Fixed Point Index for Krasnoselskii-type set-valued maps on complete ANR’s

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Among many different generalizations of the Schauder and Banach fixed point principles the following result due to Krasnoselskii played an important role.

**Theorem** Let $X$ be a nonempty closed and convex subset of a Banach space $E$. Let $K : X \to E$ be a contraction (i.e. Lipschitz with constant $k \in [0,1)$), $C : X \to E$ be a compact map and let, for all $x, y \in X$,

$$K(x) + C(y) \in X. \quad (*)$$

Then there exists $x_0 \in X$ such that

$$x_0 = K(x_0) + C(x_0).$$

This result had lost its significance when Darbo and Sadovski introduced the notions of a $k$-set-contraction and a condensing map. Namely it appears that $K + C$ is a $k$-set contraction (with respect to the Kuratowski or Hausdorff measure of noncompactness) and the existence of fixed points follows when hypothesis $(*)$ is replaced by a weaker one: for each $x \in X$, $K(x) + C(x) \in X$. However, it seems that the idea underlying the ingenious proof of the above theorem is still fruitful and has been used by different authors in order to establish some interesting generalizations of this result. One may proceed in two directions: firstly it seems important to provide set-valued generalizations of the Krasnoselskii theorem, secondly, it is reasonable to consider, instead the sum $K + C$, some more general maps by similar means, e.g. composite maps of the form $X \ni x \mapsto T(K(x), C(x))$, where a (usually nonlinear) operator $T : E \times E \to E$ replaces the sum $+ : E \times E \to E$. In the present talk we shall study both these approaches at the same time and our principal aim is to construct the homotopy invariant responsible for the existence of fixed points of (possibly) set-valued maps of the above or similar form that are defined on, no longer closed convex subsets of $E$, but on arbitrary complete absolute neighborhood retracts. In this case the techniques available for $k$-set-contractions or condensing maps are no longer applicable. We provide applications to various existence problems for constrained differential equations and inclusions. We also provide very general results concerning the topological structure of the fixed point set of a set-valued contraction with not necessarily convex values.