Regular Variation and related results for the multivariate GARCH(p,q) with constant conditional correlations.

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Multivariate GARCH (M-GARCH) modelling has been one of the most successful tools to understand and predict the temporal dependence in the second order moments of financial returns in the last two decades.
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This econometric modelling scheme captures the feature that financial volatilities move together over time across assets and markets.

Multivariate GARCH models arise naturally as an empirically more relevant explanation of this feature than working with separate univariate GARCH for each asset.
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Introduction

- One of the shortfalls of M-GARCH models is that as the dimension grows larger, the number of parameters increases dramatically.
- To avoid this issue, Bollerslev introduced the M-GARCH with constant conditional correlations (CCC-GARCH) which reduces the number of parameters involved in estimation.
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To avoid this issue, Bollerslev introduced the M-GARCH with constant conditional correlations (CCC-GARCH) which reduces the number of parameters involved in estimation.

The CCC-GARCH is thus an attractive model for empirical applications.
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Boussama proved the existence of a unique strictly stationary and ergodic solution for multivariate GARCH models which, under certain condition, is also geometrically $\beta$–mixing.

The consistency and asymptotic normality of the quasi–maximum likelihood estimator were proved by Comte and Lieberman in 2003 for square integrable BEKK–GARCH processes.
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Our results generalize the ones given for the one–dimensional case to the multidimensional CCC-GARCH model.
First, we establish the regular variation of its finite–dimensional distributions under mild assumptions on the noise distribution.
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We then obtain two expressions for the MDA of the CCC-GARCH and detail the asymptotic behavior for its autocovariance function.
Definition

Given a sequence of i.i.d. random vectors \( \{ \eta_t \} \) with mean vector 0 and covariance matrix \( R \) such that \( R(i, i) = 1 \) for all \( i = 1, 2, \ldots, d \), we say the stochastic process \( \{ X_t \} \) is a CCC-GARCH\((p,q)\) if it satisfies the equations

\[
\delta(H_t) = C + \sum_{i=1}^{p} A_i \delta(X_{t-i}X_{t-i}^T) + \sum_{j=1}^{q} B_j \delta(H_{t-j})
\]

\[
D_t = \text{diag}(H_t(1, 1)^{1/2}, H_t(2, 2)^{1/2}, \ldots, H_t(d, d)^{1/2})
\]

\[
H_t = D_t R D_t
\]

\[
X_t = D_t \eta_t
\]
This process can be represented by the equation

$$Y_t = A(\eta_t) Y_{t-1} + G$$  (2)
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Where

$$Y_t = (\delta(H_{t+1})^T, \ldots, \delta(H_{t-q+2})^T, \delta(X_tX_t^T)^T, \ldots, \delta(X_{t-p+2}X_{t-p+2}^T)^T)^T$$

$$G = (C^T, 0, \ldots, 0)^T$$

$$A(\eta_t) = \begin{bmatrix}
A_1 \text{diag}(\eta_t\eta_t^T) + B_1 & B_2 & \cdots & B_{q-1} & B_q & A_2 & A_3 & \cdots & A_p \\
I & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & I & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & I & 0 & 0 & 0 & \cdots & 0 \\
\text{diag}(\eta_t\eta_t^T) & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & I & 0 & \cdots & 0
\end{bmatrix}$$
Using this representation we prove the following

**Theorem**

The stationary CCC-GARCH\((p,q)\) process \(\{X_t\}\) is regularly varying in the following two ways

1. There exists a \(\kappa_1 > 0\) and \(w(x) > 0\) such that
   \[
   \forall x \in \mathbb{R}^d \setminus \{0\}, \quad \lim_{u \to \infty} u^{\kappa_1} P \left[ \langle x, X_t \rangle > u \right] = w(x)
   \]

2. If \(\kappa_1\) is not an even integer, then \(X_t\) is regularly varying with index \(\kappa_1\).
Main Theorem

Theorem (Hypotheses)

Provided the following conditions hold.

1. *(H1)* The parameters $A_i, B_j$ have no zero rows.
2. *(H2)* The sequence $\{\eta_t\}$
   - has a density strictly positive on $\mathbb{R}^d$,
   - for any given $\theta \geq 1$ there exists $h > 1$ s.t.

$$\theta^h \leq \mathbb{E} \left[ \eta_{t,j}^{2h} \right] \leq \infty \quad \text{for } 1 \leq j \leq n$$
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Main Theorem

Sketch of the proof

Part 1: Marginal RV of $Y_t$
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The subsequence $Y_{tm}$ of $Y_t$ satisfies the recursive equation

$$\hat{Y}_t^{(m)} := Y_{tm} = A_{tm}A_{tm-1} \cdots A_{tm-m+1} Y_{(t-1)m} + G$$

$$+ \sum_{k=1}^{m-1} A_{tm}A_{tm-1} \cdots A_{tm-k+1} G$$

$$= \hat{A}_t^{(m)} \hat{Y}_{t-1}^{(m)} + \hat{B}_t^{(m)}$$
We prove that the sequence of random matrices $\hat{A}_t^{(m)}$ satisfies all the hypotheses of Kesten’s Theorem.
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$$\forall x \in \mathbb{R}^{d(p+q-1)} \setminus \{0\}, \lim_{u \to \infty} u^{\kappa_1} \mathbb{P} [\langle x, Y_t \rangle > u] = w(x)$$

This proves the first part of the Theorem.
We prove that the sequence of random matrices $\hat{A}_t^{(m)}$ satisfies all the hypotheses of Kesten’s Theorem and conclude thus that there exists a $\kappa_1 > 0$ and $w(x) > 0$ such that

$$
\forall x \in \mathbb{R}^{d(p+q-1)} \setminus \{0\}, \quad \lim_{u \to \infty} u^{\kappa_1} \mathbb{P} [\langle x, Y_t \rangle > u] = w(x)
$$

This proves the first part of the Theorem. If $\kappa_1$ is not an even integer then the random variable $Y_{tm}$ is regularly varying.
Part 2: RV, from $Y_t$ to $X_t$
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The vector

$$\sigma_t = \delta(H_t)^{1/2}$$

is the a.s. strictly positive square root of $\delta(H_t)$, therefore RV.
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$$\sigma_t = \delta(H_t)^{1/2}$$

is the a.s. strictly positive square root of $\delta(H_t)$, therefore RV. Finally, we write 

$$X_t = diag(\eta_t)\sigma_t$$

as an independent product with a RV vector. This completes the proof.
Corollary

The finite dimensional distributions of the CCC-GARCH process \( \{X_t\} \) satisfying the hypotheses of the previous Theorem are regularly varying if the noise distribution has symmetric marginals.

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First, we show that the finite–dimensional distributions of the process \( \{ Y_t \} \) are regularly varying with index \( \kappa_1 \), extending the argument in Part 1 of the previous proof.
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We sketch the proof:
First, we show that the finite–dimensional distributions of the process \( \{Y_t\} \) are regularly varying with index \( \kappa_1 \), extending the argument in Part 1 of the previous proof.
Therefore, for any given \( k \in \mathbb{N} \) the vector

\[
(\delta(H_1), \delta(X_1X_1^T), \ldots, \delta(H_k), \delta(X_kX_k^T))^T
\]

is regularly varying.
Since for all

- $\sigma_t(i) \geq 0$,
- $X_t(i)$ is symmetric,

it follows that the square root vector

$$(\sigma_1, X_1, \ldots, \sigma_k, X_k)^T$$

is also regularly varying (with index $2\kappa_1$).
The regular variation of $X_t$ doesn't depend on symmetry.
Remark

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Main Theorem

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3. The hypothesis that the density of the noise sequence is strictly positive makes the CCC-GARCH process $\beta$–mixing (as proved by Boussama(1988)).

4. For strictly positive (or negative) noises it can be shown that the finite–dimensional distributions are regularly varying, but we may lose the $\beta$–mixing property.
Remark

1. An alternative condition for the moments of the noise distribution is the following:

   (H2’) There exists \( h_0 > 0 \) such that

   \[
   \mathbb{E} \left[ |\eta_t|^h \right] \begin{cases} < \infty, & \text{for } h < h_0, \\ = \infty, & \text{for } h = h_0. \end{cases}
   \]
Some possible noise distributions

\textit{t - distribution with } \vartheta \textit{ degrees of freedom.}

- It satisfies \( \mathcal{H}2' \), which gives us marginal RV for the GARCH.
- It is symmetric, which gives us RV in the finite–dimensional distributions.
- It has a strictly positive density, which gives us the \( \beta \)-mixing property.
logNormal distribution

- It satisfies $H2$, giving marginal RV.
- It is strictly positive, giving us finite-dimensional RV.
- We lose the $\beta$–mixing.
Main Theorem

**logNormal distribution**
- It satisfies $\mathcal{H}2$, giving marginal RV.
- It is strictly positive, giving us finite–dimensional RV.
- We lose the $\beta$–mixing.

**Normal distribution**
- Satisfies $\mathcal{H}2$, giving marginal RV.
- Symmetric, giving finite–dimensional RV.
- Strictly positive density, giving $\beta$–mixing.
Domain of attraction for the Multivariate Extremes

We focus our attention on the componentwise–maximum

\[ M_n(i) = \max\{X_t(i): t \leq n\} \]
\[ M_n = (M_n(1), \ldots, M_n(d)) \]
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and the **norm maximum**

\[ M_n^{\|X\|} = \max\{\|X_t\| : t \leq n\} \]
Description of all distribution functions such that

\[
P \left[ a_n^{-1} M_n \left\| X \right\| \leq x \right] \to G(x) \quad (4)
\]

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P \left[ a_n^{-1} M_n \leq x \right] \to H(x), \quad \text{for } x \in \mathbb{R}^d \quad (5)
\]

where \(0 < a_n \to \infty\) is the sequence of regular variation of the CCC-GARCH process.
Description of all distribution functions such that

\[ P \left[ a_n^{-1} M_n \|X\| \leq x \right] \rightarrow G(x) \quad (4) \]

\[ P \left[ a_n^{-1} M_n \leq x \right] \rightarrow H(x), \quad \text{for } x \in \mathbb{R}^d \quad (5) \]

where \( 0 < a_n \rightarrow \infty \) is the sequence of regular variation of the CCC-GARCH process.

By the \( \beta \)-mixing, we may take

\[ a_n = n^{1/\kappa_1} \]
We study the point process

\[ N_n(\cdot) = \sum_{t=1}^{n} \varepsilon(a_n^{-1} X_t, \cdot) \]  \hspace{1cm} (6)

and show that **under \mathcal{H}1 and \mathcal{H}2** there exists \( N \) such that

\[ N_t \rightarrow N \]

in law.
\( N \) is identical in law to the point process

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \epsilon(P_i Q_{i,j})
\]

where \( \sum_{i=1}^{\infty} \epsilon(P_i) \) is a Poisson process with intensity measure

\[
\nu(dy) = \theta \alpha y^{-\alpha-1} \mathbb{1}(y > 0) \, dy
\]

for some \( 0 < \theta < 1 \) and independent of the sequence of i.i.d. point processes \( \sum_{j=1}^{\infty} \epsilon(Q_{i,j}), i \geq 1 \)
Let $X_t$ be the CCC-GARCH($p,q$) model with index of regular variation $\alpha > 0$.

The real random sequence $\|X_t\|$ satisfies

$$
\mathbb{P} \left[ n^{-1/2\alpha} M_n \|X_t\| \leq x \right] \rightarrow \exp\{-\theta x^{-2\alpha}\} \mathbb{1} \ (x > 0)
$$

where $\theta < 1$ is the extremal index of $\|X_t\|$. 

**Theorem**
Theorem (Continued)

For the componentwise maximum we have

$$\mathbb{P} \left[ n^{-1/2\alpha} M_n \leq x \right] \to \exp\left\{-\lambda (\{\mu : \mu(B_x) > 0\})\right\}$$

where $\lambda$ is the canonical measure of the point process $N$ and $B_x = (-\infty, x]^c$ for any fixed $x \in \mathbb{R}_+^d$. 
Proposition

Let \( \{Y_t\} \) be the SRE of the CCC-GARCH\((p,q)\) and let \( \nu \) be such that

\[
n \mathbb{P}\left[ n^{-1/\alpha} Y_1 \in \cdot \right] \xrightarrow{\nu} \nu(\cdot)
\]

For \( x \in (0, \infty)^d \) it holds that

\[
\mathbb{P}\left[ n^{-1/\alpha} M_n^Y \leq x \right] \xrightarrow{n \to \infty} \exp \left\{ - \int_{[0,x]^c} \mathbb{P}\left[ \prod_{i=1}^j A_i y \leq x; \ j \geq 1 \right] \nu(dy) \right\}
\]
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- The second result depends only on marginal regular variation and is thus more general.
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- The first one depends on the convergence $N_n \rightarrow N$ which is a consequence of finite–dimensional regular variation. Therefore, this result depends on the symmetry of the noise distribution.

- The second result depends only on marginal regular variation and is thus more general.

- Both expressions are numerically very difficult to handle.
Remark (Continued)

The multivariate extremal index of the stationary process \( \{ Y_t \} \) may now be written as

\[
\theta(x) = \frac{\int_{[0,x]^c} \mathbb{P} \left[ \prod_{i=1}^{j} A_i y \leq x; j \geq 1 \right] \nu(dy)}{\nu([0,x]^c)}
\]

This is the only closed form for this index.
Asymptotic relationship of the sample autocovariance function

\[ \gamma_{n,X}(h) = n^{-1} \sum_{t=1}^{n-h} X_t X_{t+h}^T, \quad h \geq 0 \]

and the autocovariance function

\[ \gamma_X(h) = \mathbb{E} \left[ X_0 X_h^T \right] \]

in terms of the index of regular variation \( \alpha > 0 \).
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Sample Autocovariance Function Convergence

**Theorem**

Let \( \{X_t\} \) be a CCC-GARCH(p,q) process with RV index \( \alpha \) and the noises have symmetric distribution.

- If \( \alpha \in (0, 2) \), then

\[
(n^{1-2/\alpha} \text{vec}(\gamma_n, X(h)))_{h=0,1,...,m} \xrightarrow{d} (V_h)_{h=0,1,...,m}
\]

where \((V_0, V_1, \ldots, V_m)\) is jointly \(\alpha/2\) stable in \(\mathbb{R}^{dm}\).
If $\alpha \in (2, 4)$ then

$$
\left( n^{1-2/\alpha} \text{vec}(\gamma_n, x(h) - \gamma x(h)) \right)_{h=0,1,\ldots,m} \xrightarrow{d} (V_h)_{h=0,1,\ldots,m}
$$

(7)

where $(V_0, V_1, \ldots, V_m)$ is jointly $\alpha/2$ stable in $\mathbb{R}^{dm}$
If $\alpha \in (2, 4)$ then

$$\left(n^{1-2/\alpha}(\text{vec}(\gamma_n \chi(h) - \gamma \chi(h)))\right)_{h=0,1,...,m} \overset{d}{\rightarrow} (V_h)_{h=0,1,...,m}$$

(7)

where $(V_0, V_1, \ldots, V_m)$ is jointly $\alpha/2$ stable in $\mathbb{R}^{dm}$

If $\alpha > 4$ then equation (7) holds with normalization $n^{1/2}$, and $V$ is multivariate normal with mean zero.
Sketch of the proof

1. For $\alpha \in (0, 2)$: Immediate from point process convergence and a general Theorem for stationary and regularly varying vector sequences.
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2. **For** $\alpha \in (2, 4)$: It suffices to prove that

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \text{var} \left( n^{-2/\alpha} \sum_{t=1}^{n-h} X_{t,i} X_{t+h,j} \mathbb{I} \left( |X_{t,i} X_{t+h,j}| \leq n^{2/\alpha} \epsilon \right) \right) = 0$$

for all $i, j = 1, 2, \ldots, d$. 
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Sketch of the proof

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\]

for all \( i, j = 1, 2, \ldots, d \).

This is done using the symmetry and the regular variation
3. For $\alpha \in (4, \infty)$: The sequence

$$\{X_{t,i}^2, X_{t,i}X_{t+1,j}, \ldots, X_{t,i}X_{t+m,j}, \ i, j = 1, 2, \ldots, d\}_t$$

has a finite $2 + \theta$ moment and is geometrically $\beta$-mixing.
3. **For** $\alpha \in (4, \infty)$: The sequence

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has a finite $2 + \theta$ moment and is geometrically $\beta$-mixing. It follows that

$$(n^{1/2}(\text{vec}(\gamma_{n,X}(h) - \gamma_X(h))))_{h=0,1,\ldots,m} \xrightarrow{d} (V_h)_{h=0,\ldots,m}$$

where $V$ has a multivariate normal distribution with mean zero.
Remark

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2. For $\alpha > 2$ we have consistency, but
3. a CLT can only be proved for $\alpha > 4$. 
Thank you very much.