^x), u) F(du) ds, \quad t \geq 0,
\]
is a martingale. In this sense (3.2) is called the compensation formula.
3.3 Second exponential martingale

We are now ready to introduce our second exponential martingale, also known as the Kella-Whitt martingale. See Sect. 3.7 for historical remarks regarding its name.

Theorem 3.6. For $\theta > 0$ and $x \geq 0$,

$$M_t^x := \psi(\theta) \int_0^t e^{-\theta Y^z_s} \, ds + 1 - e^{-\theta (X_t \vee x)} - \theta (X_t \vee x), \quad t \geq 0 \tag{3.7}$$

is a $\mathbb{P}$-martingale with respect to $\mathcal{F}_t$.

Proof. Let us start by using the Markov property of $Y^z$ to write, for $x,s,t \geq 0$,

$$x \vee X_{t+s} = Y^z_{t+s} + X_{t+s} = \tilde{Y}^z_{s+} + \tilde{X}_s + X_s = (z \vee \tilde{X}_s)_{z \geq Y^z_t} + X_s,$$

where $\tilde{X}_s = X_{t+s} - X_t$ and $\tilde{Y}^z$ is independent of $\{Y^z_u : u \leq t\}$ and equal in law to $Y^z$.

Using this decomposition, it is straightforward to show that

$$\mathbb{E}[M_{t+s}^x | \mathcal{F}_t] = \psi(\theta) \int_0^t e^{-\theta Y^z_u} \, du + 1 - \theta X_t$$

$$+ \mathbb{E} \left[ \psi(\theta) \int_0^s e^{-\theta Y^z_u} - e^{-\theta \tilde{Y}^z_u} - \theta (z \vee \tilde{X}_s) \right]_{z \geq Y^z_t}.$$

The proof is thus complete as soon as we show that, for all $z,s \geq 0$,

$$\mathbb{E} \left[ \psi(\theta) \int_0^s e^{-\theta Y^z_u} - e^{-\theta \tilde{Y}^z_u} - \theta (z \vee \tilde{X}_s) \right] = -e^{-\theta z} - \theta z.$$

In order to achieve this goal, we shall develop the left-hand side above using the so-called chain rule for right-continuous functions of bounded variation (also known as an extension of the Fundamental Theorem of Calculus for the latter class). That is,

$$e^{-\theta Y^z_t} = e^{-\theta z} - \theta \int_{(0,t]} e^{-\theta Y^z_u} d(Y^z_u) + \sum_{i=1}^{N_t} [e^{-\theta Y^z_{u_i}} - e^{-\theta Y^z_{u_{i-1}}}], \tag{3.8}$$

for $z,s \geq 0$, where $(Y^z_u)_u$ is the continuous part of $Y^z$. Note, however, that

$$\int_{(0,s]} e^{-\theta Y^z_u} d(Y^z_u) = \int_{(0,s]} e^{-\theta Y^z_u} d(z \vee \tilde{X}_u) - c \int_0^s e^{-\theta Y^z_u} \, du$$

$$= \int_{(0,s]} 1_{(Y^z_u = 0)} e^{-\theta Y^z_u} d(z \vee \tilde{X}_u) - c \int_0^s e^{-\theta Y^z_u} \, du$$

$$= (z \vee \tilde{X}_s) - z - c \int_0^s e^{-\theta Y^z_u} \, du,$$

where in the second equality we have used the fact $Y^z_0 = 0$ on the set of times that the process $z \vee \tilde{X}_u$ increments. We may now take expectations in (3.8) to deduce that
\[ E \left[ e^{-\theta Y_z} + \theta (z \vee X_s) \right] = e^{-z\theta} + \theta z + E \left[ c \theta \int_0^s e^{-\theta Y_u} du + \sum_{i=1}^{N_s} e^{-\theta Y_{T_i}} - (e^{-\theta z_{T_i}} - 1) \right] \]
\[ = e^{-z\theta} + \theta z + \theta \left[ \int_0^s e^{-\theta Y_u} du \right] + \lambda \int_{(0,\infty)} (e^{-\theta x} - 1) F(dx) \left[ \int_0^s e^{-\theta Y_u} du \right] \]
\[ = e^{-z\theta} + \theta z + \psi(\theta) \left[ \int_0^s e^{-\theta Y_u} du \right], \]
where we have applied Theorem 3.4 in the second equality. The proof is now complete. \( \square \)

3.4 Duality

For our main application of the Kella-Whitt martingale, we need to address one additional property of the Cramér-Lundberg process, which follows as a consequence of the fact that it is also a Lévy process. This property concerns the issue of duality.

Lemma 3.7 (Duality Lemma). For each fixed \( t > 0 \), define the time-reversed process
\[ \{X(t-s) - X_t : 0 \leq s \leq t \} \]
and the dual process,
\[ \{-X_s : 0 \leq s \leq t \}. \]
Then the two processes have the same law under \( \mathbb{P} \).

Proof. Define the process \( R_s = X_t - X_{t-s} \) for \( 0 \leq s \leq t \). Under \( \mathbb{P} \) we have \( Y_0 = 0 \) almost surely, as \( t \) is a jump time with probability zero. As can be seen from Fig. 3.1 the paths of \( R \) are obtained from those of \( X \) by a rotation about \( 180^\circ \), with an adjustment of the continuity at the jump times, so that its paths are almost surely right-continuous with left limits. The stationary independent increments of \( X \) imply directly that the same is true of \( Y \). This puts \( R \) in the class of Lévy processes. Moreover, for each \( 0 \leq s \leq t \), the distribution of \( R_s \) is identical to that of \( X_s \). It follows that
\[ E(e^{\lambda R_s}) = e^{\psi(\lambda)s}, \]
for all \( 0 \leq s \leq t < \infty \) and \( \lambda \geq 0 \). Since clearly \( R \) belongs to the class of Lévy processes with no positive jumps, it follows from Theorem 2.2 that \( R \) has the same law as \( X \). \( \square \)

One interesting feature, that follows as a consequence of the Duality Lemma, is the relationship between the running supremum, the running infimum, the process reflected in its supremum and the process reflected in its infimum. The last four objects are, respectively,
Fig. 3.1 A realisation of the trajectory of \( \{X_s : 0 \leq s \leq t\} \) and of \( \{X_{(t-s)} : 0 \leq s \leq t\} \), respectively.

\[
\bar{X}_t = \sup_{0 \leq s \leq t} X_s, \quad \underline{X}_t = \inf_{0 \leq s \leq t} X_s
\]

\( \{\bar{X}_t - X_t : t \geq 0\} \) and \( \{X_t - \underline{X}_t : t \geq 0\} \).

Lemma 3.8. For each fixed \( t > 0 \), the pairs \((\bar{X}_t, \underline{X}_t - X_t)\) and \((X_t - \bar{X}_t, \bar{X}_t)\) have the same distribution under \(\mathbb{P}\).

Proof. Define \( R_s = X_t - X_{(t-s)} \) for \( 0 \leq s \leq t \), as in the previous proof, and write \( R_t = \inf_{0 \leq s \leq t} R_s \). Using right-continuity and left limits of paths we may deduce that

\[
(\bar{X}_t, \underline{X}_t - X_t) = (R_t - R_t, \bar{X}_t)
\]

almost surely. Now appealing to the Duality Lemma we have that \( \{R_s : 0 \leq s \leq t\} \) is equal in law to \( \{X_s : 0 \leq s \leq t\} \) under \(\mathbb{P}\) and the result follows. \(\square\)

3.5 Distribution of the minimum

We are now able to deliver the promised result concerning the law of the minimum.

Theorem 3.9. Let \( \underline{X}_t = \inf_{0 \leq u \leq t} X_u \) and suppose that \( e_q \) is an exponentially distributed random variable with parameter \( q > 0 \) independent of the process \( X \). Then for \( \theta > 0 \),

\[
\mathbb{E}(e^{\theta \underline{X}_t}) = \frac{q(\theta - \Phi(q))}{\Phi(q)(\psi(\theta) - q)}, \quad (3.9)
\]

where the right hand side is understood in the asymptotic sense when \( \theta = \Phi(q) \), i.e. \( q/\Phi(q) \psi'(\Phi(q)) \).

Proof. Let us first consider the case that \( \theta, q > 0 \) and \( \theta \neq \Phi(q) \). Let \( Y_t = y^0_t = \bar{X}_t - X_t \). Note that by an application of Fubini’s theorem together with Lemma 3.8, we have
\[\mathbb{E}\left[\int_0^{\xi_q} e^{-\theta Y_s} \, ds\right] = \int_0^\infty e^{-qs} \mathbb{E}(e^{-\theta Y_s}) \, ds = \frac{1}{q} \mathbb{E}(e^{-\theta Y_0}) = \frac{1}{q} \mathbb{E}(e^{\theta X_0}).\]

From Theorem 3.6 we have that
\[\mathbb{E}(M_0^q e^{\theta X_0}) = \mathbb{E}(M_0^q) = 0\]
and hence we obtain
\[\psi(\theta) - q \mathbb{E}(e^{\theta X_0}) = \theta \mathbb{E}(X_0) - 1.\]

Recall from Corollary 2.7 that \(X_0\) is exponentially distributed with parameter \(\Phi(q)\) and hence \(\mathbb{E}(X_0) = 1/\Phi(q)\). It follows that
\[\psi(\theta) - q \mathbb{E}(e^{\theta X_0}) = \theta \Phi(q) - \Phi(q).\]  
(3.10)

For the case that \(q > 0\) and \(\theta = \Phi(q)\), the result follows from the case that \(\theta \neq \Phi(q)\) by taking limits as \(\theta \to \Phi(q)\).

### 3.6 The long term behaviour

Let us conclude this chapter by returning to earlier remarks made in Sect. 1.2 regarding the long term behaviour of the Cramér-Lundberg process. Recall that \(\psi'(0+) = c - \lambda \mu\), where \(c\) is the premium rate, \(\lambda\) is the rate of claim arrivals and \(\mu\) is their common means. It is clear from (1.4) that when \(\psi'(0+) > 0\) we have \(\lim_{t \to \infty} X_t = \infty\) and when \(\psi'(0+) < 0\), \(\lim_{t \to \infty} X_t = -\infty\). For the remaining case, when \(\psi'(0+) = 0\), the Strong Law of Large Numbers is not as informative. We can, however, use our previous results on the law of the maximum and minimum of \(X\) to determine the long term behaviour of \(X\). Specifically, the lemma below shows that, when \(\psi'(0+) = 0\), the process \(X\) oscillates in the sense that \(\limsup_{t \to \infty} X_t = -\liminf_{t \to \infty} X_t = \infty\).

**Lemma 3.10.** We have that

(i) \(X_\infty\) and \(-X_\infty\) are either infinite almost surely or finite almost surely,
(ii) \(X_\infty = \infty\) if and only if \(\psi'(0+) \geq 0\),
(iii) \(X_\infty = -\infty\) if and only if \(\psi'(0+) \leq 0\).

**Proof.** Recall that, on account of the strict convexity \(\psi\), we have that \(\Phi(0) > 0\) if and only if \(\psi'(0+) < 0\). Hence
\[\lim_{q \to 0} \frac{q}{\Phi(q)} = \begin{cases} 0 & \text{if } \psi'(0+) \leq 0 \\ \psi'(0+) / \psi(\theta) & \text{if } \psi'(0+) > 0. \end{cases}\]

By taking \(q\) to zero in the identity (3.9) we now have that
\[\mathbb{E}(e^{\theta X_0}) = \begin{cases} 0 & \text{if } \psi'(0+) \leq 0 \\ \psi'(0+) \theta / \psi(\theta) & \text{if } \psi'(0+) > 0. \end{cases}\]  
(3.11)
In the first of the two cases above, it is clear that $\mathbb{P}(-X_\infty = \infty) = 1$. In the second case, taking limits as $\theta \uparrow \infty$, one sees that $\mathbb{P}(-X_\infty = \infty) = 0$.

Next, recall from Corollary 2.7 that $X_\infty$ is exponentially distributed with parameter $\Phi(0)$. In particular, $X_\infty$ is almost surely finite when $\psi'(0+) \geq 0$ and almost surely infinite when $\psi'(0+) < 0$.

Putting this information together, the statements (i)–(iii) are easily recovered. □

3.7 Comments

The fact that the process $Y^x$ is a Markov process for each $x \geq 0$ is well known from queuing theory, where the process $Y^x$ is precisely the workload in an $M/G/1$ queue. With a little more effort, it is not difficult to show that $Y^x$ is also a strong Markov process. Theorem 3.4 is an example of the so-called compensation formula which can be stated for general Poisson integrals. See for example Chapter XII.1 of Revuz and Yor (2004). The Kella-Whitt martingale was first introduced in Kella and Whitt (1992) in the setting of a general Lévy process. It is an extension of so-called Kennedy, or indeed of Azéma-Yor, martingales, both of which have previously been studied in the setting of Brownian motion. The Duality Lemma is also well known for (and in fact originates from) the theory of random walks, the discrete time analogue of Lévy processes, and is justified using an identical proof. See for example Chapter XII of Feller (1971).
Chapter 4
Scale functions and ruin probabilities

The two main results from the previous chapters, concerning the law of the maximum and minimum of the Cramér-Lundberg process, can now be put to use to establish our first results concerning the classical ruin problem. We shall introduce the so-called scale functions, which will prove to be indispensable, both in this chapter and later, when describing various distributional features of the ruin problem.

4.1 Scale functions and the probability of ruin

For a given Cramér-Lundberg process, $X$, with Laplace exponent $\psi$, we want to define a family of scale functions, indexed by $q \geq 0$, which we shall denote by $W(q): \mathbb{R} \rightarrow [0, \infty)$. For all $q \geq 0$ we shall set $W(q)(x) = 0$ for $x < 0$. The next theorem will also serve as a definition for $W(q)$ on $[0, \infty)$.

Theorem 4.1. For all $q \geq 0$ we may define $W(q)$ on $[0, \infty)$ as the unique non-decreasing, right-continuous function whose Laplace transform is given by

$$\int_0^\infty e^{-\beta x} W(q)(x) dx = \frac{1}{\psi(\beta) - q}, \quad \beta > \Phi(q). \quad (4.1)$$

For convenience we shall always write $W$ in place of $W(0)$. Typically we shall refer the functions $W(q)$ as $q$-scale functions, but we shall also refer to $W$ as just the scale function.

Proof (of Theorem 4.1). First assume that $\psi'(0+) > 0$. With a pre-emptive choice of notation, we shall define the function

$$W(x) = \frac{1}{\psi'(0+)} \mathbb{P}_x(X_\infty \geq 0), \quad x \in \mathbb{R}. \quad (4.2)$$

Clearly $W(x) = 0$, for $x < 0$, and it is non-decreasing and right-continuous since it is also proportional to the distribution function $P(-X_\infty \leq x)$. Integration by parts...
shows that, on the one hand,

\[
\int_0^\infty e^{-\beta x} W(x) \, dx = \frac{1}{\psi(0+)} \int_0^\infty e^{-\beta x} P(-X_\infty \leq x) \, dx
\]

\[
= \frac{1}{\psi(0+)} \int_{(0,\infty)} e^{-\beta x} P(-X_\infty \in dx)
\]

\[
= \frac{1}{\psi(0+)} \beta E(e^{\beta X_\infty}).
\] 

(4.3)

On the other hand, recalling (3.11), we also have that

\[
E(e^{\beta X_\infty}) = \psi'(0+)\beta \psi(\beta), \quad \beta \geq 0.
\]

When combined with (4.3), this gives us (4.1) as required for the case \(q = 0\) and \(\psi'(0+) > 0\).

Next we deal with the case where \(q > 0\) or where \(q = 0\) and \(\psi'(0+) < 0\). To this end, again making use of a pre-emptive choice of notation, let us define the non-decreasing and right-continuous function

\[
W(q)(x) = e^{\Phi(q)} W(q)(x), \quad x \geq 0,
\] 

(4.4)

where \(W(q)\) plays the role of \(W\) but for the process \((X, P^{\Phi(q)})\). Note in particular that, by Theorem 2.4, the latter process has Laplace exponent

\[
\psi(\Phi(q)(\theta)) = \psi(\theta + \Phi(q)) - q, \quad \theta \geq 0.
\] 

(4.5)

Hence \(\psi(\Phi(q)(0+)) = \psi'(\Phi(q)) > 0\), which ensures that \(W(q)\) is well defined by the previous part of the proof. Taking Laplace transforms we have for \(\beta > \Phi(q)\),

\[
\int_0^\infty e^{-\beta x} W(q)(x) \, dx = \int_0^\infty e^{-(\beta - \Phi(q))x} W(q)(x) \, dx
\]

\[
= \frac{1}{\psi(q)(\beta - \Phi(q))}
\]

\[
= \frac{1}{\psi(\beta) - q},
\]

thus completing the proof for the case that \(q > 0\) or that \(q = 0\) and \(\psi'(0+) < 0\).

Finally, we deal with the case that \(q = 0\) and \(\psi'(0+) = 0\). Since \(W(q)(x)\) is an increasing function, we may also treat it as a distribution function of a measure which we also, as an abuse of notation, call \(W(q)\). Integrating by parts thus gives us, for \(\beta > 0\),

\[
\int_{[0,\infty)} e^{-\beta x} W(q)(x) \, dx = \frac{\beta}{\psi(q)(\beta)}.
\] 

(4.6)

Note that the assumption \(\psi'(0+) = 0\) implies that \(\Phi(0) = 0\), and hence for \(\theta \geq 0\),
4.1 Scale functions and the probability of ruin

\[
\lim_{q \downarrow 0} \psi_{\Phi(q)}(\theta) = \lim_{q \downarrow 0} [\psi(\theta + \Phi(q)) - q] = \psi(\theta).
\]

One may appeal to the Extended Continuity Theorem for Laplace transforms, see for example Theorem XIII.1.2a of Feller (1971), and (4.6) to deduce that, since

\[
\lim_{q \downarrow 0} \int_{(0,\infty)} e^{-\beta x} W \Phi(q)(dx) = \frac{\beta}{\psi(\beta)},
\]

then there exists a measure \( W^* \) such that \( W^*(x) := W^*[0,x] = \lim_{q \downarrow 0} W \Phi(q)(x) \) and

\[
\int_{(0,\infty)} e^{-\beta x} W^*(dx) = \frac{1}{\psi(\beta)},
\]

Integration by parts shows that \( W \) satisfies

\[
\int_{0}^{\infty} e^{-\beta x} W(x) dx = \frac{1}{\psi(\beta)},
\]

for \( \beta > 0 \) as required. Note that it is clear from its definition that \( W \) is non-decreasing and right-continuous. \( \square \)

With the definition of scale functions in hand, we can return to the problem of ruin. The following corollary follows as a simple consequence of Laplace inversion of the identity in Theorem 3.9, taking account of (4.1).

**Corollary 4.2.** For \( q \geq 0 \) and \( q > 0 \),

\[
P(-X_{e^q} \in dx) = \frac{q}{\Phi(q)} W^{(q)}(dx) - q W^{(q)}(x) dx.
\]

(4.7)

Note that, in the above formula, thanks to (4.4), the function \( W^{(q)} \) is increasing and hence the measure \( W^{(q)}(dx), x \geq 0 \), makes sense. Note also that the above formula can also be stated when \( q = 0 \), providing \( \psi'(0+) > 0 \). In that case the term \( q/\Phi(q) \) should be understood, in the limiting sense, as equal to \( \psi'(0+) \).

We complete this section with our main result about ruin probabilities, using scale functions. To this end, let us define the functions

\[
Z^{(q)}(x) = 1 + q \int_{0}^{x} W^{(q)}(y) dy, \quad x \in \mathbb{R},
\]

for \( q \geq 0 \).

**Theorem 4.3 (Ruin probabilities).**

For any \( x \in \mathbb{R} \) and \( q \geq 0 \),

\[
\mathbb{E}_x \left( e^{-q \xi_0} 1_{(\xi_0 < \infty)} \right) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x),
\]

(4.8)
where we understand $q/\Phi(q)$ in the limiting sense for $q = 0$, so that
\[
P_x(\tau^-_0 < \infty) = \begin{cases} 1 - \psi'(0+)W(x) & \text{if } \psi'(0+) \geq 0 \\ 1 & \text{if } \psi'(0+) < 0 \end{cases}. \tag{4.9}
\]

**Proof.** Appealing to (4.7), we have, for $x \geq 0$,
\[
\mathbb{E}_x \left( e^{-q\tau^-_0} 1_{(\tau^-_0 < \infty)} \right) = P_x(e_q > \tau^-_0) \\
= P_x(X_{e_q} < 0) \\
= \mathbb{P}(-X_{e_q} > x) \\
= 1 - \mathbb{P}(-X_{e_q} \leq x) \\
= 1 + q \int_x^{\infty} W^{(q)}(y)dy - \frac{q}{\Phi(q)} W^{(q)}(x) \\
= Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x). \tag{4.10}
\]

Note that since $Z^{(q)}(x) = 1$ and $W^{(q)}(x) = 0$ for all $x \in (-\infty, 0)$, the statement is valid for all $x \in \mathbb{R}$. The proof is now complete for the case that $q > 0$.

In order to deal with the case $q = 0$, note that
\[
\lim_{q \downarrow 0} q/\Phi(q) = \lim_{q \downarrow 0} \psi(\Phi(q))/\Phi(q)
\]
If $\psi'(0+) \geq 0$, i.e. the process drifts to infinity or oscillates, then $\Phi(0) = 0$ and the limit is equal to $\psi'(0+)$. Otherwise, when $\Phi(0) > 0$, the aforementioned limit is zero. The proof is thus completed by taking the limit in $q$ in (4.8). \hfill \square

The last part of the above theorem can be recovered directly from the definition of $W$ in the case that $\psi'(0+) > 0$, see (4.2). Moreover, the probability of ruin when $\psi'(0+) \leq 1$ is obviously 1 given the discussion in Sect. 3.6.

### 4.2 Connection with the Pollaczek–Khintchine formula

In Theorem 1.3 we gave the classical Pollaczek-Khintchine formula for the probability of ruin in the case that $\psi'(0+) > 0$. Compared with the formula in (4.9), it is not immediately obvious how these two formulae relate to one another. Let us therefore spend a little time to make the connection between the two, first with an analytical explanation and then with a probabilistic explanation.

**Analytical explanation.** Let us start by noting that, just as in formula (4.6), we can integrate the Laplace transform of $W$ by parts to show that
\[
\int_{[0,\infty)} e^{-\beta x} W(dx) = \frac{\beta}{\psi(\beta)} \quad \beta > 0.
\]
4.2 Connection with the Pollaczek–Khintchine formula

Alternatively, this also follows from the definition \( \psi' \) and the expression for the Laplace transform of \(-X\), given in (3.11). Next note that, from the discussion following Theorem 2.2, the inequality \( \psi'(0+) > 0 \) necessarily implies that

\[
\mu := \int_{(0,\infty)} xF(dx)
\]

is finite and, moreover, that

\[
\rho := \frac{\lambda \mu}{c} < 1.
\]

This inequality also implies that

\[
\frac{\lambda \mu}{c} \int_{0}^{\infty} e^{-\beta x} \frac{1}{\mu} F(x) dx < 1.
\]

Hence, recalling the representation of \( \psi \) given in (2.4), we can write, for \( \beta > 0 \),

\[
\frac{\beta}{\psi(\beta)} = \frac{1}{c} \left( 1 - \frac{\lambda \mu}{c} \int_{0}^{\infty} e^{-\beta x} \frac{1}{\mu} F(x) dx \right) = \frac{1}{c} \sum_{k=0}^{\infty} \rho^k \left( \int_{0}^{\infty} e^{-\beta x} \frac{1}{\mu} F(x) dx \right)^k,
\]

where we recall that \( F(x) = 1 - F(x) \). Next note that

\[
\eta(dx) := \frac{1}{\mu} F(dx), \quad x \geq 0,
\]

is a probability measure. For each \( k \geq 0 \), denote by \( \eta^k(dx), x \geq 0 \), its \( k \)-fold convolution, where we understand \( \eta^0(dx) := \delta_0(dx), x \geq 0 \), the Dirac delta measure which places an atom at zero. Since, for \( \beta > 0 \) and \( k \geq 0 \),

\[
\int_{(0,\infty)} e^{-\beta x} \eta^k(dx) = \left( \int_{0}^{\infty} e^{-\beta x} \frac{1}{\mu} F(x) dx \right)^k,
\]

we may apply Laplace inversion to the right hand side of (4.11) and conclude that, for \( x \geq 0 \),

\[
W(dx) = \frac{1}{c} \sum_{k=0}^{\infty} \rho^k \eta^k(dx),
\]

which is to say, for \( x \geq 0 \),

\[
W(x) = \frac{1}{c} \sum_{k=0}^{\infty} \rho^k \eta^k(x).
\] (4.12)

Returning to the formula in (4.9) when \( \psi'(0+) = c - \lambda \mu > 0 \), we now see that
\[ 1 - P_x(\tau_0^- < \infty) = (1 - \rho) \sum_{k=0}^{\infty} \rho^k \eta^*(x), \] (4.13)

as stated in Theorem 1.3.

**Probabilistic explanation.** The Pollaczek-Khintchine formula can also be recovered by looking at the successive minima of the process \( X \). To this end, let us set \( \Theta_0 = 0 \) and sequentially define, for all \( k \geq 1 \) such that \( \Theta_{k-1} < \infty \),

\[ \Theta_k = \inf\{t > \Theta_{k-1} : X_t < X_{\Theta_{k-1}}\}, \]

with the usual understanding that \( \inf \emptyset = \infty \). As long as they are finite, the times \( \Theta_k \) are thus the times of successive new minima.

The strong Markov property implies that, for each \( k \geq 1 \) such that \( \{\Theta_{k-1} < \infty\} \), the pair \( (\Theta_k - \Theta_{k-1}, X_{\Theta_k} - X_{\Theta_{k-1}}) \) is independent of \( \mathcal{F}_{\Theta_{k-1}} \) and equal in law to the pair \( (\tau_0^-, X_{\tau_0^-}) \), where we understand \( X_{\Theta_k} = \infty \) when \( \Theta_k = \infty \) and, similarly, \( X_{\tau_0^-} = \infty \) when \( \tau_0^- = \infty \). For \( k \geq 1 \), define on \( \{\Theta_{k-1} < \infty\} \)

\[ \Delta_k = -(X_{\Theta_k} - X_{\Theta_{k-1}}). \]

Then the event \( \{\tau_0^- = \infty\} \) under \( P_x \), \( x \geq 0 \), corresponds to the event that

\[ \left\{ \sum_{n=1}^{\nu-1} \Delta_n \leq x \right\}, \]

where \( \nu = \min\{k \geq 1 : \Theta_k = \infty\} \).\(^1\) See Fig 4.1. Note however that, again by the strong Markov property, the index \( \nu \) is the time of first success in an independent sequence of Bernoulli trials with probability success \( \hat{\rho} := P(\Theta_1 = \infty) = P(\tau_0^- = \infty) \). In other words, \( \nu \) is Geometrically distributed. Moreover, \( \nu \) is independent of the outcome of each of aforesaid independent trials, each of which fail, delivering a random value which are distributed according to the measure \( \hat{\eta}(dx) := P(\Delta_1 \in dx) = P(-X_{\tau_0^-} \in dx | \tau_0^- < \infty), x > 0. \)

In conclusion, we see that

\[ 1 - P_x(\tau_0^- < \infty) = (1 - \hat{\rho}) \sum_{k=0}^{\infty} \hat{\rho}^k \eta^*(x), \quad x \geq 0. \] (4.14)

Comparing the formulae (4.13) and (4.14) when \( x = 0 \), we see that \( P(\tau_0^- < \infty) = \rho = \hat{\rho} \), and hence it follows that \( \eta = \hat{\eta} \).

Note that the following corollary falls straight out of the above comparison.

**Corollary 4.4.** When \( \psi'(0+) > 0 \),

\[^1\] We use the standard convention that \( \sum_{n=1}^{0} := 0. \)
4.3 Gambler’s ruin

A slightly more elaborate version of the ruin problem is to consider the event that a certain wealth, say $a \geq 0$, can be achieved through the surplus process before ruin. This is also known as the gambler’s ruin problem. Define the stopping times,

$$
\tau^+_a = \inf \{ t > 0 : X_t > a \}
$$

and

$$
\tau^-_0 = \inf \{ t > 0 : X_t < 0 \}.
$$

We are interested in the events \( \{ \tau^+_a < \tau^-_0 \} \) and \( \{ \tau^-_0 < \tau^+_a \} \).

**Theorem 4.5.** For all \( q \geq 0 \), \( a > 0 \) and \( x < a \),

$$
E_x \left( e^{-q \tau^+_a} \mathbf{1}_{\{ \tau^+_a < \tau^-_0 \}} \right) = \frac{W^{(q)}(x)}{W^{(q)}(a)}.
$$

**(4.15)**

**Proof.** First we deal with the case that \( q = 0 \) and \( \psi'(0+) > 0 \) as in the previous proof. Since we have identified \( W(x) = \mathbb{P}_x(X_\infty \geq 0) / \psi'(0+) \), a simple argument, using the law of total probability and the Strong Markov Property, now yields, for \( x \in [0,a] \),

$$
\mathbb{P}_x(X_\infty \geq 0)
= E_x \left( \mathbb{P}_x \left( X_\infty \geq 0 \mid F_{\tau^+_a} \right) \right)
= E_x \left( \mathbf{1}_{\{ \tau^+_a < \tau^-_0 \}} \mathbb{P}_a(X_\infty \geq 0) \right) + E_x \left( \mathbf{1}_{\{ \tau^+_a > \tau^-_0 \}} \mathbb{P}_{X_{\tau^+_a}}(X_\infty \geq 0) \right). \quad \text{(4.16)}$$
The first term on the right-hand side of (4.16) is equal to
\[ P_a(X_{\infty} \geq 0) P_x(\tau_0^+ < \tau_0^-). \]
The second term on the right-hand side of (4.16) turns out to be also equal to zero. To see why, note that \( X_{\tau_0^-} < 0 \) and the claim follows by virtue of the fact that \( P_x(X_{\infty} \geq 0) = 0 \) for \( x < 0 \). We may now deduce that
\[ P_x(\tau_0^+ < \tau_0^-) = \frac{W(x)}{W(a)}, \quad (4.17) \]
and clearly the same equality holds even when \( x < 0 \) as both left and right hand side are identically equal to zero.

Next we deal with the case \( q > 0 \). Making use of the Esscher transform and recalling that \( X_{\tau_0} = a \), we have that
\[ E_x(e^{-q \tau_0^+ 1_{\{\tau_0^+ < \tau_0^-\}}}) = E_x(e^{\Phi(q)(X_{\tau_0} - x) - q \tau_0^+ 1_{\{\tau_0^+ < \tau_0^-\}}}) e^{-\Phi(q)(a-x)} \]
\[ = e^{-\Phi(q)(a-x)} E_x^{\Phi(q)}(\tau_0^+ < \tau_0^-) \]
\[ = e^{-\Phi(q)(a-x)} W_{\Phi(q)}(x) \]
\[ = e^{-\Phi(q)(a-x)} \frac{W(x)}{W(a)}. \]

Finally, to deal with the case that \( q = 0 \) and \( \psi'(0+) \leq 0 \), one needs only to take limits as \( q \downarrow 0 \) in the above identity, making use of monotone convergence on the left hand side and continuity in \( q \) on the right hand side thanks to the Continuity Theorem for Laplace transforms.

We can also consider the converse event that ruin occurs prior to achieving a desired wealth of \( a \geq 0 \).

**Theorem 4.6.** For any \( x \leq a \) and \( q \geq 0 \),
\[ E_x(e^{-q \tau_0^- 1_{\{\tau_0^- < \tau_0^+\}}}) = Z^{(q)}(x) - Z^{(q)}(a) \frac{W^{(q)}(x)}{W^{(q)}(a)}. \quad (4.18) \]

**Proof.** Fix \( q > 0 \). We have for \( x \geq 0 \),
\[ E_x(e^{-q \tau_0^- 1_{\{\tau_0^- < \tau_0^+\}}}) = E_x(e^{-q \tau_0^- 1_{\{\tau_0^- < \infty\}}}) - E_x(e^{-q \tau_0^- 1_{\{\tau_0^- < \tau_0^+\}}}). \]
Applying the strong Markov property at \( \tau_0^+ \) and using the fact that \( X_{\tau_0^+} = a \) on \( \{ \tau_0^+ < \infty \} \), we also have that
\[ E_x(e^{-q \tau_0^- 1_{\{\tau_0^- < \tau_0^+\}}}) = E_x(e^{-q \tau_0^- 1_{\{\tau_0^- < \tau_0^+\}}}) E_a(e^{-q \tau_0^- 1_{\{\tau_0^- < \infty\}}}). \]
Appealing to (4.8) and (4.15), we now have that
\[ E_x(e^{-q\tau_0} 1_{\tau_0 < \tau}) = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x) \]
\[ - \frac{W^{(q)}(x)}{W^{(q)}(a)} \left( Z^{(q)}(a) - \frac{q}{\Phi(q)} W^{(q)}(a) \right), \]
and the required result follows in the case that \( q > 0 \). The case that \( q = 0 \) is again dealt with by taking limits as \( q \downarrow 0 \).

\[ \square \]

4.4 Comments

The name ‘scale function’ for \( W \) was first used by [Bertoin (1992)] to reflect the analogous role it plays in (4.15) to scale functions for diffusions. The gambler’s ruin problem (also known as the two-sided exit problem) has a long history, starting with the early work of Zolotarev (1964) and Takács (1966), followed by Rogers (1990), all of whom dealt with (4.15) in the case \( q = 0 \). The case that \( q > 0 \) was dealt with by [Korolyuk (1975a)], and later by [Bertoin (1997)]. Treatments of the other identities in this chapter can be found in [Korolyuk (1974), Korolyuk (1975a), Korolyuk (1975b) and Bertoin (1997)]. A recent summary of the theory of scale functions and its applications can be found in [Cohen et al. (2012)].
Chapter 5
The Gerber–Shiu measure

Having introduced scale functions, we are now ready to look at the Gerber–Shiu measure in detail. In this chapter, we shall develop an idea from the previous chapter, involving Bernoulli trials of excursions from the minimum, to provide an identity for the expected occupation of the Cramér-Lundberg process until ruin. This identity will then play a key role in identifying an expression for the Gerber-Shiu measure. In fact, our analysis will work equally well for the setting that ruin occurs before the surplus achieves a pre-specified value.

5.1 Decomposing paths at the minimum

The main objective in this section is to prove the following decoupling for the path of the Cramér-Lundberg process when sampled over an independent and exponentially distributed random time, $e^q$, with rate $q > 0$.

**Theorem 5.1.** For all $q > 0$, the pair of random variables $X_{e^q} - X_{e^q - 1}$ and $X_{e^q}$ are independent.

**Proof.** The proof mimics the way in which we gave a probabilistic explanation of the Pollaczek-Khintchine formula. Recall that, in that setting, we sequentially defined $\Theta_0 = 0$ and, for all $k \geq 1$ such that $\Theta_{k-1} < \infty$,

$$\Theta_k = \inf\{t > \Theta_{k-1} : X_t < X_{\Theta_{k-1}}\}.$$

Moreover, on the event $\{\Theta_{k-1} < \infty\}$ we defined $\Delta_k = - (X_{\Theta_k} - X_{\Theta_{k-1}})$. Although the discussion in the context of the Pollaczek-Khintchine formula focused exclusively on the case that $\psi'(0+) > 0$, the times $\Theta_k$ are still well defined when $\psi'(0+) \leq 0$. In fact, in this regime, since $X_\infty = -\infty$ almost surely, it follows that $\Theta_k < \infty$ almost surely for all $k \geq 1$.

Now suppose that $\{e^q(k) : k \geq 1\}$ is a sequence of independent and identically distributed random variables. Define
\[ \ell = \min\{k \geq 1 : \Theta_k - \Theta_{k-1} > e^{(k)}_q\}, \]

the index of the first excursion from the minimum which exceeds in duration the correspondingly indexed exponential random variable in the sequence \(\{e^{(k)}_q : k \geq 1\}\). By the lack of memory property, we can now identify the pair \(\langle X_{e_q} - X_{e_q}, -X_{e_q} \rangle\) as equal in law to the pair

\[ \left( X_{\Theta_{\ell-1} + e^{(\ell)}_q} - X_{\Theta_{\ell-1}}, \sum_{j=1}^{\ell-1} \Delta_j \right). \quad (5.1) \]

Appealing again to the concept of Bernoulli trials, it is clear that both the random variable \(\ell\) and the \(\ell\)-th excursion from the minimum will be independent from the preceding \(\ell - 1\) excursions from the minimum. In particular, this implies that the pair in (5.1) are independent, and hence so are the pair \(\langle X_{e_q} - X_{e_q}, -X_{e_q} \rangle\).

Note that the above proof allows us to say a little more in the description of \(\langle X_{e_q} - X_{e_q}, -X_{e_q} \rangle\) thanks to the representation (5.1). Indeed, \(X_{e_q} - X_{e_q}\) is equal in law to \(X_{\tau_0}\) conditional on \(e_q < \tau_0\). Moreover, each of the \(\Delta_j\) in the sum are i.i.d. and equal in distribution to \(-X_{\tau_0}\) conditional on \(\tau_0 < e_q\), and \(\ell\) is independent and geometrically distributed with parameter \(P(e_q < \tau_0)\).

### 5.2 Resolvent densities

As an intermediate step to deriving a closed-form expression for the Gerber-Shiu measure, we are interested in computing the so-called, \(q\)-resolvent measure for the Cramér–Lundberg process, \(X\), killed on exiting \([0, \infty)\). Said another way, we are interested in characterising the measure

\[ U^{(q)}(a, x, dy) := \int_0^a e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau^{[0,a]})dt, \quad y \in [0, a], \]

where \(a > 0, q \geq 0\) and

\[ \tau^{[0,a]} = \tau_0^+ \wedge \tau_0^- . \]

If, for each \(x \in [0, a]\), a density of \(U^{(q)}(a, x, dy)\) exists with respect to Lebesgue measure, then we call it the resolvent density and denote it by \(u^{(q)}(a, x, y)\). (Note that this density can only be defined Lebesgue almost everywhere). It turns out that for each Cramér–Lundberg process, not only does a potential density exist, but we can write it in semi-explicit terms with the help of scale functions. Note, in the statement of the result, it is implicitly understood that \(W^{(q)}(z)\) is identically zero for \(z < 0\).

\footnote{Formally, an excursions from the minimum may be thought of as the sequence of segments of the trajectory of \(X\) given by \(\{X_{\Theta_{k-1} : t \in (0, \Theta_k)}\}\) for all \(k \geq 1\) such that \(\Theta_{k-1} < \infty\).}
Theorem 5.2. For each $q \geq 0$ and $a > 0$, the density $u^{(q)}(a,x,y)$ exists and is equal to
\[
u^{(q)}(a,x,y) = \frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y) \quad x,y \in [0,a] \quad (5.2)
\]

Lebsegue almost everywhere.

Proof. Define, for all $x, y \geq 0$ and $q > 0$,
\[
R^{(q)}(x,dy) = \int_0^\infty e^{-qt} \mathbb{P}_x(X_t \in dy, t < \tau_x^-)dt.
\]
One may think of $R^{(q)}$ as the $q$-resolvent measure for the process $X$ when killed on exiting $[0,\infty)$. Note that, for the same parameter regimes of $x, y$ and $q$, we can also write
\[
R^{(q)}(x,dy) = \frac{1}{q} \mathbb{P}_x(X_{e_q} \in dy, -X_{e_q} \geq 0),
\]
where, as usual, $e_q$ is an independent, exponentially distributed random variable with parameter $q > 0$. Appealing to Theorem 5.1, we have that
\[
R^{(q)}(x,dy) = \frac{1}{q} \mathbb{P}(x + (X_{e_q} - X_{e_q}) + X_{e_q} \in dy, -X_{e_q} \leq x)
\]
\[
= \frac{1}{q} \int_{[x-y,x]} \mathbb{P}(-X_{e_q} \in dz)\mathbb{P}(X_{e_q} - X_{e_q} \in dy - x + z).
\]
By duality, $X_{e_q} - X_{e_q}$ is equal in distribution to $X_{e_q}$, which itself is exponentially distributed with parameter $\Phi(q)$. In addition, the law of $-X_{e_q}$ has been identified in Corollary 4.2. Using these facts we may write, for $q, x, y \geq 0$,
\[
R^{(q)}(x,dy) = \left\{ \int_{[x-y,x]} \left( \frac{1}{\Phi(q)} W^{(q)}(dz) - W^{(q)}(z)dz \right) \Phi(q)e^{-\Phi(q)(y-x+z)} \right\} dy.
\]
In particular, this shows that, for $x, q \geq 0$, there exists a density, say $r^{(q)}(x,y)$, for the measure $R^{(q)}(x,dy)$. Now note that
\[
d[e^{-\Phi(q)z}W^{(q)}(z)] = e^{-\Phi(q)z}[W^{(q)}(dz) - \Phi(q)W^{(q)}(z)],
\]
and, hence, straightforward integration gives us
\[
r^{(q)}(x,y) = e^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x-y), \quad x,y,q \geq 0.
\]
Finally, we may use the expression for $r^{(q)}$ to compute the potential density $u^{(q)}$ as follows. First note that, with the help of the strong Markov property,
\[ qU(q)(a, x, dy) = P_x(X_{e_q} \in dy, X_{e_q} \geq 0, X_{e_q} \leq a) \]
\[ = P_x(X_{e_q} \in dy, X_{e_q} \geq 0) \]
\[ - P_x(X_{e_q} \in dy, X_{e_q} > 0, X_{e_q} \leq a) \]
\[ = P_x(X_{e_q} \in dy, X_{e_q} \geq 0) \]
\[ - P_x(X_{\tau[0,a]} = a, \tau[0,a] < e_q) P_a(X_{e_q} \in dy, X_{e_q} \geq 0). \]

The first and third of the three probabilities on the right-hand side above have been computed in the previous paragraph, the second probability is equal to
\[ \mathbb{E}_x(e^{-q_{\tau[a]}}; \tau[a] < \tau) = W(q)(x). \]

In conclusion, we have that, for \( q \geq 0 \) and \( x \in [0, a] \), \( U(q)(a, x, dy) \) has a density
\[ r(q)(x, y) - W(q)(x) = \frac{W(q)(x)}{W(q)(a)} r(q)(y), \quad y \in [0, a], \]
which, after a short amount of algebra, can be shown to be equal to the right-hand side of \( (5.2) \).

To complete the proof when \( q = 0 \), one may take limits in \( (5.2) \), noting that the measure \( U(q)(a, x, dy) \) is monotone increasing in \( y \) and that the scale function is continuous in \( q \) (again, thanks to the Extended Continuity Theorem for Laplace transforms).

The above proof contains the following corollary for the \( q \)-resolvent measure \( R(q)(x, dy) \).

**Corollary 5.3.** Fix \( q \geq 0 \). The \( q \)-resolvent measure for \( X \) killed on exiting \([0, \infty)\) has density given by
\[ r(q)(x, y) = e^{-\Phi(q)y}W(q)(x) - W(q)(x-y), \]
for \( x, y \geq 0 \).

### 5.3 More on Poisson integrals

Before putting together the results in the previous section to derive an expression for the Gerber–Shiu measure, we need to briefly return to the issue of Poisson integrals. One can improve up on the result in Theorem 3.4 with a little more work.

**Theorem 5.4.** Suppose that \( f : \mathbb{R} \times [0, \infty) \times (-\infty, 0] \times (0, \infty) \to \mathbb{R} \) is bounded and measurable. Then for all \( t \geq 0 \),
5.4 Gerber–Shiu measure and gambler’s ruin

We now have all the tools we need to provide a characterisation of the Gerber–Shiu measure \( \{X_t, \overline{X}_t, \underline{X}_t\} : t \geq 0 \) in terms of scale functions. In fact, we shall establish an identity for a slightly more general measure. With a slight abuse of notation, for a bounded continuous \( 0 < x < a \), define

\[
K(a, x, dy, dz) = E_x \left[ e^{-q\tau_0^+} ; -X_{\tau_0^+} \in dy, X_{\tau_0^-} \in dz, \tau_0^- < \tau_0^+ \right], \quad z \in [0, a], y \geq 0.
\]

Note that the Gerber–Shiu measure, previously called \( K^{(q)}(x, dy, dz) \), satisfies

\[
K^{(q)}(x, dy, dz) = K^{(q)}(\infty, x, dy, dz),
\]

for \( y, z \geq 0 \). Here is our main result.

**Theorem 5.5 (Gerber–Shiu measure).** Fix \( q \geq 0 \) and \( a > 0 \). Then

\[
K^{(q)}(a, x, dy, dz) = \lambda \left\{ \frac{W^{(q)}(x)W^{(q)}(a - z) - W^{(q)}(a)W^{(q)}(x - z)}{W^{(q)}(a)} \right\} F(z + dy)dz,
\]

for \( x, z \in [0, a] \) and \( y \geq 0 \). Moreover,

\[
K^{(q)}(x, dy, dz) = \lambda \left\{ e^{-\Phi^{(q)}(x)W^{(q)}(x) - W^{(q)}(x - y)} \right\} F(z + dy)dz, \quad (5.3)
\]

for \( y, z \geq 0 \).

**Proof.** Fix \( q \geq 0 \) and \( x \in [0, a] \). For the first identity, it suffices to show that, for all bounded continuous \( f : (0, \infty) \times [0, a] \to [0, \infty) \),

\[
E_x \left[ e^{-q\tau_0^+} f(-X_{\tau_0^+}, X_{\tau_0^-}) ; \tau_0^- < \tau_0^+ \right] = \lambda \int_0^a \int_{(0, \infty)} f(y, z) u^{(q)}(a, x, y) F(z + dy)dz. \quad (5.4)
\]

To this end, note that

\[
\{ \tau_0^- < \tau_0^+ \} = \bigcup_{i=1}^{\infty} \{ X_{t_i} < 0, \overline{X}_{t_i} \leq a, \underline{X}_{t_i} \geq 0 \},
\]
where the union is taken over disjoint events. It follows with the help of Theorem 5.4 that

\[ E_x \left[ e^{-q \tau_{t_i}} f(-X_{t_i}, X_{t_i}^-); \tau_{t_i}^- < \tau_{t_i}^+ \right] \]

\[ = E_x \left[ \sum_{i=1}^{\infty} 1_{(X_{t_i}^- < \xi, t_i < 0)} 1_{(X_{t_i}^- \leq a)} 1_{(X_{t_i}^- \geq 0)} e^{-q \tau_{t_i}} f(-X_{t_i}^- + \xi, X_{t_i}^-) \right] \]

\[ = \lambda \int_{(0, \infty)} E_x \left[ \int_0^\infty 1_{(u > \xi)} 1_{(X_{t_i}^- \leq a)} 1_{(X_{t_i}^- \geq 0)} e^{-q \tau_{t_i}} f(-X_{t_i}^- + u, X_{t_i}^-) \, du \right] F(du) \]

\[ = \lambda \int_{(0, \infty)} E_x \left[ \int_0^\infty 1_{(u > \xi)} 1_{(t_i < \xi, t_i < 0)} e^{-q \tau_{t_i}} f(-X_{t_i}^- + u, X_{t_i}^-) \, du \right] F(du) \]

\[ = \lambda \int_{(0, \infty)} \int_{(0, a)} 1_{(u > \xi)} e^{-q \tau_{t_i}} f(u - z, z) \, dz \, du \, F(du). \] (5.5)

Recalling the definition of \( U^{(q)}(a, x, dz) \), it follows that

\[ E_x \left[ e^{-q \tau_{t_i}} f(-X_{t_i}, X_{t_i}^-); \tau_{t_i}^- < \tau_{t_i}^+ \right] \]

\[ = \int_{(0, \infty)} \int_0^a 1_{(u > \xi)} f(u - z, z) U^{(q)}(a, x, dz) \, du \, dz \]

The right-hand side above is equal to (5.4), after a straightforward application of Fubini’s Theorem and a change of variables.

For the second part of the theorem, use monotonicity in \( a \) on the left- and right-hand side of (5.5) and take limits as \( a \uparrow \infty \). The consequence of this is that we recover the identity

\[ E_x \left[ e^{-q \tau_{t_i}} f(-X_{t_i}, X_{t_i}^-); \tau_{t_i}^- < \infty \right] = \lambda \int_0^a \int_{(0, \infty)} f(y, z) U^{(q)}(a, x, dz) \, dy \, dz, \]

for all \( x, q \geq 0 \), from which (5.3) follows.

\[ \square \]

### 5.5 Comments

In the broader context, Theorem [5.1] is a simple example of one of the several statements that concerns the so-called Wiener-Hopf factorisation for Lévy processes. See Chapter VI of [Bertoin (1996)] or Chapter 6 of [Kyprianou (2012)]. The method of analysing the path of the Cramér-Lundberg process (and indeed any random walk) through a sequence of excursions from the minimum was largely popularised by Feller. See for example Chapter XII of [Feller (1971)]. Many of the computations concerning resolvent densities are taken directly from [Bertoin (1997)] who deals
with general spectrally negative Lévy processes. However, older literature dealing
with the current setting also exists; see for example [Suprun (1976)].
Chapter 6
Reflection strategies

Let us now return to the first of the three cases in which we perturb the path of the Cramér–Lundberg process through the payments of dividends. Recall that a refraction strategy consists of paying dividends in such a way that, for a fixed threshold $a > 0$, any excess of the surplus above this level is instantaneously paid out. The cumulative dividend stream is thus given by $L_t := (a \vee X_t) - a$, for $t \geq 0$. The resulting trajectory satisfies the dynamics

$$U_t := X_t - L_t = X_t + a - (a \vee X_t), \quad t \geq 0,$$

with probabilities $\{P_x : x \in [0, a]\}$. The net present value of dividends paid until ruin is thus given by

$$\int_0^\varsigma e^{-qt} dL_t,$$

where $\varsigma = \inf\{t > 0 : U_t < 0\}$.

It is more convenient to measure the reflected process from the threshold $a$. In which case $\{a - U_t : t \geq 0\}$, under $P_{a-x}$, is equal in law to $\{Y^x_t : t \geq 0\}$, under $P$, where we recall that $Y^x_t := (x \vee X_t) - X_t, t \geq 0$. Moreover, from this point of view, the dividends that are paid out under $P_{a-x}$ are equal in law to the process $\{(x \vee X_t) : t \geq 0\}$ under $P$ and the time ruin corresponds to

$$\sigma_a = \inf\{t > 0 : Y^a_t > a\}.$$

The key object of interest in this chapter is the net present value of the dividends paid until ruin:

$$\int_0^{\sigma_a} e^{-qt} d(x \vee X_t) = \int_0^{\sigma_a} e^{-qt} 1_{X_t \geq a} dX_t,$$

(6.1)

where $q \geq 0$ is the force of interest.
6.1 Perpetuities

Suppose that \( N = \{ N_t : t \geq 0 \} \) is a Poisson process with rate \( \alpha > 0 \), \( \{ \xi_i : i \geq 0 \} \) is a sequence of i.i.d. random variables with common distribution function \( G \) and \( b > 0 \) is a constant. The process

\[
\gamma_t := bt + \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0
\]

is a compound Poisson process with positive jumps and positive drift. Then in the spirit of (2.3) we can compute its Laplace exponent as follows:

\[
E(e^{-\theta \gamma_t}) = e^{-\phi(\theta) t}, \quad t, \theta \geq 0,
\]

where

\[
\phi(\theta) = b\theta + \alpha \int_{(0,\infty)} (1 - e^{-\theta x}) G(dx).
\]

The key mathematical object that will help us analyse the net present value of dividends paid until ruin is the so-called perpetuity

\[
\int_0^{e_p} e^{-q \gamma_t} \, d\gamma_t,
\]

where, as usual, \( e_p \) is an independent exponential random variable with rate \( p \geq 0 \) and \( q \geq 0 \). We have the following main result which characterises its moments.

**Theorem 6.1.** For all \( n \in \mathbb{N} \),

\[
E \left[ \left( \int_0^{e_p} e^{-q \gamma_t} \, d\gamma_t \right)^n \right] = n! \prod_{k=1}^{n} \frac{1}{p + \phi(qk)}.
\]

**Proof.** Define, for \( t \geq 0 \),

\[
J_t = \int_t^{e_p} e^{-q \gamma_u} 1_{(u < e_p)} \, du.
\]

Our objective is to compute, for \( n \in \mathbb{N} \),

\[
\Psi_n := E(J_t^n).
\]

To this end, note that

\[
\frac{d}{dt} J_t^n = -nJ_t^{n-1} e^{-q \gamma_t} 1_{(t < e_p)},
\]

we obtain

\[
J_0^n - J_t^n = n \int_t^e e^{-q \gamma_u} 1_{(u < e_p)} J_u^{n-1} \, du.
\]

(6.3)
6.2 Decomposing paths at the maximum

Using that \( \{ \gamma_t : t \geq 0 \} \) has stationary and independent increments together with the lack of memory property, we have

\[
J_t = e^{-qt} \mathbf{1}_{(t < e^p)} f_0^t,
\]

where \( f_0^t \) is independent of \( \{ \gamma_u : u \leq t \} \) and has the same distribution as \( f_0 \). In conclusion, taking expectations in (6.3) we find that

\[
\Psi_n \left( 1 - \mathbb{E}(e^{-nq\gamma_t} \mathbf{1}_{(t < e^p)}) \right) = n^r \Psi_{n-1} \int_0^t \mathbb{E}(e^{-nq\gamma_u} \mathbf{1}_{(u < e^p)}) du.
\]

(6.4)

Since

\[
\mathbb{E}(e^{-nq\gamma_t} \mathbf{1}_{(t < e^p)}) = \exp\{-(p + \phi(nq))t\},
\]

it follows that

\[
\Psi_n = \frac{n}{p + \phi(nq)} \Psi_{n-1}.
\]

Iterating gives the result. \( \square \)

6.2 Decomposing paths at the maximum

Let us now look at the process \( \{ X_t - X_S : t \geq 0 \} \) and consider how the time it spends in the state zero, as well as the excursions it makes from the state zero, will reveal yet another path decomposition in the spirit of “Bernoulli trials” that we have already seen twice previously.

For convenience write \( Y_t \) in place of \( Y_0^t, t \geq 0 \). Define \( S_0 = 0 \) and

\[
S_0^* = \inf \{ t > 0 : Y_t > 0 \}.
\]

Then continue recursively, so that, for \( k \in \mathbb{N} \), on \( \{ S_{k-1} < \infty \} \),

\[
S_k = \inf \{ t > S_{k-1}^* : Y_t = 0 \}
\]

and \( S_k^* = \inf \{ t > S_k : Y_t > 0 \} \)

and set

\[
h_k = \sup_{s \in [S_{k-1}^*, S_k]} Y_s.
\]

Finally, define

\[
\nu_a = \min \{ k \geq 1 : h_k > a \}.
\]

In words, for \( k \geq 1 \), the intervals \( [S_{k-1}, S_{k-1}^*] \) are the successive periods that the process \( Y \) spends at the origin. Equivalently, they are the intervals of time that the process \( X \) is increasing. The intervals \( [S_k^*, S_k] \) are the successive periods that the process \( Y \) undertakes an excursion from the origin. Equivalently they are the intervals of time that the process \( X \) does not increase (and hence \( X \) makes an excursion away from \( \mathbb{X} \)). Moreover, appealing again to the logic of Bernoulli trials, \( \nu_a \) is the
index of the first excursion from the maximum which exceeds a height $a$. Accordingly, $\upsilon_a$ is geometrically distributed with rate

$$r_a = \int_{(0,a]} F(dx) P_x(\tau^- < \tau^+_0).$$

Next, again appealing to the concept of Bernoulli trials, the triples $\{(S_k^* - S_k, S_{k+1} - S_k^*, h_k) : k = 0, \ldots, \upsilon_a - 1\}$ are i.i.d., as well as being independent of $\upsilon_a$, and have the same law as the triple $(e^\lambda, \tau^+_0 - X^\sigma_{\tau^+_0})$ under the measure $\int_{(0,a]} F(dx) P_x(\cdot | \tau^+_0 < \tau^+_a)$. Finally, this sequence of triples is also independent of the triple $(S_{\upsilon_a} - S_{\upsilon_a}, S_{\upsilon_a+1} - S_{\upsilon_a}, h_{\upsilon_a})$, which itself is also independent of $\upsilon_a$ and has the law of $(e^\lambda, \tau^+_0 - X^\sigma_{\tau^+_0})$ under the measure $\int_{(0,a]} F(dx) P_x(\cdot | \tau^+_0 < \tau^+_a)$.

![Fig. 6.1 Decomposing the path of $X$ into excursions from its maximum.](image)

Now note that $X$ increases if and only if $Y$ is zero. Moreover, when it increases, it does so at a rate $c$. In other words,

$$X_t = c \int_0^t 1_{(Y_s=0)} ds. \quad (6.5)$$

With this in mind, note that the sequence of positive random variables
\{X_{S_k} - X_{S_0} : k = 0, \ldots, \nu_a - 1\} = \{c(S_k^* - S_k) : k = 0, \ldots, \nu_a - 1\}

is nothing more than an independent geometric number of independent exponential random variables, each with rate \(\lambda/c\). Therefore, the maximum height reached by \(X\) when it first drops a distance \(a\) below its previous maximum, \(X_{S_0} = c\sum_{k=0}^{
u_a-1} (S_k^* - S_k)\), is exponentially distributed with rate \(\lambda r_a/c\). In fact, if we define \(\chi_k = X_{S_k^* - 1}, k = 1, \ldots, \nu_a\), then the pairs \(\{(\chi_k, h_k) : k = 1, \ldots, \nu_a\}\) are times of arrival and the marks of a marked Poisson process up until the first mark exceeding \(a\) in value, where the arrival rate is \(\lambda/c\) and the marks are distributed according to

\[
H(dy) = \int_{(0,\infty)} F(dx)^{\mathbb{P}_{-a}}(-X_{\tau_0^+} \in dy), \quad y \geq 0.
\]

Appealing to the Poisson thinning theorem, this means that

\[
\{\{(\chi_k, h_k) : k = 1, \ldots, \nu_a - 1\}\}
\]

is equal in law to the times of arrival and the marks of a Poisson processes, say \(N_a = \{N_a^0 : t \geq 0\}\), with arrival rate \(\alpha_a = \lambda(1 - r_a)c\) and mark distribution

\[
\int_{(0,a]} F(dx)^{\mathbb{P}_{-a}}(-X_{\tau_0^+} \in dy|\tau_0^+ < \tau_1^+)\), \quad y \in [0, a],
\]

when sampled up to an independent and exponentially distributed random time, \(e_{p_a}\), with parameter \(p_A = \lambda r_a/c\). Note in particular, for \(t > 0\), on the event \(\{e_{p_a} > t\}\) the process \(N_a^0\) counts the number of excursions (of height less than \(a\)) that have occurred until the process \(X\) reaches a height \(t\).

Fix \(t > 0\). On the event \(\{t < e_{p_a}\}\), let us write

\[
\gamma := \inf\{s > 0 : X_s > t\}.
\]

That is, the amount of time it takes for \(X\) to climb to the level \(t\). We can split the time horizon \([0, \gamma]\) into the time that \(Y\) spends at zero (i.e. the time that \(X\) is climbing) plus the time that \(Y\) undertakes excursions from the origin (i.e. the time that \(X\) is stationary). On the one hand, thanks to the relation (6.5), the time that \(Y\) spends at zero until \(X\) reaches the level \(t\) is given by

\[
\int_0^\gamma 1_{[Y_s=0]} ds = \frac{1}{c} X_{\tau_a} = \frac{t}{c}.
\]
On the other hand, there are, in total, \( N^a \) excursions of \( Y \) from zero, say \( \{ \zeta_i : i = 1, \ldots, N^a \} \), where \( \zeta_i = S_i - S_{i-1} \). Moreover, the \( \zeta_i \) are i.i.d. with common distribution given by

\[
G_a(du) = \int_{(0,a]} F(d\tau) \mathbb{P}_{-\tau}(\tau_0^+ < \tau_a^+), \quad u \geq 0.
\]

In conclusion, we have that, on \( \{ t < e_{p_a} \} \),

\[
\gamma_t = bt + \sum_{i=1}^{N^a} \zeta_i, \quad t \geq 0,
\]

where \( b = 1/c \).

Putting all these pieces together, we develop the expression for net present value of dividends paid until ruin (6.1) for the special case that \( x = 0 \). Indeed, by making a simple change of variables, \( t \mapsto \gamma_s \), we have

\[
\int_0^{\sigma_a} e^{-q \gamma_s} dX_t = \int_0^{\sigma_a} \mathbb{I}_{(t < S_{\gamma_a})} e^{-q \gamma_s} dX_t = \int_0^{\sigma_a} \mathbb{I}_{(u < X_{S_{\gamma_a}})} e^{-q \gamma_s} du = \int_0^{e_{p_a}} e^{-q \gamma_u} du.
\]

We therefore see that, when the surplus process starts at the barrier \( a \), equivalently \( x = 0 \) in (6.1), the net present value of dividends until ruin is equal distribution to the perpetuity (6.2) for appropriate choices of \( p_a, b, \alpha_a \) and \( G_a \), as given above.

If we would be able to say a little more about these quantities, then we will be able to develop an expression for the \( n \)-th moments of the net present value of dividends paid at ruin, at least when the surplus process starts at the barrier, by using Theorem 6.1. That is, we would be able to give an explicit expression for

\[
\mathbb{E} \left[ \left( \int_0^{\sigma_a} e^{-q \gamma_s} dX_t \right)^n \right].
\]

Once again, scale functions come to our assistance.

### 6.3 Derivative of the scale function

Before we can proceed to develop identities using scale functions, we need to say a few words about differentiability as derivatives of the scale functions will appear in the forthcoming analysis. Recall that, for each \( q \geq 0 \), the function \( W^{(q)} \) is monotone.

**Lemma 6.2.** For \( x \geq 0 \),

\[
W^{(q)}(dx) = \frac{1}{c} \delta_0(dx) + \frac{\lambda + q}{c} W^{(q)}(x)dx - \left( \frac{\lambda}{c} \int_{(0,x]} W^{(q)}(x-y)F(dy) \right) dx. \quad (6.6)
\]
In particular,

$$W'(q)(x) = \frac{(\lambda + q)}{c} W(q)(x) - \left( \frac{\lambda}{c} \int_{0,x} W(q)(x-y)F(dy) \right),$$  \hspace{1cm} (6.7)$$

for $x > 0$, where $W'(q)(x)$ is the right derivative of $W(q)$ at $x$. Moreover, $W(q)$ is continuously differentiable on $(0, \infty)$ if and only if $F$ has no atoms.

Proof. Note also that, where as $W(q)(0^-) = 0$, we have that, for $q \geq 0$,

$$W(q)(0) = \lim_{\beta \rightarrow 0} \int_{0}^{\infty} e^{-\beta x} W(q)(x)dx$$

$$= \lim_{\beta \rightarrow \infty} \frac{\lambda}{\psi(\beta) - q}$$

$$= \lim_{\beta \rightarrow \infty} \frac{1}{c - \int_{0}^{\infty} e^{-\beta x} F(dx) - q/\beta}$$

$$= \frac{1}{c}.$$  

It follows that $W(q)dx$ has an atom at zero of size $1/c$. Integrating (4.1) by parts, we have, on the one hand,

$$\int_{0, \infty} e^{-\beta x} W(q)(dx) = \int_{0, \infty} e^{-\beta x} W(q)(dx) + 1/c = \frac{\beta}{\psi(\beta) - q}.$$  

On the other hand,

$$\frac{\lambda + q}{c} \int_{0}^{\infty} e^{-\beta x} W(q)(x)dx - \lambda \int_{0}^{\infty} e^{-\beta x} \int_{0,x} W(q)(x-y)F(dy)dx$$

$$= \frac{1}{c} (\lambda + q) - \frac{1}{c} \int_{0, \infty} e^{-\beta x} F(dx)$$

$$= \frac{1}{c} \lambda \int_{0, \infty} (1 - e^{-\beta x}) F(dx) + q$$

$$= \frac{1}{c} \frac{\beta}{\psi(\beta) - q} + q$$

$$= \frac{1}{c} \frac{\beta}{\psi(\beta) - q} - \frac{1}{c}.$$  

Comparing transforms, it follows that (6.6) holds. In particular, this implies that $W(q)$ is an absolutely continuous (and hence a continuous) function on $(0, \infty)$.

Thanks to the just-proved fact that $W(q)$ is a continuous function on $(0, \infty)$, inspecting the right hand side of (6.6), we see that its density with respect to Lebesgue measure is, in general, right-continuous. The formula for $W(q)'(x)$ on $(0, \infty)$ in (6.7) thus follows. Moreover, it is continuous if and only if the convolution is continuous. This is equivalent to requiring that $F$ has no atoms. In that case, on $(0, \infty)$, we
have that \( W(q) \) is absolutely continuous with a continuous version of its density, thus making it continuously differentiable.

### 6.4 Net-present-value of dividends at ruin

Let us use a completely different technique to examine the first moments of the net present value of dividends paid at ruin when the initial value of the surplus is \( a \). This will give us an expression from which we can glean the desired information about the quantities \( p \) and \( \phi \) mentioned in the previous section. Thereafter we can return to the more general problem of the \( n \)-th moments of the net present value of dividends paid until ruin when the initial value of the surplus is \( x \in [0, a] \).

**Lemma 6.3.** For all \( q \geq 0 \),

\[
E_a \left[ \int_0^\sigma e^{-qt} \, dL_s \right] = E \left[ \int_0^\sigma e^{-qt} \, dX_t \right] = W(q)(a) \frac{W'(q)}{W(q)(a)}. \tag{6.8}
\]

**Proof.** The proof works by splitting the integral \( \int_0^\sigma e^{-qt} \, dX_t \) at the time of the first jump of \( X \). Note that \( X \) initially increases at rate \( c \) for an independent and exponentially period of time, with rate \( \lambda \), and then undertakes a jump of size \( \xi_1 \) downwards. This jump kick-starts an excursion from the maximum. This excursion will either cause ruin, or its height will remain below the level \( a \) and \( X \) will return to its previous maximum, during which time no dividends will have been paid. Thereafter, by the strong Markov property, the process we have just describes begins again, except that future payments are additionally discounted by the time that has already lapsed. Let

\[
V_a = E \left[ \int_0^\sigma e^{-qt} \, dX_t \right].
\]

Following the description given above, we have, with the help of Theorem 4.5

\[
V_a = E \left[ \int_0^{\xi_1} e^{-qt} \, du \right] + E \left[ e^{-q\xi_1} E_{\xi_1} \left[ e^{-q\tau_1} \mathbf{1}_{\tau_1 < \tau_{1\lambda}} \right] \mathbf{1}_{\xi_1 \leq a} V_a \right]
\]

\[
= \frac{c}{q} E (1 - e^{-q\xi_1}) + E (e^{-q\xi_1}) \int_{[0,a]} \frac{W(q)(a-y)}{W(q)(a)} F(dy) V_a
\]

\[
= \frac{c}{\lambda + q} + \frac{\lambda}{\lambda + q} V_a \int_{[0,a]} \frac{W(q)(a-y)}{W(q)(a)} F(dy). \tag{6.9}
\]

Substituting (6.7) into (6.9) and solving for \( V_a \) now gives the statement of the theorem.

Taking account of the discussion in Sect. 6.2 as well as the conclusion of Theorem 6.1 we see that
6.5 Comments

As alluded to in the introduction, reflection strategies for Cramér–Lundberg processes emerge naturally from the control problem (1.7). Higher moments of the net present value of reflection strategies were first considered in Dickson and Waters (2004) for the case of exponentially distributed jumps. The connection with scale functions was made simultaneously in Renaud and Zhou (2007) and Kyprianou and Palmowski (2007). The latter of these two identifies the net present value of dividends paid until ruin as a perpetuity The review article Bertoin and Yor (2005) gives an interesting overview of perpetuities for Lévy processes in general. Decomposition the paths of the Cramér–Lundberg process at the maximum and identifying excursions through a marked Poisson process is an idea that goes back to Greenwood (1980). Ultimately, however, it is the natural continuous-time analogue of

\[ V_a = \frac{1}{p_a + \phi_a(q)} = \frac{W(q)(a)}{W_+(q)(a)}, \]

where \( \phi_a(\theta) = b\theta + \alpha_a \int_{(0,\infty)} (1 - e^{-\alpha x}) G_a(dx), \ \theta \geq 0. \) It follows that, for \( n \geq 1, \)

\[ E_a \left[ \left( \int_0^\xi e^{-q t} dL_t \right)^n \right] = E \left[ \left( \int_0^\xi e^{-q t} dX_t \right)^n \right] \]

\[ = n! \prod_{k=1}^n \frac{1}{p_a + \phi_a(qk)} \]

\[ = n! \prod_{k=1}^n \frac{W(qk)(a)}{W_+(qk)(a)}. \]

We are almost done. All we need now is to deal with the case that the surplus process starts from any \( x \in [0,a]. \) This turns out to be a minor adjustment to the formula above. Indeed, note that, when the Cramér–Lundberg process is issued from \( x \in [0,a], \) dividends are not paid until \( X \) reaches the threshold \( a, \) providing ruin does not occur beforehand. In that case, future dividend payments are discounted by the time elapsed until reaching the threshold. Hence, by the strong Markov property,

\[ E_x \left[ \left( \int_0^\xi e^{-q t} dL_t \right)^n \right] = E \left[ e^{-qn \tau^a} \mathbf{1}_{[\tau^a < \tau^c]} \right] V_a \]

In conclusion, taking account of Theorem 4.5, we have the following main result for this chapter.

**Theorem 6.4.** For \( q \geq 0 \) and \( x \in [0,a], \)

\[ E_x \left[ \left( \int_0^\xi e^{-q t} dL_t \right)^n \right] = n! \frac{W(qn)(x)}{W(qn)(a)} \prod_{k=1}^n \frac{W(qk)(a)}{W_+(qk)(a)}. \]
the discrete analysis that occurs when one decomposes paths at the minimum, and therefore relates back to Feller’s ideas on the renewal process of maxima (or indeed minima) for random walks. For further information on the smoothness of scale functions, the reader is referred to [Cohen et al. (2012)].
Chapter 7
Perturbation-at-maxima strategies

In this chapter, we dig a little deeper in the decomposition discussed in Sect. 6.2. In particular, we bring out in more detail the characterisation of the marked Poisson process of excursion heights in terms of scale functions. This will give us help us analyse the case where tax is paid in proportion to increments of the maximum.

Recall that if $\gamma : [0, \infty) \to [0, \infty)$, then the surplus process, when taxed at rate $\gamma$ with respect to its maximum process, yields an aggregate

$$U_t = X_t - \int_{(0,t]} \gamma(X_u) dX_u, \quad t \geq 0.$$  

We restrict ourselves to the cases mentioned in the introduction: the heavy-tax regime, for which $\gamma : [0, \infty) \to (1, \infty)$ and the light-tax regime, for which $\gamma : [0, \infty) \to (0, 1)$. The case of a reflection strategy, when $\gamma(x) = 1_{x \geq a}$, sits between these two regimes.

7.1 Re-hung excursions

Let us start by noting that, for all $x \geq 0$, under $P_x$,

$$U_t = A_t - Y_t, \quad t \geq 0,$$  

where we recall that $Y_t = X_t - X_t$ and

$$A_t = X_t - \int_0^t \gamma(X_u) dX_u = \bar{\gamma}_t(X_t), \quad t \geq 0,$$

with

$$\bar{\gamma}_{x}(s) := s - \int_x^s \gamma(y) dy = x + \int_x^s [1 - \gamma(y)] dy, \quad s \geq x.$$
The last equality shows us that process $A = \{A_t : t \geq 0\}$ is strictly increasing (resp. decreasing) when $\gamma$ belongs to the light-tax (resp. heavy-tax) regime on account of the fact that the same is true $\bar{\gamma}_x$. (This observation will also be important later as we will use the inverse function $\bar{\gamma}_x^{-1}$, which, accordingly, is well defined in both regimes.) Moreover, $A$ is stationary in value if and only if the process $Y$ is non-zero valued. As a consequence we may interpret (7.1) as a path decomposition in which excursions of $X$ from its maximum (equivalently excursions of $Y$ away from zero) are ‘hung’ off the trajectory of $A$ during its stationary periods. See Fig. 7.1 for a visual interpretation of this heuristic.

$$U_t$$

\[ x \]

Fig. 7.1 Re-hanging excursions from the process $A$ in the heavy-tax case.

In the light-tax regime, since $A$ is an increasing process which is stationary whenever the process $Y$ is non-zero valued, we have that

$$\tilde{U}_t := \sup_{s \leq t} U_s = A_{g_t} = A_t,$$

where $g_t = \sup\{s \leq t : Y_s = 0\}$. Hence, unless it is assumed that

$$\int_x^\infty (1 - \gamma(s))ds = \infty,$$  \hspace{1cm} (7.2)

in the light-tax regime, the perturbed process $U$ will have an almost surely finite global maximum.
In contrast, in the heavy-tax regime, so that $A$ is a decreasing, similar reasoning shows that

$$\bar{U}_t := \sup_{s \geq t} U_s = A_{d_t} = A_t,$$

where $d_t = \inf\{s > t : Y_s = 0\}$. Hence the process $U$ is always bounded by its initial value $x$.

Henceforth, our analysis of the process $U$ will centre around further analysis of the excursions of $X$ from its maximum $\bar{X}$. In particular, their representation through a marked Poisson process, such as we saw in Sect. 6.2, will proved to be extremely useful. In the next section, we shall revisit this marked Poisson process and look at a more detailed characterisation of its parameters in terms of scale functions.

### 7.2 Marked Poisson process revisited

Recall from the previous chapter that we write $Y_t = X_t - X_t$, for $t \geq 0$. Moreover, we recursively defined $S_0 = 0$,

$$S^+_0 = \inf\{t > 0 : Y_t > 0\},$$

and, for $k \in \mathbb{N}$, on $\{S_{k-1} < \infty\}$,

$$S^+_k = \inf\{t > S^+_{k-1} : Y_t = 0\} \text{ and } S^*_k = \inf\{t > S_k : Y_t > 0\}.$$

The $k$-th excursions from the maxima occurs over the time intervals $[S^+_{k-1}, S_k]$, and the height of the excursion is denoted by

$$h_k = \sup_{s \in [S^+_{k-1}, S_k]} Y_s.$$

Now let

$$\nu_\infty = \min\{k \geq 1 : S_k = \infty\},$$

the index of the first excursion which is infinite in length. Note, by Lemma 3.10 that if $\psi'(0+) \geq 0$, then $\nu_\infty = \infty$ almost surely. Moreover, if $\psi'(0+) < \infty$ then $\nu_\infty$ is geometrically distributed with parameter

$$r_\infty = \int_{(0,\infty)} F(dx) \mathbb{P}_{-x}(\tau^+_0 = \infty).$$

In both cases it is clear that we may equivalently

$$\nu_\infty = \min\{k \geq 1 : h_k = \infty\}.$$

Recall also that $\chi_k = \bar{X}_{S^+_{k-1}}$, where now, we allow the index to run from 1 to $\nu_\infty$. In the spirit of the reasoning in Sect. 6.2 we also note that
\{(X_k, h_k) : k = 1, \cdots, \nu_\infty\}

is equal to times of arrival and the marks of a marked Poisson process, say \(N = \{N_t : t \geq 0\}\), with rate \(\lambda / c\) up until the first mark which is infinite in value. Moreover, marks are i.i.d. (and independent of \(N\)) with common distribution

\[G(dy) := \int_{(0,\infty)} F(dx) \mathbb{P}(-X_{\tau_0} \in dy), \quad y \in [0, \infty].\]

Let us return to the event \(\{\tau^+_a < \tau^-_a\}\), for \(a > 0\). Note that, for \(0 \leq x \leq a\),

\[\mathbb{P}_x(\tau^+_a < \tau^-_a) = \mathbb{P}(h_k \leq x + X_k \text{ for } k = 1, \cdots, N_{a-x}).\]

Fig. 7.2 gives a visual explanation of this equality.

Classical theory for Poisson processes tells us that, conditional on \(\{N_{a-x} = n\}\),
the arrival times \(\{X_1, \cdots, X_n\}\) are equal in law to an ordered i.i.d. sample of uniformly distributed points on \([0, a-x]\). Then,
7.3 Gambler’s ruin for the perturbed process

\[
\mathbb{P}(h_k \leq x + X_t \text{ for } k = 1, \ldots, N_{a-x}) = \sum_{n=0}^{\infty} e^{-\frac{x}{n!}} \left( \frac{\lambda(a-x)}{c} \right)^n \left( \int_0^{a-x} G(x+t) \frac{1}{(a-x)} dt \right)^n = \exp \left\{ \frac{\lambda}{c} \int_0^a (G(x+t) - 1) dt \right\} = \exp \left\{ \frac{\lambda}{c} \int_0^a \overline{G}(y) dy \right\},
\]

where \( \overline{G}(y) = 1 - G(y) \), for \( y \geq 0 \).

It was proved earlier, however, that \( \mathbb{P}_x(\tau_+^a < \tau_0^-) = W(x)/W(a) \), where \( W \) is the scale function associated to \( X \). This leads us to the identity

\[
\frac{W(x)}{W(a)} = \exp \left\{ -\frac{\lambda}{c} \int_x^a \overline{G}(y) dy \right\}.
\]

(7.3)

Note that this confirms the conclusion of Lemma 9.23 that \( W \) is almost everywhere differentiable. Taking account of the above discussion, we come to rest at the following important result.

**Theorem 7.1.** Recall that \( W'_+ \), the right derivative of \( W \) on \((0, \infty)\). Then, for all \( x > 0 \),

\[
\frac{\lambda}{c} \overline{G}(x) = \frac{W'_+(x)}{W(x)}.
\]

(7.4)

Later on in this chapter, when using this result, the quantity \( \overline{G} \) will always appear in the context of a Lebesgue integral. In that case, it suffices to write \( W'/W \) on the right hand side of (7.4), without needing to refer to the right derivative of \( W \).

7.3 Gambler’s ruin for the perturbed process

Let

\[
T_0^- \ := \ \inf \{ t > 0 : U_t < 0 \},
\]

where we understand, as usual, \( \inf \emptyset := \infty \). We shall also use the earlier introduced stopping time

\[
\tau_+^a = \inf \{ t > 0 : X_t > a \} = \inf \{ t > 0 : \overline{X}_t > a \}.
\]

In the light-tax case, the function \( \overline{\gamma} \), we may write for all values \( b \) in the range of \( \overline{\gamma} \),

\[
\tau_+^{\overline{\gamma}^{-1}(b)} = T_{b^+},
\]

(7.5)

where

\[
T_{b^+} = \inf \{ t > 0 : U_t > b \}.
\]
Theorem 7.2. Fix $x > 0$ and assume (7.2) in the case of the light-tax regime. In the case of the heavy-tax regime, define

$$S^*(x) = \inf\{s \geq x : \bar{\gamma}_x(s) < 0\}.$$  

Then, for any $q \geq 0$, and $0 \leq x \leq a$ in the case of light-tax, resp. $0 \leq x < a < S^*(x)$ in the case of heavy-tax, we have

$$E_x[e^{-qT^+} 1_{\{T^+ < T^-\}}] = \exp \left( - \int_x^a \frac{W'(q)(\bar{\gamma}_x(s))}{W(q)(\bar{\gamma}_x(s))} \, ds \right),  \quad (7.6)$$

Before moving to the proof of this result, let us remark that, taking account of the equivalence (7.5) in the light-tax regime, (7.6) can be more conveniently written as

$$E_x[e^{-qT^+} 1_{\{T^+ < T^-\}}] = \exp \left( - \int_x^{\gamma^{-1}(a)} \frac{W'(q)(\bar{\gamma}_x(s))}{W(q)(\bar{\gamma}_x(s))} \, ds \right).$$

Proof (of Theorem 7.2). The proof does not distinguish between the two different regimes of light- and heavy-tax. All that is required in what follows is that $\gamma^{-1}(a) < \infty$. Thereafter, the proof needs little more than to recycle a number of existing computations we have already seen.

Using the Esscher transform and appealing to similar reasoning found in the proof of Theorem 7.1, we have

$$E_x[e^{-qT^+} 1_{\{T^+ < T^-\}}] = e^{-(a-x)\Phi(q)} E_x^{\Phi(q)}(T^+_a < T^-_0) = e^{-(a-x)\Phi(q)} \mathbb{P}^{\Phi(q)}(T^+_a < T^-_0),$$

where $e^{-(a-x)\Phi(q)} \mathbb{P}^{\Phi(q)}(T^+_a < T^-_0)$ for $k = 1, \ldots, N_{a-x}$

$$= e^{-(a-x)\Phi(q)} \sum_{n=0}^{\infty} e^{-\frac{1}{2}(a-x)} \frac{1}{n!} \left( \frac{\lambda(a-x)}{c} \right)^n \left( \int_0^{a-x} \frac{G^{\Phi(q)}(\bar{\gamma}_x(x+t))}{(a-x)} \, dt \right)^n$$

$$= \exp \left( - \int_0^{a-x} \Phi(q) + \frac{\lambda}{cG^{\Phi(q)}}(\bar{\gamma}_x(x+t)) \, dt \right),  \quad (7.7)$$

where $G^{\Phi(q)}$ plays the role of the quantity $G$, but under the measure $\mathbb{P}^{\Phi(q)}$ and $G^{\Phi(q)} = 1 - G^{\Phi(q)}$. In particular, recalling the conclusion of Theorems 2.4 and 7.1 we have for Lebesgue almost every $x \geq 0$,

$$\frac{\lambda}{cG^{\Phi(q)}}(x) = \frac{W'(\Phi(q))(x)}{W(\Phi(q))(x)}.$$  

Moreover, recalling from the definition (4.4) that $W(q)(x) = e^{\Phi(q)x}W(\Phi(q))(x)$, $x \geq 0$, we have that, for Lebesgue almost every $x \geq 0$,

$$\frac{W'(q)(x)}{W(q)(x)} = \Phi(q) + \frac{W'(\Phi(q))(x)}{W(\Phi(q))(x)}.$$
Putting the pieces together in (7.7) produces the required identity. □

Theorem 7.2 motivates some interesting observations concerning the event of ruin, \{T_0^- < \infty\}. First, suppose that we are in the heavy-tax regime and \(s^*(x) < \infty\). In that case
\[
\mathbb{P}_x(T_0^- < \infty) \geq \mathbb{P}_x(\tau_{s^*(x)}^+ < \infty) \lor \mathbb{P}_x(\tau_0^- < \infty).
\]
Indeed, on the event \{\tau_{s^*(x)}^+ < \infty\}, we have \(\bar{X}_{s^*(x)} - X_{s^*(x)} = 0\) and hence \(U_{s^*(x)} = A_{s^*(x)} = \bar{P}(s^*(x)) = 0\). Moreover, since \(U_t \leq X_t\) for all \(t \geq 0\), it follows that \(\{\tau_0^- < \infty\} \subseteq \{T_0^- < \infty\}\). In the event that \(\limsup\tau_{1 \infty} X_t = \infty\) almost surely, we have \(\mathbb{P}_x(\tau_{s^*(x)}^+ < \infty) = 1\). Otherwise, it follows that \(\mathbb{P}_x(\tau_0^- < \infty) = 1\). Either way, \(\mathbb{P}_x(T_0^- < \infty) = 1\).

Remaining in the heavy-tax regime, suppose that \(s^*(x) = \infty\). Then from (7.6), by taking limits as \(a \uparrow \infty\), we get an expression for the ruin probability,
\[
\mathbb{P}_x(T_0^- < \infty) = 1 - \exp\left(-\int_x^\infty \frac{W'(\bar{P}(s))}{W(\bar{P}(s))} \, ds\right).
\]
However, the right-hand side above turns out to be equal to 1. Recalling that \(W'(x)/W(x) = \lambda \overline{G}(x)/c\) for Lebesgue almost every \(x > 0\), since \(\overline{G}(x)\) is non-increasing on \((0, \infty)\) and \(\bar{P}(s) \leq x\) for all \(s \geq 0\), the claim follows.

Finally, in the light-tax regime, where necessarily \(s^*(x) = \infty\), the reasoning that leads to (7.8) still applies, from which one may deduce that this probability need not be unity. Indeed, suppose that we take the function \(\gamma\) to be simply constant in value, also denoted by \(\gamma \in (0, 1)\), and \(\psi'(0+ > 0\). In that case, \(\bar{P}_t(s) = (s - x)(1 - \gamma) + x\), and hence, noting that \(d[\log W(x)]/dx = W'(x)/W(x)\) Lebesgue almost everywhere on \((0, \infty)\), after a change of variables, we have
\[
\mathbb{P}_x(T_0^- < \infty) = 1 - \exp\left(-\frac{1}{(1 - \gamma)} \int_x^\infty \frac{W'(u)}{W(u)} \, du\right) = 1 - \left(\psi'(0+)W(x)\right)^{1/(1 - \gamma)},
\]
where we have used, from (4.2), that \(W(\infty) = 1/\psi'(0+)\).

### 7.4 Continuous ruin with heavy tax

Although the perturbed process is almost surely ruined in the heavy-tax regime, it is interesting to note that, unlike a Cramér-Lundberg process, there are two different ways to become ruined. The first, i.e. by a jump downwards, is a property inherited from the underlying process, \(X\). The other way of becoming ruined, which we refer to as continuous ruin, is the result of continuously passing the origin at the moment in time that an increment in \(X\) brings \(U\) along the curve \(\gamma\), just as it intersects the
Perturbation-at-maxima strategies

Said another way, type II creeping corresponds to the event that \( \{ \tau^+_x = T^+_0 \} \), in which case, as remarked upon above, \( U_{\tau^+_x} = 0 \). This can only happen with positive probability if \( s^*(x) < \infty \).

The following result is a corollary to Theorem 7.2 on account of the fact that its proof is identical, albeit that one replaces \( a \) by \( s^*(x) \).

**Corollary 7.3.** Fix \( x > 0 \), and suppose that \( \gamma : [0, \infty) \to (1, \infty) \) such that \( s^*(x) < \infty \). Then, for all \( q \geq 0 \),

\[
\mathbb{E}_x \left[ e^{-qT^+_0} 1_{\{ T^+_0 = \tau^+_x \}} \right] = \exp \left( - \int_x^{s^*(x)} \frac{W^{(q)}(\bar{\gamma}(s))}{W^{(q)}(\bar{\gamma}(s))} ds \right).
\]

If we take the case that \( \gamma(s) \) is a constant valued in \((1, \infty)\), again denoted by \( \gamma \), then we may simplify the formula in the above corollary. Indeed, we have \( \bar{\gamma}(s) = x - (s-x)(\gamma - 1) \) and hence \( s^*(x) = \gamma x / (\gamma - 1) \). Moreover, a straightforward computation, in a similar vein to the computation in (7.10), gives us

\[
\mathbb{P}_x(\text{continuous ruin}) = \exp \left( - \frac{1}{(\gamma - 1)} \int_0^x \frac{W'(u)}{W(u)} du \right) = \left( \frac{1}{cW(x)} \right)^{1/(\gamma - 1)},
\]

where we have used, from Lemma 9.23, that \( W(0) = 1/c \).

### 7.5 Net-present-value of tax

In the spirit of the Gerber–Shiu-type results presented in the previous sections, our final theorem for perturbed processes (with either light- or heavy-tax) considers the net-present-value of dividends paid until ruin.

**Theorem 7.4.** Fix \( x \geq 0 \) and assume (7.2) in the case of the light-tax regime. Then, for \( q \geq 0 \),

\[
\mathbb{E}_x \left[ \int_0^{T^+_0} e^{-qu} \gamma(X_u) dX_u \right] = \int_x^{s^*(x)} \exp \left( - \int_x^t \frac{W^{(q)}(\bar{\gamma}(s))}{W^{(q)}(\bar{\gamma}(s))} ds \right) \gamma(t) dt.
\]

**Proof.** Appealing to a straightforward change of variables and Fubini’s Theorem, we have

\[
\mathbb{E}_x \left[ \int_0^{T^+_0} e^{-qu} \gamma(X_u) dX_u \right] = \mathbb{E}_x \left[ \int_x^{s^*(x)} \mathbb{1}_{\{ u < T^+_0 \}} e^{-qu} \gamma(X_u) dX_u \right]
\]

\[
= \mathbb{E}_x \left[ \int_x^{s^*(x)} \mathbb{1}_{\{ \tau^+_x < T^+_0 \}} e^{-q\tau^+_x} \gamma(t) dt \right]
\]

\[
= \int_x^{s^*(x)} \mathbb{E}_x \left[ e^{-q\tau^+_x} \mathbb{1}_{\{ \tau^+_x < T^+_0 \}} \right] \gamma(t) dt.
\]
The proof is completed by taking advantage of the identity in (7.6).

We again get a simplification of this formula in the case that we take the function $\gamma(s)$ to be a constant, for either the light-tax or heavy-tax regime. For example when $\gamma(s)$ is equal to the constant $\gamma \in (0,1)$, one gets, for $x, q \geq 0$,

$$
E_x \left[ \int_0^{T^0} e^{-qu} \gamma(X_u) \text{d}X_u \right] = \frac{\gamma}{1 - \gamma} \int_x^\infty \left( \frac{W(q)(x)}{W(q)(z)} \right)^{1/(1-\gamma)} \text{d}z.
$$

7.6 Comments

The representation of the scale function in the form (7.3) is lifted from Theorem 8 of Chapter VII in [Bertoin, 1996]. This representation and the observation that excursions are re-hung from the process $A$ form a key part of the analysis in [Albrecher et al., 2008], [Kyprianou and Zhou, 2009] and [Kyprianou and Ott, 2012], in increasing degrees of generality, respectively.
Chapter 8
Refraction strategies

Let us return to the case of the refracted Cramér-Lundberg process. That is, the solution to the stochastic differential equation

\[ Z_t = X_t - \alpha \int_0^t \mathbf{1}_{\{Z_s > b\}} \, ds, \quad t \geq 0, \]  

also written as

\[ dZ_t = dX_t - \alpha \mathbf{1}_{\{Z_t > b\}} \, dt, \quad t \geq 0. \]

We shall charge ourselves with the task of providing identities for the probability of ruin as well as the net-present-value of dividends paid until ruin. As we have seen earlier for the case of a Cramér-Lundberg process, it turns out to be convenient to first derive an identity for the resolvent of the refracted process until first exiting a finite interval. It turns out that all identities can be written in terms of two scale functions for two different Cramér-Lundberg processes. As one might expect, however, these identities are somewhat more complicated however.

8.1 Pathwise existence and uniqueness

Before we can look at functionals which pertain to Gerber-Shiu theory, we are confronted with the more pressing issue of whether a solution to this SDE exists. As the reader might already suspect, problems my occur when \( \alpha \geq c \), as, in that case, when dividends are paid, it is at a higher rate than the premiums collected. We therefore assume throughout that

\[ 0 < \alpha < c. \]

**Theorem 8.1.** The SDE \[8.1\] has a unique pathwise solution.

**Proof.** Define the times \( T_n^\uparrow \) and \( T_n^\downarrow \) recursively as follows. We set \( T_0^\downarrow = 0 \) and, for \( n = 1, 2, \ldots \),
Refraction strategies

The difference between the two consecutive times $T^\downarrow_n$ and $T^\uparrow_n$ is strictly positive. Moreover, $\lim_{n \to \infty} T^\downarrow_n = \lim_{n \to \infty} T^\uparrow_n = \infty$, almost surely. Now we construct a solution to (8.1), $U = \{Z_t : t \geq 0\}$, as follows. The process is issued from $X_0 = x$ and

$$Z_t = \begin{cases} \ X_t - \alpha \sum_{i=1}^{n-1} (T^\downarrow_i - T^\uparrow_i), \quad & \text{for } t \in [T^\uparrow_n, T^\downarrow_{n+1}) \text{ and } n \geq 0, \\ \ X_t - \alpha \sum_{i=1}^{n-1} (T^\downarrow_i - T^\uparrow_i) - \alpha(t - T^\downarrow_n), \quad & \text{for } t \in [T^\uparrow_n, T^\downarrow_n) \text{ and } n \geq 1. \end{cases}$$

Note that the times $T^\uparrow_n$ and $T^\downarrow_n$, for $n = 1, 2, \ldots$, can then be identified as

$$T^\uparrow_n = \inf\{t > T^\downarrow_{n-1} : Z_t > b\}, \quad T^\downarrow_n = \inf\{t > T^\downarrow_{n-1} : Z_t < b\}.$$ 

Hence

$$Z_t = X_t - \alpha \int_0^t 1_{\{Z_s > b\}} \, ds, \quad t \geq 0.$$ 

For uniqueness of this solution, suppose that $\{Z^{(1)}_t : t \geq 0\}$ and $\{Z^{(2)}_t : t \geq 0\}$ are two pathwise solutions to (8.1). Then, writing

$$\Delta_t = U^{(1)}_t - U^{(2)}_t = -\alpha \int_0^t (1_{\{U^{(1)}_s > b\}} - 1_{\{U^{(2)}_s > b\}}) \, ds,$$

it follows from integration by parts that

$$\Delta^2_t = -2\alpha \int_0^t \Delta_s (1_{\{U^{(1)}_s > b\}} - 1_{\{U^{(2)}_s > b\}}) \, ds.$$ 

Thanks to the fact that $1_{\{x > b\}}$ is an increasing function, it follows from the above representation, that, for all $t \geq 0$, $\Delta^2_t \leq 0$ and hence $\Delta_t = 0$ almost surely. This concludes the proof of existence and uniqueness amongst the class of pathwise solutions.

Let us momentarily return to the reason why $U$ is referred to as a refracted Lévy processes. A simple sketch of a realisation of the path of $U$ (see for example Fig. 8.1) gives the impression that the trajectory of $U$ “refracts” each time it passes continuously from $(-\infty, b)$ into $(b, \infty)$, much as a beam of light does when passing from one medium to another.

The construction of the unique pathwise solution described above clearly shows that $U$ is adapted to the natural filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$ of $X$. Conversely, since, for all $t \geq 0, X_t = Z_t + \alpha \int_0^t 1_{\{Z_s > b\}} \, ds$, it is also clear that $X$ is adapted to the natural filtration of $U$. We can use this observation to reason that $U$ is a strong Markov process.
Fig. 8.1 A sample path of $U$ when the driving Lévy process is a Cramér-Lundberg process. Its trajectory “refracts” as it passes continuously above the horizontal dashed line at level $b$.

To this end, suppose that $T$ is a stopping time with respect to $\mathcal{F}$. Then define a process $\tilde{Z}$ whose dynamics are those of $\{Z_t : t \leq T\}$ issued from $x \in \mathbb{R}$ and, given $\mathcal{F}_T$, on the event that $\{T < \infty\}$, it continues to evolve on the time horizon $[T, \infty)$ as the unique solution, say $\tilde{Z}$, to (8.1) driven by the Lévy process $\tilde{X} = \{X_{T+s} - X_T : s \geq 0\}$ and issued from $Z_T$. Note that by construction, on $\{T < \infty\}$, the dependence of $\{\hat{Z}_t : t \geq T\}$ on $\{\hat{Z}_t : t \leq T\}$ occurs only through the value $\hat{Z}_T = Z_T$. Note also that for $t > 0$,

$$\hat{Z}_{T+t} = \tilde{Z}_t$$

$$= \tilde{Z}_T + \tilde{X}_t - \alpha \int_0^t 1_{\{\tilde{Z}_s > b\}} \mathrm{d}s$$

$$= x + X_T - \alpha \int_0^T 1_{\{Z_s > b\}} \mathrm{d}s + (X_{T+t} - X_T) - \alpha \int_0^t 1_{\{\hat{Z}_s > b\}} \mathrm{d}s$$

$$= x + X_T + t - \alpha \int_0^{T+t} 1_{\{\hat{Z}_s > b\}} \mathrm{d}s,$$

thereby showing that $\tilde{Z}$ solves (8.1) issued from $x$. Since (8.1) has a unique pathwise solution, this solution must be $\tilde{Z}$ and therefore possesses the strong Markov property.

8.2 Gambler’s ruin and resolvent density

Let us now introduce the stopping times for $Z$,

$$\kappa_a^+ := \inf\{t > 0 : Z_t > a\} \text{ and } \kappa_0^- := \inf\{t > 0 : Z_t < 0\},$$

where $a > 0$. We are interested in studying the ruin probability
For convenience, we will write Theorem 8.3.

Proof (of Theorem 8.2).

Write

\[ \int_0^{\kappa_0^-} e^{-qt} 1_{\{Z_t > b\}} dt. \]

Not unlike our treatment of the analogous object for the Cramér–Lundberg process, it turns out to be more convenient to first study the seemingly more complex two-sided exit problem. To this end, let \( \Lambda = \{ \Lambda_t : t \geq 0 \} \), where \( \Lambda_t = X_t - at \) and denote by \( P \) the law of the process \( \Lambda \) when issued from \( x \) (with \( E_x \) as the associated expectation operator). For each \( q \geq 0 \), \( W^{(q)} \) and \( Z^{(q)} \) denote, as usual, the \( q \)-scale functions associated with \( X \). We shall write \( W^{(q)} \) for the \( q \)-scale function associated with \( \Lambda \).

For convenience, we will write

\[ W^{(q)}(x; y) = W^{(q)}(x - y) + \alpha 1_{(x, y] > 0} \int_y^x (W^{(q)}(z - y) dz, \]

for \( x, y \in \mathbb{R} \) and \( q \geq 0 \). We have two main results concerning the gambler’s ruin problem, from which more can be said about the quantities (8.2) and (8.3).

Theorem 8.2. For \( q \geq 0 \) and \( 0 \leq x, b \leq a \) we have

\[ E_x \left( e^{-q\kappa_0^-} 1_{\{\kappa_0^- < \kappa_0^+\}} \right) = \frac{w^{(q)}(x; 0)}{w^{(q)}(a; 0)} \]

(8.4)

Theorem 8.3. For \( q \geq 0 \) and \( 0 \leq x, b \leq a \),

\[ \int_0^\infty e^{-qt} P_x (Z_t \in dy, t < \kappa_0^- \wedge \kappa_0^+) dt \]

\[ = 1_{[y \in [b, a)]} \left\{ \frac{w^{(q)}(x; 0)}{w^{(q)}(a; 0)} \left[ W^{(q)}(a - y) - W^{(q)}(x - y) \right] \right\} dy \]

\[ + 1_{[y \in [0, b)]} \left\{ \frac{w^{(q)}(x; 0)}{w^{(q)}(a; 0)} w^{(q)}(a; y) - w^{(q)}(x; y) \right\} dy. \]

Although appealing to relatively straightforward methods, the proofs are quite long, requiring a little patience.

Proof (of Theorem 8.2). Write \( p(x, \alpha) = E_x (e^{-q\kappa_0^-} 1_{\{\kappa_0^- < \kappa_0^+\}}) \). Suppose that \( x \leq b \). Then, by conditioning on \( \mathcal{F}_{\tau_0^+} \), we have

\[ p(x, \alpha) = E_x \left( e^{-q\tau_0^+} 1_{\{\tau_0^+ > \tau_0^-\}} \right) p(b, \alpha) = \frac{W^{(q)}(x)}{W^{(q)}(b)} p(b, \alpha), \]

(8.6)

where, in the last equality, we have used Theorem 4.5. Suppose now that \( b \leq x \leq a \). Using, Theorem 4.5, the strong Markov property (8.6) and the Gerber-Shiu measure.
(in the context of the gambler’s ruin problem) from Theorem 4.5, we have
\[ p(x, \alpha) \]
\[ = E_x \left( e^{-q_{\alpha}^+} 1_{\{T_{\alpha}^+ > \zeta_+^+\}} \right) + E_x \left( e^{-q_{\alpha}^-} 1_{\{T_{\alpha}^- < \zeta_+^+\}} \right) \]
\[ = \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)} + \frac{p(b, \alpha)}{W^{(q)}(b)} E_x \left( e^{-q_{\alpha}^-} 1_{\{T_{\alpha}^- < \zeta_+^+\}} \right) W^{(q)}(A_{\alpha}^-) \]
\[ = \frac{W^{(q)}(x-b)}{W^{(q)}(a-b)} + \frac{p(b, \alpha)}{W^{(q)}(b)} h(a, b, x), \]  
(8.7)
where
\[ h(a, b, x) \]
\[ = \int_0^{a-b} \left( (c - \alpha) \frac{W^{(q)}(a-b)}{W^{(q)}(a-b)} \right) \times \left[ \frac{W^{(q)}(x-b) W^{(q)}(a-b-y)}{W^{(q)}(a-b)} - \frac{W^{(q)}(x-b-y)}{W^{(q)}(a-b)} \right] F(d\theta) dy. \]

By setting \( x = b \) in (8.7) and recalling that \( W^{(q)}(0) = 1/(c - \alpha) \), we can now solve for \( p(b, \alpha) \). Indeed, we have

\[ p(b, \alpha) = W^{(q)}(b) \left\{ (c - \alpha) W^{(q)}(a-b) W^{(q)}(b) \right. \]
\[ \left. - \int_0^{a-b} \int_{(y, \infty)} W^{(q)}(b+y-\theta) W^{(q)}(a-b-y) F(d\theta) dy \right\}^{-1}. \]  
(8.8)

Next, we want to simplify the term involving the double integral in the above expression.

To this end, noting that for \( \alpha = 0 \) (the case that there is no refraction) we have, again by Theorem 4.5, that, for all \( x \geq 0 \),

\[ p(b, 0) = E_{b}\left( e^{-q_{\alpha}^+} 1_{\{T_{\alpha}^+ > \zeta_+^+\}} \right) = \frac{W^{(q)}(b)}{W^{(q)}(a)}. \]  
(8.9)

It follows, by comparing (8.8) (for \( \alpha = 0 \)) with (8.9), that

\[ \int_0^{a-b} \int_{(y, \infty)} W^{(q)}(b+y-\theta) W^{(q)}(a-b-y) F(d\theta) dy \]
\[ = c W^{(q)}(b) W^{(q)}(a-b) - W^{(q)}(a). \]  
(8.10)

As \( a \geq b \) is taken arbitrarily, we may take Laplace transforms in \( a \) on the interval \((b, \infty)\) of both sides of the above expression. Denote by \( \mathcal{L}_b \) the operator which satisfies \( \mathcal{L}_b f[\lambda] := \int_b^\infty e^{-\lambda x} f(x) dx \) and let \( \lambda > \Phi(q) \). For the left-hand side of (8.10),
we get with the help of Fubini’s Theorem
\[
\int_{b}^{\infty} e^{-\lambda x} \int_{0}^{\infty} W^{(q)}(b + y + \theta)W^{(q)}(x - b - y)dy F(d\theta) dx
= \frac{e^{-\lambda b}}{\psi(\lambda) - q} \int_{0}^{\infty} e^{-\lambda y} W^{(q)}(b + y - \theta)F(d\theta) dy.
\]
For the right-hand side of \[8.10\] we get
\[
\int_{b}^{\infty} e^{-\lambda x} \left(W^{(q)}(x-b)\circ W^{(q)}(b) - W^{(q)}(x)\right)dx
= \frac{e^{-\lambda b}}{\psi(\lambda) - q} c W^{(q)}(b) - \int_{b}^{\infty} e^{-\lambda x} W^{(q)}(x) dx,
\]
and so
\[
\int_{0}^{\infty} \int_{(y, \infty)} e^{-\lambda y} W^{(q)}(b + y - \theta)F(d\theta) dy
= c W^{(q)}(b) - (\psi(\lambda) - q)e^{\lambda b} L_{b}W^{(q)}[\lambda],
\]
for \(\lambda > \Phi(q)\). Our objective is now to use \[8.11\] to show that for \(q \geq 0\) and \(x \geq b\), we have
\[
\int_{0}^{\infty} \int_{(y, \infty)} W^{(q)}(b + y - \theta)\hat{W}^{(q)}(x - b - y)F(d\theta) dy
= -W^{(q)}(x) + (c - \alpha)W^{(q)}(b)\hat{W}^{(q)}(x - b)
- \alpha \int_{b}^{x} \hat{W}^{(q)}(x - y)W^{(q)}(y) dy.
\]
We will do this by taking Laplace transforms of \[8.12\] on both sides in \(x\) on \((b, \infty)\). To this end note that, by \[8.11\], it follows, with the help of Fubini’s Theorem, that the Laplace transform of the left-hand side of \[8.12\] equals
\[
\int_{b}^{\infty} e^{-\lambda x} \int_{0}^{\infty} W^{(q)}(b + y - \theta)\hat{W}^{(q)}(x - b - y)F(d\theta) dy dx
= \frac{e^{-\lambda b}}{\psi(\lambda) - \alpha \lambda - q} \left(c W^{(q)}(b) - (\psi(\lambda) - q)e^{\lambda b} L_{b}W^{(q)}[\lambda]\right),
\]
where \(\lambda > \varphi(q)\) and, for \(q \geq 0\), \(\varphi(q) = \text{sup}\{\theta \geq 0 : \psi(\theta) - c\theta = q\}\). (Note that \(\varphi\) is the right inverse of the Laplace exponent of \(\Lambda\).) Since
\[
L_{b}\left(\int_{b}^{x} f(x-y)g(y) dy\right)[\lambda] = (L_{b}f)[\lambda](L_{b}g)[\lambda]
\]
and, for \(\lambda > \Phi(q)\),
\( \mathcal{L}_b W^{(q)}[\lambda] = \lambda \mathcal{L}_b W^{(q)}[\lambda] - e^{-\lambda b} W^{(q)}(b) \)

(which follows from integration by parts), we have that the Laplace transform of the right-hand side of (8.12) is equal to the right-hand side of (8.13), for all sufficiently large \( \lambda \). Hence (8.12) holds for almost every \( x \geq b \). Because both sides of (8.12) are continuous in \( x \), we finally conclude that (8.12) holds for all \( x \geq b \).

To complete the proof, it suffices to plug (8.12) and the expression for \( h(a,b,x) \) into (8.7) and the desired identity follows after straightforward algebra. \( \Box \)

In anticipation of the proof of Theorem 8.3, we shall note here a particular identity which follows easily from (8.12). That is, for \( v \geq u \geq m \geq 0 \),

\[
\int_0^\infty \int_{(z,\infty)} W^{(q)}(z - \theta + m) \times \left[ \frac{\mathbb{P}(q)(v-m-z)}{\mathbb{P}(q)(v-m)} - \mathbb{P}(q)(u-m) \right] F(d\theta)dz \\
= -\frac{\mathbb{P}(q)(u-m)}{\mathbb{P}(q)(v-m)} \left( W^{(q)}(v) + \alpha \int_m^v \mathbb{P}(q)(v-z)W^{(q)r}(z)dz \right) \\
+ W^{(q)}(u) + \alpha \int_m^u \mathbb{P}(q)(u-z)W^{(q)r}(z)dz. \quad (8.14)
\]

**Proof (of Theorem 8.3).** Define for Borel \( B \subseteq [0,a] \) and \( x, q \geq 0 \),

\[ V^{(q)}(x,a,B) = \int_0^\infty e^{-qt} \mathbb{P}_x(Z_t \in B, t < \kappa_0^- \wedge \kappa_0^+)dt. \]

For \( x \leq b \), by the strong Markov property, Theorem 4.3 and Theorem 5.2, we have

\[
V^{(q)}(x,a,B) = \mathbb{E}_x \left( \int_0^{\tau_b^+} e^{-qt} 1_{\{Z_t \in B, t < \kappa_0^- \wedge \kappa_0^+, \tau_b^+ < \tau_0^+\}} dt \right) \\
+ \mathbb{E}_x \left( \int_{\tau_b^+}^{\tau_0^-} e^{-qt} 1_{\{Z_t \in B, t < \kappa_0^- \wedge \kappa_0^+, \tau_b^+ < \tau_0^+\}} dt \right) \\
= \mathbb{E}_x \left( \int_0^{\tau_b^+} e^{-qt} 1_{\{X_t \in B\}} dt \right) \\
+ \mathbb{E}_x \left( e^{-q\tau_b^+} 1_{\{\tau_b^+ < \tau_0^+\}} \right) V^{(q)}(b,a,B) \\
= \int_B \left( \frac{W^{(q)}(b-y)}{W^{(q)}(b)} - W^{(q)}(x-y) \right) dy \\
+ \frac{W^{(q)}(x)}{W^{(q)}(b)} V^{(q)}(b,a,B). \quad (8.15)
\]
Moreover, for \( b \leq x \leq a \) we have, using similar arguments,

\[
V^{(q)}(x, a, B) = \int_0^{\infty} e^{-qt} P_x(\Lambda_t \in B \cap [b, a], t < \tau^+_\theta) \, dt
\]

\[
= \int_{B \cap [b, a]} \left( \frac{\mathbb{W}^{(q)}(a - z)}{\mathbb{W}^{(q)}(a - b)} \mathbb{W}^{(q)}(x - b) - \mathbb{W}^{(q)}(x - z) \right) \, dz
\]

\[
+ \int_0^\infty \int_{(-\infty, -z]} \left\{ \int_B \left[ \frac{W^{(q)}(b - y)}{W^{(q)}(b)} W^{(q)}(z + \theta + b) - W^{(q)}(z + \theta + b - y) \right] \, dy
\]

\[
+ \frac{V^{(q)}(b, a, B)}{W^{(q)}(b)} W^{(q)}(z + \theta + b)
\}

\times \left[ \frac{\mathbb{W}^{(q)}(a - b - z)}{\mathbb{W}^{(q)}(a - b)} \mathbb{W}^{(q)}(x - b) - \mathbb{W}^{(q)}(x - b - z) \right] \Pi(d\theta) \, dz,
\]

where in the first equality we have used the strong Markov property and in the second equality we have again used the Gerber-Shiu measure from Theorem 2.2.

Next, we shall apply the identity (8.15) twice in order to simplify the expression for \( V^{(q)}(x, a, B) \), \( a \geq x \geq b \). We use it once by setting \( m = b, u = x, v = a \) and once by setting \( m = b - y \) and \( u = x - y, v = a - y \) for \( y \in [0, b] \). We obtain

\[
V^{(q)}(x, a, B)
\]

\[
= \int_{B \cap [b, a]} \left( \frac{\mathbb{W}^{(q)}(a - z)}{\mathbb{W}^{(q)}(a - b)} \mathbb{W}^{(q)}(x - b) - \mathbb{W}^{(q)}(x - z) \right) \, dz
\]

\[
+ \int_{B \cap [0, b]} \left\{ \frac{W^{(q)}(b - y)}{W^{(q)}(b)} \left( - \frac{\mathbb{W}^{(q)}(x - b)}{\mathbb{W}^{(q)}(a - b)} w^{(q)}(a; 0) + w^{(q)}(x; 0) \right)
\]

\[
- \left( - \frac{\mathbb{W}^{(q)}(x - b)}{\mathbb{W}^{(q)}(a - b)} w^{(q)}(a; y) + w^{(q)}(x; y) \right) \right\} \, dy
\]

\[
+ \frac{V^{(q)}(b, a, B)}{W^{(q)}(b)} \left( - \frac{\mathbb{W}^{(q)}(x - b)}{\mathbb{W}^{(q)}(a - b)} w^{(q)}(a; 0) + w^{(q)}(x; 0) \right).
\]

Setting \( x = b \) in (8.16), we get an expression for \( V^{(q)}(b, a, B) \) in terms of itself. Solving this and then putting the resulting expression for \( V^{(q)}(b, a, B) \) back in (8.15) and (8.16) leads to (8.5) which completes the proof. □
8.3 Resolvent density with ruin

The two expressions we are interested in, namely the ruin probability and the expected net present value of dividends paid until ruin, can both be extracted from the identity for the potential measure of $U$ on $[0,\infty)$,

$$\int_0^\infty P_x(Z_t \in B, t < \kappa^-_0) \, dt = \lim_{a \uparrow \infty} V(x, a, B),$$

where $B$ is any Borel set in $[0,\infty)$. Note that the limit is justified by monotone convergence. In order to describe this potential measure, let us introduce some more notation. Recall that $\varphi$ was defined as the right inverse of the Laplace exponent of $\Lambda$, so that

$$\varphi(q) = \sup\{\theta \geq 0 : \psi(\theta) - \alpha \theta = q\}.$$

Corollary 8.4. For $x, y, b \geq 0$ and $q \geq 0$

$$\int_0^\infty e^{-qt} P_x(Z_t \in dy, t < \kappa^-_0) \, dt = 1_{\{y \in (b,\infty)\}}$$

$$\left\{ \frac{W(q)(x,0)}{\alpha \int_0^b e^{-\varphi(q)z} W(q)(z) \, dz} e^{-\varphi(q)y} - \frac{W(q)(x-y)}{W(q)(x)} \right\} dy$$

$$+ 1_{\{y \in [0,b)\}} \left\{ \frac{\int_b^\infty e^{-\varphi(q)z} W(q)(z-y) \, dz}{\int_0^b e^{-\varphi(q)z} W(q)(z) \, dz} - \frac{W(q)(x,0) - W(q)(x,y)}{W(q)(x)} \right\} dy. \quad (8.17)$$

Proof. Assume that $q > 0$. We begin by noting that, from the representation of $W(q)$ in (4.4), it is straightforward to deduce that, for all $x, q > 0$,

$$\lim_{a \uparrow \infty} \frac{W(q)(a-x)}{W(q)(a)} = e^{-\varphi(q)x}.$$

Note that, for each $q \geq 0$, $\varphi(q) \geq \Phi(q)$ and hence, appealing to the same representation in (4.4) for both $W(q)$ and $\Phi(q)$, it also follows that, for all $q, x > 0$,

$$\lim_{a \uparrow \infty} \frac{W(q)(a-x)}{W(q)(a)} = 0.$$

For $q > 0$, the result we are after is obtained by dividing the numerator and denominator of each of the first terms in the curly brackets of (8.5) by $W(q)(a)$ and taking limits as $a \uparrow \infty$, making use of the above two observations. The case that $q = 0$ is handled by taking limits as $q \downarrow 0$ in (8.17). \qed

Now we are in a position to derive expressions for (8.2) and (8.3).

Corollary 8.5. For $x \geq 0$, if $E(X_1) \leq \alpha$ then $P_x(\kappa^-_0 < \infty) = 1$. Otherwise, when $E(X_1) > \alpha$, we have
Proof. Let $U_q = \inf_{t \leq T} e_t$ and, as usual, $e_q$ denotes an independent and exponentially distributed random variable with mean $1/q$. Note that for $q > 0$,

$$
\mathbb{P}_x (k_0 - \infty) = 1 - \frac{E(X_1) - \alpha}{1 - \alpha W(b)} \left( W(x) + \alpha 1_{(x \geq b)} \int_b^x W(y) W'(y) dy \right). \tag{8.18}
$$

Computing the integral above from (8.17) is relatively straightforward and gives us, for $x, b \geq 0$ and $q > 0$,

$$
\mathbb{E}_x (e^{-qk_0} 1_{(k_0 < \omega)}) = z^{(q)}(x) - q \int_0^x e^{-\varphi(q)y} W^{(q)}(y) dy \int_b^y e^{-\varphi(q)W^{(q)}(y)} dy W^{(q)}(x;0) + q \int_b^x W^{(q)}(x-z) dz + q \int_b^b W^{(q)}(x-z) dz - q \int_0^x W^{(q)}(z) dz - q \alpha \int_b^x W^{(q)}(x-z) W^{(q)}(z-b) dz, \tag{8.19}
$$

where

$$
z^{(q)}(x) = Z^{(q)}(x) + \alpha q \int_b^x \psi^{(q)}(x-z) W^{(q)}(z) dz, \quad x \in \mathbb{R}, q \geq 0.
$$

The details of the computation are left to the reader.

Although it is not immediately obvious, it turns out that the last four terms in (8.19) sum to precisely zero. Indeed, in the case that $x \leq b$, this observation is straightforward, noting that the two integrals from $b$ to $x$ are identically zero and the second integral may be replaced by $\int_b^x W^{(q)}(x-z) dz = \int_b^x W^{(q)}(z) dz$ on account of the fact that $W^{(q)}$ is identically zero on $(-\infty, 0)$. In the case that $x > b$, the last four terms of (8.19) can be easily rearranged to be equal to $h(x-b)$, where

$$
h(u) = q \int_0^u \psi^{(q)}(z) dz - q \int_0^u W^{(q)}(z) dz - q \alpha \int_0^u \psi^{(q)}(z) W^{(q)}(u-z) dz.
$$

Taking Laplace transforms of $h$ and using (4.1), we easily verify that $h$ is identically zero.

In conclusion, we have that, for $x, b \geq 0$ and $q > 0$,

$$
\mathbb{E}_x (e^{-qk_0} 1_{(k_0 < \omega)}) = z^{(q)}(x) - q \int_0^\infty e^{-\varphi(q)y} W^{(q)}(y) dy \int_b^x e^{-\varphi(q)W^{(q)}(y)} dy W^{(q)}(x;0).
$$
The expression for the ruin probability in (8.18) by taking limits on the left- and right-hand side above as \( q \downarrow 0 \). On the left-hand side, thanks to monotone convergence, the limit is equal to \( P_x(\kappa_q^{-} < \infty) \). Computing the limits on the right-hand side is relatively straightforward, taking account of the fact that

\[
\int_0^\infty e^{-\varphi(q)z}W^{(q)}(z)dz = \frac{1}{\varphi(q)\alpha} \quad (8.20)
\]

and the fact that

\[
\lim_{q \downarrow 0} \frac{q}{\varphi(q)} = \lim_{q \downarrow 0} \frac{\psi(\varphi(q)) - \alpha \varphi(q)}{\varphi(q)} = 0 \lor (E(X_1) - \alpha).
\]

The details are again left to the reader.

**Corollary 8.6.** For \( x \geq 0 \),

\[
\mathbb{E}_x \left( \int_0^{\kappa_q^-} e^{-q't} \alpha 1_{[z_t > b]} dt \right) = -\alpha \int_b^x \varphi(q)^{-1}(z-b)dz + \frac{W^{(q)}(x) + \alpha 1_{[x > b]} \frac{\varphi(q)^{-1}}{\varphi(q)} \int_b^x \varphi(q)^{-1}(x-y)W^{(q)}(y)dy}{\varphi(q) \int_0^\infty e^{-\varphi(q)y}W^{(q)}(y+b)dy}.
\]

**Proof.** The proof is a simple exercise in integrating the potential measure (8.17) over \((b, \infty)\).

---

### 8.4 Comments

The terminology “refraction” comes from [Gerber (2006b)](http://example.com). See also [Gerber and Shiu (2006a)](http://example.com). The majority of the arguments in this chapter are taken from [Kyprianou and Loeffen (2010)](http://example.com), where \( X \) is taken to be a general spectrally negative Lévy process. The question of existence and uniqueness for the SDE (8.1) when \( X \) is a general Lévy process has not yet been handled in the literature. In particular, when \( X \) has no Gaussian component, although still a simple-looking SDE, (8.1) lies outside of the realms of standard theory.
Chapter 9
Concluding discussion

On the one hand, the use of scale functions would appear to have made many of
the problems we have looked at solvable. On the other hand, one may also question
the extent to which we have solved the posed problems as our scale functions are
only defined up to a Laplace transform. Therefore we have arguably only provided
a solution “up to the inversion of a Laplace transform”. It would be nice to have
some concrete examples. It turns out that few concrete examples are known and
they are quite difficult to produce in general. Nonetheless, we shall show that there
is still sufficient analytical structure known for a general scale function to justify
their use, in particular when moving to a bigger class of processes to model the
surplus process.

9.1 Mixed-exponential jumps

In general, it is quite hard to construct the scale function $W^{(q)}$ for a Cramér–
Lundberg process. There is one family of Cramér–Lundberg processes, however,
for which there is a reasonable degree of tractability as far scale functions are con-
cerned. In order to specify these processes, let us define the arrival rate

$$\lambda = \sum_{j=1}^{m} a_j/\rho_j$$  \hspace{1cm} (9.1)

and the claims distribution by

$$F(dx) = \frac{1}{\lambda} \left( \sum_{j=1}^{m} a_j e^{-\rho_j x} \right) dx, \hspace{1cm} x > 0,$$  \hspace{1cm} (9.2)

where $m \in \mathbb{N}$, and, for $j = 1, \cdots, m$, the coefficients $a_j$ and $\rho_j$ are strictly positive.
For convenience, we also assume that the $\rho_j$ are arranged in increasing order. The
premium rate $c$ may be taken valued in $(0, \infty)$ without restriction.
It is easily verified that the Laplace exponent is given by

\[ \psi(\theta) = c\theta - \theta \sum_{j=1}^{m} \frac{a_j}{\rho_j(\rho_j + \theta)}, \quad \theta \geq 0. \]

A little thought reveals that the exponent \( \psi \) is in fact well defined as a mapping from \( \mathbb{C} \) to \( \mathbb{C} \), provided one is prepared to see it as a meromorphic function which has only negative poles at points \( z = -\rho_j \). Henceforth, we shall treat \( \psi \) in this broader sense, as a complex valued function. We know from our previous analysis that, for \( q > 0 \), the equation \( \psi(\theta) = q \) has a unique solution \( z = \Phi(q) \) in the half-plane \( \text{Re}(\theta) > 0 \), and we also know that this solution is real. It can be proved (see Sect. 9.5) that if we look for other roots of the equation \( \psi(\theta) = q \) on the complex plane, then there are precisely \( N \) of them and they are all to be found on the negative part of the real axis. If we write these solutions as \( \theta = -\zeta_j \), for \( j = 1, \ldots, m \), then it also turns out that they satisfy the interlacing property

\[ 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \cdots < \zeta_m < \rho_m. \tag{9.3} \]

The following lemma gives an explicit formula for the scale function when expressed in terms of the roots \( \zeta_j \) and the first derivative of the Laplace exponent.

**Lemma 9.1.** If \( X \) is a Cramér–Lundberg process with \( \lambda \) and \( F \) given by (9.1) and (9.2), respectively, then, for all \( q > 0 \), the scale function is given by

\[ W(q)(x) = e^{\Phi(q)x} \frac{\psi'(\Phi(q))}{\psi'(q)} + \sum_{j=1}^{m} e^{-\zeta_j x} \frac{\psi'(q)}{\psi'(q)} - \frac{\psi'(q)}{\psi'(q)}, \quad x \geq 0. \tag{9.4} \]

**Proof.** We give a sketch proof. Knowing that \( \{-\zeta_m, \ldots, -\zeta_1, \Phi(q)\} \) are all roots of the equation \( \psi(\theta) = 0 \) in \( \mathbb{C} \), the basic idea is to use partial fractions to write

\[ \frac{1}{\psi(\theta) - q} = \frac{c_0}{(\theta - \Phi(q))} + \sum_{j=1}^{m} \frac{c_j}{(\theta + \zeta_j)}, \quad \theta \in \mathbb{C}. \tag{9.5} \]

To determine the constants \( c_0 \), note that, for example,

\[ \frac{1}{\psi'(q)} = \lim_{\theta \to \Phi(q)} \frac{(\theta - \Phi(q))}{\psi(\theta) - q} = c_0 + \sum_{j=1}^{m} \lim_{\theta \to \Phi(q)} \frac{c_j (\theta - \Phi(q))}{(\theta + \zeta_j)} = c_0. \]

One derives \( c_1, \ldots, c_m \) similarly. The identity (9.4) now follows by inverting (9.5) in a straightforward way.

\[ ^1 \text{This is the part of the proof that we have not justified.} \]
9.2 Spectrally negative Lévy processes

One of the advantages working with scale functions is that all of the results, as well as many of their proofs, are practically identical if we replace the Cramér–Lundberg process by a general spectrally negative Lévy process. Membership of this class of Lévy process has the simple requirement that, in the almost sure sense, there are no positive jumps and that paths are not monotone.

A simple example of a spectrally negative Lévy process would be the resulting object we would get from adding a Cramér–Lundberg process to a (scaled) independent Brownian motion, i.e.

\[ X_t = \sigma B_t + ct - \sum_{i=1}^{N_t} \xi_i, \quad t \geq 0, \quad (9.6) \]

where \( \{B_t : t \geq 0\} \) is a standard independent Brownian motion and \( \sigma \geq 0 \).

In general, spectrally negative Lévy processes can be characterised through their Laplace exponent, also denoted by \( \psi \), which satisfies

\[ \psi(\theta) := \frac{1}{t} \log \mathbb{E}[e^{\theta X_t}], \]

and is well defined for \( \theta \geq 0 \). The Lévy-Khintchine formula, which is normally cited for the characteristic exponent of a Lévy process, also identifies the Laplace exponent in its general form as

\[ \psi(\theta) = a\theta + \frac{1}{2} \sigma^2 \theta^2 + \int_{(0,\infty)} (e^{-\theta x} - 1 + \theta x 1_{(x<1)}) \nu(dx), \quad \theta \geq 0, \quad (9.7) \]

where \( a \in \mathbb{R}, \sigma^2 \geq 0 \) and the so-called Lévy measure \( \nu \) is a (not-necessarily finite) measure concentrated on \((0,\infty)\) which satisfies the integrability condition

\[ \int_{(0,\infty)} (1 \wedge x^2) \nu(dx) < \infty. \]

Although \( \psi \) looks more complicated than for the case of a Cramér–Lundberg process, its shape is essentially the same. Indeed, it is not difficult to show, again by differentiation, that \( \psi \) is a strictly convex function which satisfies \( \psi(0) = 0 \) and \( \psi(\infty) = \infty \). Moreover, just as with the Cramér–Lundberg process, we have \( \mathbb{E}(X_1) = \psi'(0+) \in [-\infty, \infty) \). Accordingly, we may also work with the right inverse of \( \psi \),

\[ \Phi(q) := \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad q \geq 0. \]

Just as is the case with Cramér–Lundberg processes, the quantity \( \Phi(0) \) is strictly positive if and only if \( \psi'(0+) < 0 \) and otherwise, it is zero. In this respect, Fig. 2.1 could equally depict the Laplace exponent of a general spectrally negative Lévy process.
We should note that the class of spectrally negative Lévy processes contains the class of Cramér–Lundberg processes. Indeed, this can be seen by taking \( \nu(dx) = \lambda F(dx) \) on \((0, \infty)\) and \(\sigma = 0\). In that case, as \( F \) is now a finite measure, (9.7) can be written

\[
\psi(\theta) = \left( a + \lambda \int_{(0,1)} xF(dx) \right) \theta - \lambda \int_{(0,\infty)} (1 - e^{-\theta x}) F(dx), \quad \theta \geq 0.
\]

The exclusion of monotone paths from the definition of spectrally negative Lévy processes would force us to take

\[
c := a + \lambda \int_{(0,1)} xF(dx) > 0,
\]

which brings us back to the class of Cramér–Lundberg processes. Indeed, requiring that \( \nu \) is a finite measure and that the quantity \( c \), as defined in (9.8), is strictly positive is a necessary and sufficient condition for a spectrally negative Lévy process to be a Cramér–Lundberg process.

Describing the paths of the Lévy process \( X = \{X_t : t \geq 0\} \) associated to \( \psi \) is not as straightforward as in the case of a Cramér–Lundberg process. It is clear that the quadratic term \( a\theta + \sigma^2 \theta^2/2 \) is the result of an independent linear Brownian component \( \{at + \sigma B_t : t \geq 0\} \) in \( X \). The integral term in \( \psi \) can be written

\[
\int_{(0,\infty)} (e^{-\theta x} - 1 - \theta x 1_{(x<1)}) \nu(dx) = \sum_{n \geq 1} \left\{ c_n \theta - \lambda_n \int_{(2^{-n}, 2^{-n-1})} (1 - e^{-\theta x}) \frac{\nu(dx)}{\lambda_n} \right\}
\]

\[
-\lambda_0 \int_{(0,\infty)} (1 - e^{-\theta x}) \frac{\nu(dx)}{\lambda_0}, \quad \theta \geq 0;
\]

where \( \lambda_0 = \nu((1, \infty)) \) and, for \( n \geq 1 \),

\[
c_n = \int_{(2^{-n}, 2^{-n-1})} x\nu(dx) \quad \text{and} \quad \lambda_n = \nu((2^{-n}, 2^{-(n-1)}))
\]

Note that, if \( \lambda_n = 0 \) for some \( n \geq 1 \), then we should understand the relevant term on the right-hand side of (9.9) as absent.

The decomposition (9.9) gives us the intuitive understanding that, for the given triple \((a, \sigma, \nu)\), the associated spectrally negative Lévy process may be seen as the independent sum of a linear Brownian component, a series of Cramér–Lundberg processes and the negative of a compound Poisson process. The special choice of \( c_n \), for \( n \geq 1 \), means that each of the Cramér–Lundberg processes have zero mean (in fact they are martingales). Moreover the \( n \)-th Cramér–Lundberg process experiences jumps whose magnitude falls strictly into the interval \((2^{-n}, 2^{-(n-1)})\). Meanwhile, the compound Poisson process experiences jumps which are of magnitude strictly greater than 1.
9.2 Spectrally negative Lévy processes

The resulting path of the superimposition of these processes can be quite varied. Indeed, almost surely over any finite time horizon, there will be an infinite (albeit countable) number of jumps if and only if \( \nu \) is an infinite measure. Moreover, \( X \) has paths of bounded variation (almost surely over each finite time horizon) if and only if \( \int_{(0,1)} x \nu(dx) < \infty \) and \( \sigma = 0 \).

For the class of spectrally negative Lévy processes, we may also define scale functions \( W^{(q)}(x) \), \( q \geq 0, x \in \mathbb{R} \), in exactly the same way as we did for Cramér–Lundberg processes. In particular, for \( q \geq 0 \), \( W^{(q)}(x) = 0 \) for \( x < 0 \) and otherwise, on \([0, \infty)\), it is the unique right-continuous increasing function whose Laplace transform satisfies

\[
\int_{0}^{\infty} e^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q). \quad (9.10)
\]

The majority of the main results presented in the previous chapters and indeed many of their proofs, are still valid when the setting of a Cramér–Lundberg process is replaced by a general spectrally negative Lévy process. We are thus brought again to the question of the existence of concrete examples of scale functions.

Not surprisingly, it is also difficult to find closed form examples of scale functions for a spectrally negative Lévy process that is not a Cramér–Lundberg process. Here are a couple of related examples however.

A spectrally negative \( \alpha \)-stable process, for \( \alpha \in (1, 2) \), has Lévy measure

\[
\nu(dx) = \frac{k_\alpha}{x^{1+\alpha}} dx, \quad x > 0,
\]

where \( k_\alpha \) is a constant that can be chosen appropriately so that \( \psi(\theta) = \theta^\alpha \), \( \theta \geq 0 \).

Note that \( \psi'(0+) = 0 \) and hence the \( \alpha \)-stable process oscillates.

Denote by

\[
\mathcal{E}_{\alpha, \beta}(x) = \sum_{n \geq 0} \frac{x^n}{\Gamma(n\alpha + \beta)}, \quad x \in \mathbb{R}, \quad (9.11)
\]

the two-parameter Mittag-Leffler function. It is characterised by its Fourier transform. Specifically, for \( \lambda \in \mathbb{R}, \theta \in \mathbb{C} \) and \( \Re(\theta) > \lambda^{1/\alpha} \), we have

\[
\int_{0}^{\infty} e^{-\theta x} x^{\beta-1} \mathcal{E}_{\alpha, \beta}(\lambda x^{\alpha}) dx = \frac{\theta^{-\beta}}{\theta^{\alpha} - \lambda}. \quad (9.12)
\]

We recognise immediately that

\[
W^{(q)}(x) = x^{\alpha-1} \mathcal{E}_{\alpha, \alpha}(qx^{\alpha}), \quad q, x \geq 0.
\]

Suppose that we look at the \( \alpha \)-stable process under the Escher transform (2.7). As alluded to above, the main part of this result is still valid for general spectrally
negative Lévy processes. In particular, we note that the class of spectrally negative Lévy processes is closed under the Escher transformation. For each $\gamma \geq 0$ and $\alpha \in (1, 2)$, if $(X, \mathbb{P})$ is an $\alpha$-stable process then $(X, \mathbb{P}^\gamma)$ is a spectrally negative Lévy process with Laplace exponent

$$\psi(\theta) = (\theta + \gamma)^\alpha - \gamma^\alpha, \quad \theta \geq -\gamma.$$ 

This process is also known as a tempered stable process. Like the $\alpha$-stable process, it has no Gaussian component. Similarly to the conclusion of Theorem 2.4 for Cramér–Lundberg processes, the effect of the Esscher transform is to transform the Lévy measure to

$$\nu(\gamma)(dx) = k\alpha e^{-\gamma x}x^{1+\alpha}1_{x>0}.$$ 

Note also that $\psi'(0+) = \psi'(\gamma) > 0$ and hence the process drifts to $+\infty$.

Just as above, we may note derive by inspection of (9.11) the following identity for $W^{(q)}$, the $q$-scale function of $(X, \mathbb{P}^\gamma)$:

$$W^{(q)}(x) = e^{-\gamma x}x^{\alpha-1}e_{\alpha,\alpha}(q + \gamma^\alpha)x^\alpha, \quad x \geq 0.$$ 

9.3 Analytic properties of scale functions

Despite the fact that, for any given spectrally negative Lévy process, we are generally unable to invert the transform (9.10), we can nonetheless get a general understanding of the shape of the general scale function. Here is a summary of some of the known facts, most of which can be derived from the Laplace transform (9.10).

Continuity at the origin

For all $q \geq 0$,

$$W^{(q)}(0+) = \begin{cases} 0 & \text{if } \sigma > 0 \text{ or } \int_{(0,1)} x \nu(dx) = \infty, \\ c^{-1} & \text{if } \sigma = 0 \text{ and } \int_{(0,1)} x \nu(dx) < \infty, \end{cases}$$

(9.13)

where $c = -a - \int_{(-1,0)} x \nu(dx)$.

Derivative at the origin

For all $q \geq 0$,
9.3 Analytic properties of scale functions

\[ W^{(q)'}(0+) = \begin{cases} 
\frac{2}{\sigma^2} & \text{if } \sigma > 0 \\
\infty & \text{if } \sigma = 0 \text{ and } \nu((0,\infty)) = \infty \\
(q + \nu(-\infty,0))/c^2 & \text{if } \sigma = 0 \text{ and } \nu((0,\infty)) < \infty.
\end{cases} \quad (9.14) \]

**Behaviour at \(+\infty\) for \(q = 0\)**

As \(x \uparrow \infty\) we have

\[ W(x) \sim \begin{cases} 
1/\psi'(0+) & \text{if } \psi'(0+) > 0 \\
e^{\Phi(0)x}/\psi'(\Phi(0)) & \text{if } \psi'(0+) < 0.
\end{cases} \quad (9.15) \]

When \(E(X_1) = 0\) a number of different asymptotic behaviours may occur. For example, if \(\phi(\theta) := \psi(\theta)/\theta\) satisfies \(\phi'(0+) < \infty\) then \(W(x) \sim x/\phi'(0+)\) as \(x \uparrow \infty\).

**Behaviour at \(+\infty\) for \(q > 0\)**

As \(x \uparrow \infty\) we have

\[ W^{(q)}(x) \sim e^{\Phi(q)x}/\psi'(\Phi(q)) \quad (9.16) \]

and thus there is asymptotic exponential growth.

**Smoothness**

It is known that if \(X\) has paths of bounded variation then, for all \(q \geq 0\), \(W^{(q)}|_{(0,\infty)} \in C^1(0,\infty)\) if and only if \(\nu\) has no atoms. In the case that \(X\) has paths of unbounded variation, it is known that, for all \(q \geq 0\), \(W^{(q)}|_{(0,\infty)} \in C^1(0,\infty)\). Moreover if \(\sigma > 0\) then \(C^1(0,\infty)\) may be replaced by \(C^2(0,\infty)\). Clearly this picture is incomplete.

Taking account of the fact that the Laplace transform of \(W^{(q)}\) is expressed in terms of \(\psi'\), which, itself, can be considered as a type of analytical transform of the measure \(\nu\), it is not surprising that there is an intimate connection between the smoothness of the scale functions and the Lévy measure. Whilst there are a number of existing results results connecting the two (see Sect. 9.5), a general result remains at large. The following result has been conjectured by Ron Doney.

**Conjecture 9.2.** Let \(\nu(x) = \nu((x,\infty))\) for \(x > 0\). For \(k = 0, 1, 2, \ldots\)

1. if \(\sigma^2 > 0\) then
   \[ W \in C^{k+3}(0,\infty) \iff \nu \in C^k(0,\infty), \]
2. if \(\sigma = 0\) and \(\int_{(0,1)} x\nu(dx) = \infty\) then
   \[ W \in C^{k+2}(0,\infty) \iff \nu \in C^k(0,\infty), \]
3. if \(\sigma = 0\) and \(\int_{(0,1)} x\nu(dx) < \infty\) then
Concluding discussion

\[ W \in C^{k+1}(0, \infty) \iff \nabla \in C^k(0, \infty). \]

Concavity and convexity

If \( x \mapsto \nabla(x), x > 0, \) is a completely monotone function\(^2\) then, for all \( q > 0, W^{(q)'}(x), x > 0, \) is convex. Note in particular, the latter implies that there exists an \( a^* \geq 0 \) such that \( W^{(q)} \) is concave on \( (0, a^*) \) and convex on \( (a^*, \infty) \). In the case that \( \psi'(0+) \geq 0 \) and \( q = 0 \) the same conclusion holds with \( a^* = \infty \), which is to say that \( W \) is a concave function. More generally, we have the following result.

**Theorem 9.3.** Suppose that \( \nabla \) is log-convex. Then for all \( q \geq 0, W^{(q)} \) has a log-convex first derivative.

(Note that a completely monotone function is log-convex and a log-convex function is also convex.)

**9.4 Engineered scale functions**

Within the class of spectrally negative Lévy process, there are a number of methods for generating examples of scale functions with \( q = 0 \). Rather than trying to invert (9.10) for a given \( \psi \), the idea is to construct a \( \psi \) corresponding to a given \( W \). We outline one method here, which is based around the *Wiener-Hopf factorisation*.

For the purpose of this discussion, the Wiener-Hopf factorisation concerns the Laplace exponent of \( X \) and takes the form:

\[ \psi(\theta) = (\theta - \Phi(0))\phi(\theta), \quad \theta \geq 0, \tag{9.17} \]

where \( \phi \), is a so-called *Bernstein function*. Specifically, the term \( \phi(\theta) \) must necessarily take the form

\[ \phi(\theta) = \kappa + \delta \theta + \int_{(0, \infty)} (1 - e^{-\theta x}) \Upsilon(dx), \quad \theta \geq 0, \tag{9.18} \]

where \( \kappa, \delta \geq 0 \) and \( \Upsilon \) is a measure concentrated on \( (0, \infty) \) which satisfies \( \int_{(0, \infty)} (1 \wedge x) \Upsilon(dx) < \infty \). Roughly speaking, this factorisation can be proved by recalling that \( \psi(\Phi(0)) = 0 \) and then manually factoring out \( (\theta - \Phi(0)) \) from \( \psi \) by using integration by parts to deal with the integral part of (9.7). Thereby, it turns out that

\[^2\] A function \( f : (0, \infty) \to [0, \infty) \) is completely monotone if, for all \( n \in \mathbb{N} \),

\[ (-1)^n \frac{d^n f(x)}{dx^n} \geq 0. \]
\[ \Upsilon((x, \infty)) = e^{\Phi(0)x} \int_x^\infty e^{-\Phi(0)u} \psi(u) du \quad \text{for } x > 0, \]  

(9.19)

\[ \delta = \sigma^2/2 \text{ and } \kappa = \psi'(0+) \vee 0. \]

It is remarkable that \( \psi \) is also the Laplace exponent of a subordinator\(^3\) which is sent to the cemetery state +\( \infty \) after an independent an exponentially distributed random time with rate \( \kappa \). If we write this process \( H = \{H_t : t \geq 0\} \) and let \( \zeta = \inf\{t > 0 : H_t = +\infty\} \), then, for all \( t \geq 0 \),

\[ \phi(\theta) = -\frac{1}{\theta} \log E \left( e^{-\theta H_1} I_{(t<\zeta)} \right), \quad \theta \geq 0. \]

What is even more remarkable is that the range of \( \{H_t : t < \zeta\} \) agrees precisely with the range of the process \( \{X_t : t \geq 0\} \). Accordingly we call \( H \) the descending ladder height process of \( X \).

In the special case that \( \Phi(0) = 0 \), that is to say, the process \( X \) does not drift to \(-\infty\) or equivalently that \( \psi'(0+) \geq 0 \), it can be shown that the scale function \( W \) describes the renewal measure of \( H \). Indeed, recall that the renewal measure of \( H \) is defined by

\[ V(dx) = \int_0^\infty dt \cdot P(H_t \in dx, t < \zeta), \quad \text{for } x \geq 0. \]  

(9.20)

Calculating its Laplace transform we get the identity

\[ \int_0^\infty e^{-\theta x} V(dx) = \frac{1}{\phi(\theta)} \quad \text{for } \theta > 0. \]  

(9.21)

Recall that we can integrate (9.10) by parts and get

\[ \int_{(0,\infty)} e^{-\theta x} W(dx) = \frac{\theta}{\psi(\theta)} = \frac{1}{\phi(\theta)}, \quad \theta \geq 0. \]

Hence, it appears that \( W \) agrees precisely with the renewal function, \( V \), of the subordinator \( H \) that appears in the Wiener-Hopf factorisation.

It can be shown similarly that when \( \Phi(0) > 0 \), the scale function is related to the renewal measure of \( H \) by the formula

\[ W(x) = e^{\Phi(0)x} \int_0^x e^{-\Phi(0)y} V(dy), \quad x \geq 0. \]  

(9.22)

This relationship between scale functions and renewal measures of subordinators lies at the heart of the approach we shall describe in this section for engineering scale functions. A key to the method is the fact that one can find in the literature several subordinators for which the renewal measure is known explicitly. Should these subordinators turn out to be the descending ladder height process of a spectrally neg-

\(^3\) A subordinator is a Lévy process with non-decreasing paths.

\(^4\) Recall our convention that an exponential random variable with rate 0 is defined to be the infinite valued with probability 1.
ative Lévy process, then this would give an exact expression for its scale function. Said another way, we can build scale functions using the following approach.

\textbf{Step 1.} Choose a subordinator, say $H$, with Laplace exponent $\phi$, for which one knows its renewal measure, $V$, or equivalently, in light of (9.21), one can explicitly invert the Laplace transform $1/\phi(\theta)$.

\textbf{Step 2.} Choose a constant $\varphi \geq 0$ and verify whether the relation

$$\psi(\theta) := (\lambda - \varphi)\phi(\theta), \quad \theta \geq 0,$$

defines the Laplace exponent of a spectrally negative Lévy process.

\textbf{Step 3.} Once steps 1 and 2 are verified, then the scale function of the spectrally negative Lévy process we have generated is given by (9.22).

Of course, for this method to be useful we should first provide necessary and sufficient conditions for the pair $(H, \varphi)$ to belong to the Wiener-Hopf factorisation of a spectrally negative Lévy process.

The following theorem shows how one may identify a spectrally negative Lévy process $X$ (called the \emph{parent process}) for a given pair $(H, \varphi)$. The proof follows by a straightforward manipulation of the Wiener-Hopf factorization (9.17).

\textbf{Theorem 9.4.} Suppose that $\hat{H}$ is a subordinator, killed at rate $\kappa \geq 0$, with drift $\delta \geq 0$ and Lévy measure $\Upsilon$ which is absolutely continuous with non-increasing density. Suppose further that $\varphi \geq 0$ is given such that $\varphi \kappa = 0$. Then there exists a spectrally negative Lévy process $X$, henceforth referred to as the ‘parent process’, whose descending ladder height process is precisely the process $H$. The Lévy triple $(a, \sigma, \nu)$ of the parent process is uniquely identified as follows. The Gaussian coefficient is given by $\sigma = \sqrt{2\delta}$. The Lévy measure is given by

$$\nu(x) = \varphi \Upsilon(x, \infty) + \frac{d\Upsilon}{dx}(x).$$

Finally, $a$ can be chosen such that

$$\psi(\theta) = (\theta - \varphi)\phi(\theta),$$

for $\theta \geq 0$ where $\phi(\theta) = -\log \mathbb{E}(e^{-\theta H_1})$.

Conversely, the killing rate, drift and Lévy measure of the descending ladder height process associated to a given spectrally negative Lévy process $X$ satisfying $\varphi = \Phi(0)$ are also given by the above formulae.

Let us conclude this section by presenting a concrete example of how this methodology works in practice. Consider a spectrally negative Lévy process which is the parent process of a (killed) tempered stable process. That is to say a subordinator with Laplace exponent given by
\[
\phi(\theta) = \kappa - c \Gamma(-\alpha) \left( (\gamma + \theta)^\alpha - \gamma^\alpha \right),
\]

where \( \alpha \in (-1,1) \setminus \{0\} \), \( \gamma \geq 0 \) and \( c > 0 \).

The corresponding Lévy measure is given by

\[
\nu(dx) = c \frac{\phi + \gamma}{x^{\alpha+1}} e^{-\gamma x} dx + c \frac{\alpha + 1}{x^{\alpha+2}} e^{-\gamma x} dx, \quad x > 0.
\]

(9.25)

This indicates that the jump part of the parent process is the result of the independent sum of two spectrally negative tempered stable processes with stability parameters \( \alpha \) and \( \alpha + 1 \). We also note from Theorem 9.4 that \( \sigma = \sqrt{2 \zeta} = 0 \), indicating the presence of a Gaussian component.

If \( 0 < \alpha < 1 \) the jump component is the sum of an infinite activity negative tempered stable subordinator and an independent spectrally negative tempered stable process with infinite variation. If \( -1 \leq \alpha < 0 \) the jump part of the parent process is the independent sum tempered stable subordinator with stability parameter \( 1+\alpha \) and exponential parameter \( \gamma \), and an independent negative compound Poisson subordinator with jumps from a gamma distribution with \( -\alpha \) degrees of freedom and exponential parameter \( \gamma \).

One easily deduces the following transformations as special examples of (9.12) for \( \theta, \lambda > 0 \),

\[
\int_0^\infty e^{-\theta x} x^{\alpha-1} \mathcal{E}_{\alpha,\alpha}(\lambda x^{\alpha}) dx = \frac{1}{\theta^\alpha - \lambda},
\]

(9.26)

and

\[
\int_0^\infty e^{-\theta x} x^{-\alpha-1} \mathcal{E}_{-\alpha,-\alpha}(\lambda^{-1} x^{-\alpha}) dx = \frac{\lambda}{\lambda - \theta^\alpha} - 1,
\]

(9.27)

valid for \( \alpha > 0 \) and \( \alpha < 0 \), respectively. Together with the well-known rules for Laplace transforms concerning primitives and tilting, we may quickly deduce the following expressions for the scale functions associated to the parent process with Laplace exponent given by (9.24) such that \( \kappa \phi = 0 \).

If \( 0 < \alpha < 1 \) then, for \( x \geq 0 \),

\[
W(x) = \frac{e^{\theta x}}{\kappa + c \Gamma(-\alpha) \gamma^{\alpha}} \int_0^x e^{-(\gamma + \phi) y} y^{\alpha-1} \mathcal{E}_{\alpha,\alpha} \left( \frac{\kappa + c \Gamma(-\alpha) \gamma^{\alpha}}{\kappa + c \Gamma(-\alpha) \gamma^{\alpha}} \right) dy.
\]

If \( -1 < \alpha < 0 \), then, for \( x \geq 0 \),

\[
W(x) = \frac{e^{\theta x}}{\kappa + c \Gamma(-\alpha) \gamma^{\alpha}}
\]

\[
+ \frac{c \Gamma(-\alpha) e^{\theta x}}{(\kappa + c \Gamma(-\alpha) \gamma^{\alpha})^2} \int_0^x e^{-(\gamma + \phi) y} y^{\alpha-1} \mathcal{E}_{-\alpha,-\alpha} \left( \frac{c \Gamma(-\alpha) \gamma^{-\alpha}}{\kappa + c \Gamma(-\alpha) \gamma^{\alpha}} \right) dy.
\]
9.5 Comments

The case of a Cramér–Lundberg process with mixed exponential jumps (sometimes called hyperexponential jumps) can be generalised by taking jumps whose distribution has a Laplace transform that is the ratio of two polynomial functions of finite degree (also called a rational Laplace transform). Another favourite class in the family of jump distributions with rational Laplace transform (which also contains the class of mixed exponential distributions) is the phase-type distributions. The tractability of the class of processes with jumps having rational transform can be routed back to early work of Borovkov (1976) concerning the Wiener-Hopf factorisation. Many authors have worked on these types of Cramér–Lundberg processes and it would be impossible to give a complete list here. We cite instead two of the most recent references which give a good overview in the context of Gerber-Shiu type problems. These are Kuznetsov and Morales (2013) and Egami and Yamazaki (2012). The idea of engineering scale functions through the Wiener-Hopf factorisation comes from Hubalek and Kyprianou (2010) and Kyprianou and Rivero (2008). Analytical properties of scale functions have been described in a variety of different papers. A recent summary of these and many more facts can be found in the review on scale functions found in Cohen et al. (2012).
References