Lecture Notes: Kahler geometry and diffeomorphism groups
Lecture 1.

The theme of these lectures will be the study of Kahler metrics of constant scalar curvature. We begin with some background. In any co-ordinate system, the Gauss curvature of a surface in 3-space is given by a complicated formula involving the coefficients of the first fundamental form (induced Riemannian metric). A basic theorem asserts that we can always find “isothermal” coordinates, so the induced metric is

\[ e^f (du^2 + dv^2), \]

and in these coordinates the Gauss curvature has the simple representation

\[ K = -\frac{e^{-f}}{2} (f_{uu} + f_{vv}). \]

The existence of isothermal coordinates means that any surface in 3-space can be regarded as a Riemann surface, with an atlas of charts which differ by conformal, or equivalently holomorphic, maps. In this way, Riemann surfaces arise as differential geometric objects—i.e., 2-dimensional oriented Riemannian manifolds modulo conformal equivalence. On the other hand, Riemann surfaces also arise as algebro-geometric objects, for example as complex plane curves given by equations \( p(z_1, z_2) = 0 \), where \( p \) is a polynomial in two variables. Actually we will consider projective curves in complex projective space \( \mathbb{CP}^n \). (Recall that this is the quotient of \( \mathbb{C}^{n+1} \setminus \{0\} \) by the action of scalar multiplication, and may be regarded as \( \mathbb{C}^n \) compactified by the addition of points at infinity.)

Moving closer to our main theme: the starting point is a fact known from the end of the 19th. century:

Any compact Riemann surface has a metric of constant Gauss curvature (and this is essentially unique).

We want to extend these ideas to higher dimensions, in the framework of Kahler geometry. To explain what this is, recall some linear algebra. On a real, even dimensional, vector space we can consider three kinds of algebraic structure

- A complex structure;
- A Euclidean metric;
- A symplectic (i.e., skew-symmetric) form

Any two of these three structures, which are suitably compatible, define the third one. A Kahler manifold is an even dimensional manifold with each of these algebraic structures on its tangent spaces, compatible in a natural way. There are at least three different routes one can take to motivate the definition.
• In Riemannian geometry, a Kahler manifold is a Riemannian manifold whose holonomy is contained in the unitary group.

• Kahler metrics naturally occur on complex projective varieties. For example there is a standard “Fubini-Study” metric $g_{FS}$ on $\text{CP}^n$ and if $X \subset \text{CP}^n$ is a complex submanifold the restriction of $g_{FS}$ to $X$ is Kahler.

• Starting with a symplectic manifold, a Kahler structure appears as a “complex quantisation” in that we can associate to it a complex vector space (Hilbert space) $\mathcal{H}_X$ of holomorphic functions. More precisely, we should talk about holomorphic sections of a line bundle $L \to X$ and write $\mathcal{H}_{X,L}$.

The scalar curvature $S$ of a Kahler manifold is given, in local complex coordinates $z_a$ by

$$S = i \sum_{a,b} g^{ab} \frac{\partial^2 f}{\partial z_a \partial \bar{z}_b},$$

where $g_{ab}$ is the (Hermitian) matrix representing the metric, $g^{ab}$ is its inverse, and $e^f$ is the volume form (i.e. the determinant of the matrix $(g_{ab})$).

We have now set up the background for our problem:

If $X$ is a complex projective manifold, does it have a Kahler metric of constant scalar curvature? If so, is the metric unique?

(Note that a special case of this discussion is the problem of existence of Kahler-Einstein metrics, where the Ricci tensor is constant. Much more is known about this, through renowned work of Calabi, Aubin, Yau, Tian,... .)

More precisely, in our problem, we want to fix the “Kahler class” $[\omega] \in H^2(X)$. In a fixed class, the general Kahler metric is represented by a Kahler potential $\phi$,

$$\omega_\phi = \omega_0 + i \sum \frac{\partial^2 \phi}{\partial z_a \partial \bar{z}_b} dz_a d\bar{z}_b.$$  

Thus the scalar curvature of $\omega_\phi$ depends on the first four derivatives of $\phi$ and our problem is asking about the solution of a (highly nonlinear) 4th order PDE.

While the full solution of the problem above seems a long way off, we will outline in these lectures:

• A general conceptual scheme in which to fit the problem.
• A conjecture about the correct answer.
• The uniqueness of solutions
• A partial existence theory in the case of “toric varieties”
• An algorithm for constructing solutions numerically when these exist.

To end this first lecture we will discuss the 4th and 5th items above briefly.

**4th item.** Consider a convex polytope \( P \subset \mathbb{R}^n \). Suppose given a positive measure \( d\sigma \) on the boundary of \( P \) and let \( d\mu \) be standard Lebesgue measure on \( \mathbb{R}^n \). Let \( u \) be a convex function on \( P \) so the Hessian \( u_{ij} \) is a positive definite matrix at each point of \( P \). Put

\[
\mathcal{F}(u) = -\int_P \log \det(u_{ij}) \, d\mu + \int_{\partial P} u \, d\sigma - A \int_P u \, d\mu,
\]

where

\[
A = \frac{\text{Mass}(\partial P, d\sigma)}{\text{Mass}(P, d\mu)}.
\]

Thus we have defined a functional \( \mathcal{F} \) on the space of convex functions on \( P \).

**Question:** when does this functional achieve a minimum ? We will explain later that this question is equivalent to our problem in the case of toric varieties.

**5th. item**

Recall the “quantisation” vector space \( \mathcal{H}_{X,L} \) i.e. \( H^0(X;L) \). A kahler potential \( \phi \) can be thought of more geometrically as a Hermitian metric on the fibres of \( L \). This defines a Hermitian metric on \( \mathcal{H}_{X,L} \) by the standard \( L^2 \) norm:

\[
\|s\|^2 = \int_X |s|^2 \phi^m.
\]

On the other hand (provided \( L \) is “very ample”) we get a projective embedding \( X \to \mathbb{C}P^n = P(\mathcal{H}_{X,L}) \) defined by the sections of \( L \) and a metric on \( \mathcal{H}_{X,L} \) defines a Fubini-Study metric on \( \mathbb{C}P^n \) and hence on \( X \).

Let \( \mathcal{M}_{X,L} \) denote the set of Hermitian metrics on \( \mathcal{H}_{X,L} \). Putting these constructions together we get a map

\[
\Psi : \mathcal{M}_{X,L} \to \mathcal{M}_{X,L}.
\]

We an replace \( L \) by \( L^k \) throughout, where \( k \) is large. (This corresponds to the “classical limit” \( h \to 0 \).) Thus we get \( \Psi_k \) on \( \mathcal{M}_{L^k}X \). We let \( \Psi_k^{(n)} \) denote the \( n \)-fold composite of \( \Psi_k \).

**Proposition** Suppose \( X \) has a constant scalar curvature metric \( \omega \). Then for large enough \( k \) and any initial \( H \) the composites \( \Psi_k^{(n)}(H) \) converge as \( n \to \infty \) to some limit, which corresponds to a metric \( \omega_k \). Now

\[
\omega = \lim_{k \to \infty} \frac{\omega_k}{k}.
\]