Notes on Higgs bundles and D-branes

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1 A brief introduction to D-branes

The concept of a D-brane (Dirichlet brane) in string theory had evolved considerably [**?**] in recent years:

Original point of view: A D-brane is a special locus of space-time on which open strings can end. The Dirichlet boundary conditions for the propagation of the strings are encoded in a $U(n)$-gauge theory supported on the brane and the scalar fields live in the normal bundle to the brane.

Current point of view: A D-brane is a cycle in a generalized cohomology theory (e.g. in differential K-theory) or, more generally, an object in a triangulated or an A_∞ category parameterizing geometric objects (e.g. the derived category of coherent sheaves or the Fukaya category). The strings stretching between two D-branes are the morphism in this category.

When string theory is compactified on a Calabi-Yau space, the D-branes become loci with extra structure sitting inside the Calabi-Yau background. The condition that half of the supersymmetry of the model is preserved is translated into a geometric constrain on the branes which is typically a calibration condition (in the sense of differential geometry).

Example 1.1 The D-branes in a 3-dimensional Calabi-Yau background X of a type II string theory come in two flavors:

A-branes These are the D-branes in the topological IIA theory. They are odd dimensional and are supersymmetric 3-cycles on X.

B-branes These are the D-branes of the topological IIB theory. They are even dimensional and are holomorphic vector bundles, supported on complex analytic submanifolds of X. More generally the B-branes are (complexes) of coherent sheaves on X.

Additionally, in the presence of a non-trivial B-field one may need to twist (or deform in a non-commutative direction) these geometric objects, in order to obtain the actual D-branes.

Remark 1.2 **(i)** It is important to be able to recognize the D-branes that correspond to actual physical states. In mathematical language the property of a D-brane being 'physical' is expressed in the *stability* of the corresponding geometric object. Finding the right stability notion and describing it in purely geometric terms is somewhat of a puzzle at the moment. Recently interesting progress in that direction was made by M.Douglas [Dou02, AD02] and T.Bridgeland [Bri07, Bri08]

(ii) Mathematically, a physical A-brane is captured by the notion of a supersymmetric cycle. On a Calabi-Yau n-fold Z this notion depends on the choice of a Kähler form ω on Z as well as the choice of a *complex orientation* i.e. a trivialization $\Omega \in H^0(Z, \Omega_Z^n)$ of the canonical line bundle on Z. A supersymmetric cycle then is a pair $(\Sigma, \mathcal{L})$ where:

- $\Sigma \subset Z$ is a submanifold which is Lagrangian for ω and is callibrated by Ω i.e.

$$\mathrm{Im}(\Omega)_{|\Sigma} = 0 \text{ and } \mathrm{Re}(\Omega)_{|\Sigma} = \mathrm{vol}_g(\Sigma).$$

  Such a Σ called a special Lagrangian submanifold.

- $\mathcal{L} \to \Sigma$ is a complex rank k local system on Σ. In other words, $\mathcal{L} = (L, A)$ where L is a C^∞ rank k complex vector bundle on Σ, and A is a flat connection on L.

Specifically one can argue [PW11] that a SUSY configuration of n $D6$ branes wrapping Σ is described *classically* by a pair (L, A) as above. The derivation of this statement is based on the assumption that when quantum corrections are suppressed the moduli space of A-branes can be described to first order near any point as the moduli of A-branes on the symplectic linearization of Z near Σ, i.e. on $\mathrm{tot}(T^\vee\Sigma)$ taken with its standard symplectic structure. In this setup, the data describing n $D6$ branes wrapping Σ is interpretted as the result of a dimensional reduction of a Hermitian Yang-Mills instanton on $\mathrm{tot}(T^\vee\Sigma)$ along the leaves of the foliation $T^\vee\Sigma \to \Sigma$ The resulting object is a triple (L, a, φ), where:

- L is a complex vector bundle of rank n;
- a is a connection on L preserving some Hermitian metric h;
- $\varphi \in \Gamma(\Sigma, \mathrm{ad}(L) \otimes A^1_\Sigma)$, satisfying $\varphi^* = -\varphi$ where φ^* is the adjoint of φ w.r.t. h;

Furthermore (L, a, φ) must satisfy two systems of PDE:

(F-flatness condition) $F_a = -\varphi \wedge \varphi, \qquad D_a\varphi = 0.$

(D-flatness condition) $D_a * \varphi = 0.$

Given (a, φ) satisfying the F-flatness condition, it is clear that the combination $A = a + \sqrt{-1}\varphi$ is a complex flat connection on L. Thus it is natural to try and understand how one can ensure the D-flatness condition starting simply with a complex flat connection.

Note: If we start with a hermitian flat connection A, then

$$A = a, \qquad \varphi = 0$$

and the D-flatness holds automatically. In this way we recover the standard description of n-tuples of A-branes as pairs (Σ, A) consisting of a slag submanifold and a unitary flat connection A on a rank n complex vector bundle on Σ.

More generally, if A is any complex flat connection on a vector bundle L and if h is a Hermitian metric on L, then A decomposes canonically as $A = a + \sqrt{-1}\varphi$, where a is a connection on V which preserves h. Now:

F-flatness of $(a, \varphi) \Longleftrightarrow$ flatness of A.

D-flatness for $(a, \varphi) \Longleftrightarrow$ a constraint on the metric h.

Metrics h satisfying the D-flatness constraint on a flat complex bundle (L, A) are called *harmonic metrics* have been studied in detail Corlette and Donaldson [Cor88, **?**]:

Theorem [K.Corlette] *Suppose Σ is a compact Riemannian manifold and let L be a complex rank n vector bundle on Σ equipped with a flat connection A. Then:*

- *If A has reductive monodromy, then a harmonic metric on (L, A) exists;*
- *If A has irreducible monodromy, then the harmonic metric h on (V, A) is unique. Furthermore the h-preserving piece a of A is also flat.*

Thus, for an irreducible (L, A) we get a unique decomposition $A = a + \sqrt{-1}\varphi$ so that

$$\begin{aligned} a \text{ is Hermitian }, &\qquad \varphi^* = -\varphi \\ F_a = 0, &\qquad \varphi \wedge \varphi = 0 \\ D_a\varphi = 0, &\qquad D_a * \varphi = 0. \end{aligned}$$

In other words - the data (L, a, φ) describing n type A-branes wrapping the manifold Σ should be thought of as the analogue of a Higgs bundle in Riemannian geometry.

This analogy can be made even more geometric and leads to some interesting insights on the Strominger-Yau-Zaslow [SYZ96] picture of mirror symmetry and suggests the following

Questions

• *Is there a quantum deformation of the harmonicity equation on the metric h?*

• *Is there a deformation of the notion of a complex local system that captures the quantum corrected configurations of n branes wrapping Σ?*

(iii) A more careful analysis of the supersymmetry preservation conditions for A-branes shows that the special Lagrangian condition is sufficient but not necessary. In particular, Kapustin and Orlov have argued in [KO01] that there exist supersymmetric A-branes which are pairs $(\Sigma, \mathcal{L})$ consisting of a ω-coisotropic subvariety $\Sigma \subset X$ equipped with a $U(1)$ gauge field $\mathcal{L}$ so that:

- The null foliation of Σ has a transversal holomorphic structure, and
- $\mathcal{L}$ is flat along the leaves of the foliation and holomorphic in the transversal direction.

The moduli space of D-branes in a given string compactification is a part of the quantum moduli space of the theory. We can extract information about the string background by studying the geometry of the D-brane moduli and our primary goal in these notes is to understand the symmetries acting on D-branes.

The symmetries can come from two sources:

Physical: U-duality, T-duality, quantum mirror symmetry, Seiberg duality, M-theory/IIA duality, ...

Mathematical: Fourier-Mukai transforms, Koszul duality, the Atiyah-Ward correspondence, geometric Langlands duality ...

We will explain how these mathematical symmetries act on D-brane moduli spaces and how they relate to the string dualities. The prototype for most of these duallities is the *spectral correspondence.*

2 Higgs bundles

2.1 Review of spectral covers

Common strategy in mathematics: search for a duality operation that will simplify a given problem.

Example 2.1 Fourier transform for functions on a locally compact abelian group. Gives a way of converting between continuous and discrete data.

The *spectral cover construction* is another example. Rougly this is a duality operation which aims to replace a linear operator by its spectrum.

Simplest setup: Let V be a finite dimensional $\mathbb{C}$-vector space and let $\phi : V \to V$ be an endomorphism.

When ϕ is generic (=diagonalizable) one can describe ϕ via its spectral data, i.e. by giving:

- the eigenvalues of ϕ;
- the decomposition of V into a direct sum of ϕ-eigenlines;
- a mathcing between eigenvalues and eigenlines.

If $\dim_{\mathbb{C}} V = n$ this means that we are specifying n complex numbers

$$\lambda_1, \ldots, \lambda_n \in \mathbb{C} \quad (= \text{ spectrum of } \phi)$$

and to each such number we are prescribing a line $L_i \subset V$, so that $L_1 \oplus \ldots \oplus L_n = V$.

The spectral covers appear when we let this picture vary in families.

If $\phi_s : V \to V$ is a family of endomorphisms parameterized by $s \in S$, then by repeating the construction for each s we get a subvariety $\overline{S} \subset S \times \mathbb{C}$, where

$$\overline{S} = \{(s, \lambda) \mid \lambda \text{ is an eigenvalue of } \phi_s\}$$

If all ϕ_s have distinct eigenvalues we also get a family of eigenlines $L_{(s,\lambda)}$ parameterized by the points of $\overline{S}$.

The space $\overline{S}$ is called the *spectral cover* corresponding to the family $\{\phi_s\}_{s\in S}$. Under the genericity assumption $\overline{S}$ is an unramified n-sheeted cover of S and it carries a line bundle consisting of all eigenvalues of the ϕ_s's.

Note: The data $(\overline{S} \to S, L \to \overline{S})$ completely reconstructs the family $\{\phi_s\}_{s\in S}$.

The correspondence $\{\phi_s\}_{s\in S} \leftrightarrow (\overline{S}, L)$ is not very useful under the genericity assumption. In applications one needs to deal with ϕ_s which have repeated eigenvalues. In this case $\overline{S} \to S$ becomes ramified over $s \in S$ and the fibers of $L \to \overline{S}$ may jump at the multiple valued points.

Thus one expects some kind of a sheaf structure for L along the ramification locus of $\overline{S} \to S$.

Important special case: Allow ϕ_s to have multiple eigenvalues but require that there is exactly one Jordan block per eigenvalue.

Such an endomorphism of V is called regular. It carries a single eigenline per eigenvalue. In particular if all ϕ_s, $s \in S$ are regular we get again a line bundle $L \to \overline{S}$ on the spectral cover $\overline{S}$.

More invariantly, consider the polynomial map

$$\begin{array}{rccc} h: & \mathrm{End}(V) & \longrightarrow & \mathbb{C}^n \\ & \phi & \longmapsto & (a_1(\phi), \ldots, a_n(\phi)), \end{array}$$

where the $a_i(\phi)$'s are the coefficients

$$\det(t \cdot \mathrm{id}_V - \phi) = t^n + a_1(\phi) t^{n-1} + \ldots + a_n(\phi).$$

of the characteristic polynomial of ϕ.

The spectrum of ϕ depends only on $h(\phi)$ and so $\overline{S}$ is just the pullback via the map

$$\begin{array}{ccc} S & \longrightarrow & \mathbb{C}^n \\ \phi_s & \longrightarrow & h(\phi_s). \end{array}$$

of the obvious cover

$$\overline{\mathbb{C}}^n \subset \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n,$$

given by the equation $t^n + a_1 t^{n-1} + \ldots + a_n = 0$ in the coordinates $(a_1, \ldots, a_n; t) \in \mathbb{C}^n \times \mathbb{C}$.

The fibers of $h : \mathrm{End}(V) \to \mathbb{C}^n$ are invariant under conjugation action of $GL(V)$ and in fact

$$\mathbb{C}^n = \mathrm{End}(V)//GL(V)$$

is the GIT quotient.

Explanation: The orbits of the $GL(V)$-action on $\mathrm{End}(V)$ are not all closed and so the natural topology on the set of orbits $\mathrm{End}(V)/GL(V)$ will not be Hausdorff. To remedy that one looks for a space $\mathrm{End}(V)//GL(V)$ parameterizing the closures of $GL(V)$-orbits in $\mathrm{End}(V)$.

For a general (regular and semisimple) ϕ in $\mathrm{End}(V)$ the $GL(V)$-orbit is closed and in a neighborhood of such ϕ the quotients

$$\mathrm{End}(V)/GL(V) \quad \text{and} \quad \mathrm{End}(V)//GL(V)$$

coincide.

When ϕ is arbitrary, then $\overline{GL(V) \cdot \phi}$ contains a unique closed and a unique open orbit. The closed one is the orbit of a semisimple (diagonalizable) endomorphism and the open one is the orbit of a regular endomorphism. This leads to

Two interpretations for $\mathrm{End}(V)//GL(V)$*:* either as the space parameterizing semisimple endomorphisms modulo conjugation, or as the space parameterizing all regular endomorphisms modulo conjugation.

Both interpreations are useful but the one for which the eigenlines vary 'continuosly' is the interpretation via regular endomorphisms.

Example 2.2 Let $\dim_{\mathbb{C}}(V) = 2$. Use $SL(V)$ instead of $GL(V)$. Then we have $h : SL(V) \to \mathbb{C}$, $h(\phi) = \det \phi$, and if $\det \phi \neq 0$, then ϕ is regular and semisimple. If $\det \phi = 0$, then ϕ is nilpotent and then

$$h^{-1}(0) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \coprod \left\{ SL(V) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$$

Extensions and generalizations

To make the simple-minded spectral cover construction useful in applications one needs to extend it in two ways:

(i) Allow for abitrary groups G, not only for $GL(V)$;

(ii) Allow for twisted versions of ϕ.

For **(i)**: Instead of looking at elements $\phi \in \mathrm{End}(V)$ we take elements $\phi \in \mathfrak{g}$ for $\mathfrak{g} := \mathrm{Lie}(G)$ of some complex semi-simple group G. The spectral cover construction in this case is somewhat subtler since it has to reflect the complexity of the group G. I will not discuss this part of the story. For more details see Ron Donagi's papers [Do95], [DG02], [Do98].

For **(ii)**: The 'twisting' of the $\{\phi_s\}_{s \in S}$ can be achieved in two ways. Firstly, one can allow for the vector space V to vary with the point $s \in S$. This is easily realized by replacing $S \times V$ by a non-trivial vector bundle E on S. In this setup the family of endomorphisms naturally should be viewed as a section $\phi \in \Gamma(S, \mathrm{End}(E))$. Secondly, one can allow for ϕ to have nontrivial coefficients in some coefficient object K.

The freedom of choosing K is essential in the applications. Since the elements in K can be thought of as the matrix coefficients of ϕ, it is natural to require that K has an abelian group structure. Possible natural choices for K are: a vector bundle on X, a family of affine tori on X, a family of abelian varieties on X or more generally a family of commutative group stacks over X.

We will see examples of most of these choices later on and will relate them to D-brane moduli and dualities. The simplest choice is to take K to be a vector bundle. This leads to the classical notion of a Higgs bundle.

2.2 Higgs bundles with linear coefficients

Let S be a a complex algebraic variety and let K be a fixed algebraic vector bundle of rank n on S. Consider a vector bundle $E \to S$ of rank r and an $\mathcal{O}_S$-linear map

$$\phi : E \to E \otimes K.$$

We would like to take the 'spectrum' of ϕ and recast the data (E, ϕ) in terms of a spectral cover C of S possibly decorated with some additional structure (e.g. a coherent sheaf).

Problem The spectrum may not be well defined for a general ϕ.

Indeed, if we trivialize K locally on S, i.e. if we choose a local frame $K_{|V} \cong \mathbb{C}^n \otimes \mathcal{O}_V$, then we see that locally ϕ comprises n endomorphims

$$\phi_{|V} = (\phi_1, \dots, \phi_n), \text{ with } \phi_i \in \Gamma(V, \mathrm{End}(E)).$$

We can apply the naive spectral construction to each ϕ_i but the collection of spectral covers we will get this way will depend on the trivialization of K.

To fix that one may look only at ϕ's for which all the ϕ_i's behave in the same way e.g. are simultaneously diagonalizable. More generally we can require that $[\phi_i, \phi_j] = 0$ for all i, j, i.e. that the ϕ_i's generate a commutative subalgebra in $\mathrm{End}(E)$. The latter condition is clearly equivalent to requiring that

$$\phi \wedge \phi = 0 \in \Gamma(S, \mathrm{End}(E) \otimes \bigwedge^2 K).$$

This motivates the following

Definition 2.3 *A K-valued Higgs bundle on an algebraic variety S is a pair*

$$(E, \phi : E \to E \otimes K)$$

satisfying $\phi \wedge \phi = 0$.

Similarly one defines a Higgs coherent sheaf on S.

Observe that for a Higgs bundle (E, ϕ), the Higgs field ϕ can be interpreted as a map $K^\vee \otimes E \to E$ and so generates an action $\boldsymbol{T}K^\vee \otimes E \to E$ of the sheaf of tensor algebras $\boldsymbol{T}K^\vee := \oplus_i (K^\vee)^{\otimes i}$ on E. The

condition $\phi \wedge \phi = 0$ is equivalent to saying that this action descends to an action

$$S^\bullet K^\vee \otimes E \to E$$

of the symmetric algebra $S^\bullet K^\vee$ on E.

This fact admits a geometric interpretation. Consider the total space $X := \mathrm{tot}(K)$ of the vector bundle K. Let $p : X \to S$ be the natural projection. Then p is an affine map and

$$p_* \mathcal{O}_X = S^\bullet K^\vee, \qquad X = \mathrm{Spec}(S^\bullet K^\vee).$$

In particular, a quasi-coherent sheaf $\mathscr{E}$ on X is the same thing as a quasi-coherent sheaf $E (= p_* \mathscr{E})$ on S together with a $S^\bullet K^\vee$-action.

Note that since p is affine, an $S^\bullet K^\vee$-module E which is coherent as a sheaf on S will correspond to a coherent sheaf $\mathscr{E}$ on X which is *finite* over S. In other words we have an equivalence of categories

$$p_* : \left\{ \begin{matrix} \text{quasi-coherent} \\ \text{sheaves on } X \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Sheaves of } S^\bullet K^\vee \text{ modules on } S, \text{ quasi-} \\ \text{coherent as sheaves of } \mathcal{O}_S\text{-modules} \end{matrix} \right\}$$

which restricts to an equivalence

$$p_* : \left\{ \begin{matrix} \text{coherent sheaves} \\ \text{on } X \text{, finite over } S \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{Higgs coherent} \\ \text{sheaves on } S \end{matrix} \right\}.$$

This is the *K-valued spectral correspondence*. It converts spectral data (= coherent sheaves on $\mathrm{tot}(K)$ whose support is finite over S) to Higgs data (= K-twisted families of endomorphisms on S).

Remark 2.4

- The Higgs sheaf (E, ϕ) corresponding to a sheaf $\mathscr{E}$ on X can be described explicitly: $E = p_*\mathscr{E}$ is the pushforward of $\mathscr{E}$, $\phi : E \to E \otimes K$ is the pushforward of $\mathscr{E} \xrightarrow{\otimes\lambda} \mathscr{E} \otimes p^*K$ where, $\lambda \in \Gamma(\mathrm{tot}(K), p^*K)$ is the tautological section.

- If a sheaf $\mathscr{E}$ on X corresponds to a Higgs bundle (E, ϕ) of rank r, then the spectral cover for (E, ϕ) is defined as the subscheme $\mathrm{Supp}(\mathscr{E}) \subset X$ which maps onto S and is finite of degree r over S. It is given explicitly as the zero locus of the section

$$\det(\lambda \cdot \mathrm{id} - p^*\phi) \in \Gamma(X, p^*S^rK).$$

- When K is the trivial line bundle on S, then $X = S \times \mathbb{C}$ and we recover the old definition of a spectral cover for a family of endomorphisms.

2.3 D-branes and Higgs bundles

The spectral correspondence is a simple geometric duality which can be used to describe D-branes.

Setup 1: Let (S, g) be a compact Kähler manifold with g real-analytic, $K = \Omega^1_S$ the holomorphic cotangent bundle of S. The total space $X = \mathrm{tot}(K)$ of K carries a holomorphic symplectic form - the exterior derivative $\Omega = d\lambda$ of the tautological one form λ on X. It is known [Fei01, Kal99] that a tubular neighborhood of the zero section $S \subset X$ of K supports a unique hyper-Kähler metric which is compatible with Ω and restricts to g on S. Thus X is a non-compact, non-complete physicists Calabi-Yau manifold which can be taken as a string background.

The B-branes on X are coherent sheaves on X with compact support, i.e. coherent sheaves $\mathscr{E}$ on X which are finite over S. By the spectral correspondence one can describe the moduli space of such $\mathscr{E}$ as the moduli space of Higgs bundles $(E, \phi : E \to E \otimes \Omega^1_S)$ on S. If the Chern classes of $\mathscr{E}$ are chosen so that $c_1(E) = 0$ and $c_2(E) = 0$, then all such Higgs bundles correspond to representations of $\pi_1(S)$ by C.Simpson's theory [Sim97]. This gives a concrete description of a component of the moduli space of B-branes on X.

Setup 2: Let Z be a three dimensional compact Calabi-Yau manifold. Let $C \subset Z$ be a smooth rigid curve in Z. In M-theory one is interested in the moduli space $\mathrm{BPS}(Z, C, r)$ of

BPS states on Z of charge $r\cdot[C] \in H_2(Z,\mathbb{Z})$. Geometrically $\mathrm{BPS}(Z,C,r)$ should parameterize torsion sheaves on Z whose support represents the homology class $r\cdot[C]$.

Note: This problem is not very well posed - the corresponding quot scheme is not of finite type.

On the other hand Gopakumar-Vafa [GC98] gave an explicit formula expressing the Euler characteristic of the space $\mathrm{BPS}(Z,C,r)$ in terms of finitely many GW invariants in the homology class $r\cdot[C]$.

Question: What is this formula really calculating?

One possible answer suggested by the perturbative nature of the Gopakumar-Vafa calculation is that in fact the space $\mathrm{BPS}(Z,C,r)$ should be linearized in a suitable way before we start counting.

Proposal: Replace the global Calabi-Yau Z by its linearization near C, i.e. by the local Calabi-Yau

$$X := \mathrm{tot}(N_{C/Z}),$$

with C sitting inside X as the zero section.

Now we have a natural projection $p : X \to C$ and the space $\mathrm{BPS}(X,C,r)$ is just the moduli space of coherent sheaves on X, which are finite and of degree r over C. By the spectral correspondence we can identify $\mathrm{BPS}(X,C,r)$ with the moduli of rank r Higgs bundles $(E,\phi : E \to E\otimes N_{C/Z})$ on C which is much simpler. In particular, one can utilize the natural $\mathbb{C}^\times$-action on Higgs bundles:

$$t\cdot(E,\phi) := (E,t\phi), \text{ for all } t\in\mathbb{C}^\times,$$

and try to localize the calculation of the Euler characteristic of $\mathrm{BPS}(X,C,r)$ at the fixed locus of $\mathbb{C}^\times$.

3 Koszul duality

3.1 Quadratic duality for graded algebras

To put things in context, let us exhibit the spectral correspondence as a special case of a general class of duality transformations known as *Koszul dualities*.

Let

$$A = \bigoplus_{i\geq 0} A_i, \quad A_i\cdot A_j \subset A_{i+j}$$

be a non-negatively graded unital complex associative algebra. For simplicity we will also assume that:

- A is connected, i.e. $A_0 = \mathbb{C}$ and the unit 1_A satisfies $1_A = 1 \in \mathbb{C} = A_0 \subset A$.

- A is locally finite dimensional, i.e. $\dim_{\mathbb{C}} A_i < \infty$.

Note that every such A has a natural augmentation with augmentation ideal $A_+ := \bigoplus_{i>0} A_i$.

Definition 3.1 *A graded algebra A is called* quadratic *if A is generated by A_1 and has relations of degree two.*

This means that the canonical map

$$\text{can} : \boldsymbol{T}A_1 \to A$$

is surjective and that ker(can) is generated by its degree two component

$$R_A := \ker(\text{can}) \cap (A_1 \otimes A_1)$$

as a two-sided ideal in $\boldsymbol{T}A_1$.

Equivalently, a quadratic algebra is uniquely determined by a space of generators $V := A_1$ and an arbitrary subspace $R \subset V \otimes V$ of quadratic relations.

Notation: $A = \{V, R\}$.

Definition 3.2 *The* quadratic dual *of a quadratic algebra $A = \{V, R\}$ is the quadratic algebra $A^! := \{V^\vee, R^\perp\}$.*

The quadratic duality functor

$$! : (\text{quadAlg}_{\mathbb{C}}) \to (\text{quadAlg}_{\mathbb{C}})^{op}$$

is an involutive equivalence of categories, i.e. $(A^!)^! = A$.

Example 3.3 • Let V be a finite dimensional $\mathbb{C}$-vector space. Then $\boldsymbol{T}V = \{V, 0\}$ is quadratic and $(\boldsymbol{T}V)^! = \mathbb{C} \oplus V^\vee \oplus 0 \oplus \ldots$.

• The algebras $S^\bullet V$ and $\wedge^\bullet V$ are obviously quadratic. Moreover $(S^\bullet V)^! = \wedge^\bullet V^\vee$.

We also have quadratic duality for modules over dual quadratic algebras:

Let A be a quadratic algerba and let $M = \bigoplus_{i \geq 0} M_i$ be a graded left module over A. We say that M is *quadratic* if there is a subspace $L \subset A_1 \otimes M_0$ so that

$$M = (A \otimes M_0)/(A \cdot L).$$

If M is a quadratic module, then its *quadratic dual* is a module $M^!$ over the quadratic dual algebra $A^!$ defined by

$$M^! := (A^! \otimes M_0^\vee)/(A^! \cdot L^\perp),$$

where $L^\perp \subset A_1^\vee \otimes M_0^\vee$ is the annihilator of L.

Again the duality functor

$$! : (A - \text{quadmod}) \to (A^! - \text{quadmod})^{op}$$

establishes an involutive equivalence of categories.

Example 3.4 ! converts free A-modules into trivial $A^!$-modules and vice versa. In other words we have isomorphisms of $A^!$-modules:

$$\mathbb{C}^! \cong A^! \quad \text{and} \quad A^! \cong \mathbb{C}.$$

The Koszul algebras and modules are particular instances of quadratic algebras and modules. They have the property that the quadratic dual can be described completely in cohomological terms.

Definition 3.5 *A quadratic algebra A is called Koszul if we have an isomorphism of graded algebras $\mathrm{ext}_A^\bullet(\mathbb{C},\mathbb{C}) \cong A^!$. A quadratic module M over a Koszul algebra A is called a Koszul module if we have an isomorphism of $A^!$ modules: $M^! \cong \mathrm{ext}_A^\bullet(M,\mathbb{C})$.*

Here $\mathrm{ext}_A^\bullet$ denotes the Yoneda exts in the category of graded left A-modules. Note that these ext groups are naturally bigraded and so, in particular, if A is Koszul we have that $\mathrm{ext}_A^{ij}(\mathbb{C},\mathbb{C}) = 0$ for all $i \neq j$. It turns out that this is also sufficient for Koszulity. Similarly M is Koszul iff $\mathrm{ext}_A^{ij}(M,\mathbb{C}) = 0$ for $i \neq j$.

Remark 3.6 If A is a finite dimensional Koszul agebra, then the quadratic duality functor for modules can be extended to give an equivalence of derived categories of graded modules over A and $A^!$ respectively. More precisely one has the so-called Koszul duality functor

$$D^b(A - \text{mod}^{fg}) \longrightarrow D^b((A^! - \text{mod}^{fg})^{op})$$

$$M \longrightarrow \text{cobar}(A, M).$$

Here $\text{cobar}(A)$ is the cobar algebra associated to A, namely $\text{cobar}(A)$ is a dg algebra, which is free as a graded algebra, i.e. $\text{cobar}(A) = \boldsymbol{T}(A_+^\vee[-1])$, and has a differential which is uniquely determined by the property that its restriction to $A_+^\vee$ is the dual $A_+^\vee \to A_+^\vee \otimes A_+^\vee$ to the multiplication $A_+ \otimes A_+ \to A_+$.

Similarly $\text{cobar}(A, M) = \text{cobar}(A) \otimes M^\vee$ as a complex of graded modules with a differential extending the action of A on M.

Example 3.7 (Bernstein-Gelfand-Gelfand) If V is a finite dimensional vector space, then the derived categories of modules over $S^\bullet V$ and $\wedge^\bullet V^\vee$ are equivalent.

3.2 Filtered quadratic duality

In order to make the connection with the Higgs bundles (and their generalizations) we will need a filtered version of quadratic duality which in its most general form is due to Leonid Positselski.

Let

$$
\begin{aligned}
&F^0A \subset F^1A \subset F^2A \subset \ldots A,\\
&F^iA \cdot F^jA \subset F^{i+j}A
\end{aligned}
$$

be a filtered unital associative algebra over $\mathbb{C}$. We assume that:

- A is connected, i.e. $F^0A = \mathbb{C}$ and the unit satisfies $1_A = 1 \in \mathbb{C} = F^0A \subset A$;

- A is locally finite dimensional, i.e. $\mathrm{gr}_F(A)$ is a locally finite dimensional graded algebra.

Definition 3.8 *A filtered algebra A is called a* (filtered) quadratic *algebra if A is generated by F^1A and has relations in degree $\leq$ two.*

This can be spelled out as follows. Consider the reduced tensor algebra

$$\boldsymbol{T}(1_A \in F^1A) := \boldsymbol{T}(F^1A)/\langle 1_{\boldsymbol{T}} - 1_A\rangle$$

generated by the two step filtration

$$[\mathbb{C} \cdot 1_A \subset F^1A].$$

This is a filtered algebra with a filtration

$$\boldsymbol{T}_0(1_A \in F^1A) \subset \boldsymbol{T}_1(1_A \in F^1A) \subset \ldots,$$

given by $\boldsymbol{T}_i(1_A \in F^1A) := \mathrm{im}(F^1A)^{\otimes i}$. Now A is a filtered quadratic algebra if the canonical map

$$\mathrm{can} : \boldsymbol{T}(1_A \in F^1A) \to A$$

is surjective and if ker(can) is generated by its subspace

$$J_A := \ker(\mathrm{can}) \cap \boldsymbol{T}_2(1_A \in F^1A)$$

as a two-sided ideal in $\boldsymbol{T}(1_A \in F^1A)$.

Equivalently, a filtered quadratic algebra is uniquely determined by:

- a finite dimensional vector space W (generators);

- a fixed vector $e \in W$ (unit);

- a subspace $J \subset \boldsymbol{T}_2(e \in W)$ (relations).

Indeed, given A we can take $W = F^1A$, $e = 1_A$ and $J = J_A$. Conversely, given W, e and J we define $A := \boldsymbol{T}(e \in W)/\langle J \rangle$.

Notation: $A = \{e \in W, J\}$.

Note: A filtered quadratic algebra $A = \{e \in W, J\}$ has an associated ordinary quadratic algebra

$$A^{(0)} := \{W/\mathbb{C} \cdot e, J \bmod \boldsymbol{T}_1(e \in W)\}.$$

Moreover $A^{(0)}$ coincides with the quadratic part $q\,\mathrm{gr}_F(A)$ of the associated graded algebra $\mathrm{gr}_F(A)$.

We say that A is a Koszul algebra if $A^{(0)}$ is a Koszul algebra. It turns out that if A is Koszul, then $\mathrm{gr}_F(A)$ is in fact isomorphic to $A^{(0)}$.

Remark 3.9 For any graded algebra $A = \oplus_{i \geq 0} A_i$, there is a uniquely defined quadratic algebra qA together with a canonical map $qA \to A$ which is an isomorphism in degree one and a monomorphism in degree two. Explicitly we have

$$qA = \{A_1, \ker(\boldsymbol{T}A_1 \to A) \cap A_1 \otimes A_1)\}.$$

The algebra qA is called *the quadratic part* of A.

For any graded algebra A and any graded A-module M concentrated in non-negative degrees, there is a uniquely defined quadratic module q_AM over the quadratic part qA of A, together with a morphism $q_AM \to M$ of qA-modules which is an isomorphism in degree zero and a monomorphism in degree one. Explicitly

$$q_AM = \{M_0, \ker(qA \otimes M_0 \to M) \cap A_1 \otimes M_0\}_{qA}.$$

The module q_AM is called the *quadratic part* of M.

Hope: In the Koszul case, we may be able to define a quadratic dual of a filtered A by endowing the dual graded algebra $(A^{(0)})^!$ with some extra data remembering the extensions corresponding to the filtration $F^\bullet A$. It turns out that the extra data needed is a differential on the graded algebra $(A^{(0)})^!$.

Definition 3.10 *A* curved differential graded algebra *is a triple* $\boldsymbol{B} = (B, d_B, h_B)$, *where* $B = \oplus_{i=0}^{\infty} B_i$ *is a graded algebra,* $d_B : B_i \to B_{i+1}$ *is an odd derivation, and* $h_B \in B_2$ *is an element such that* $d_B^2 x = [h_B, x]$ *for all* x *and* $d_B h_B = 0$.

A morphism $\boldsymbol{g} : \boldsymbol{B} \to \boldsymbol{C}$ *of curved dg algebras is a pair* $\boldsymbol{g} = (g : B \to C, \alpha \in C_1)$*, satisfying*

$$g(d_B x) = d_C g(x) + [\alpha, g(x)]$$
$$g(h_B) = h_C + d_C\alpha + \alpha^2.$$

A left curved dg module over $\boldsymbol{B}$ *is a pair* $\boldsymbol{N} = (N, d_N)$ *consisting of a graded* B*-module* N *and an odd derivation* $d_N : N_i \to N_{i+1}$ *compatible with* d_B *and such that* $d_N^2 u = h_B u$*.*

A curved dg algebra is called *quadratic* or *Koszul* if the underlying graded algebra is of the corresponding type. A curved dg algebra witha zero curvature $h_B = 0$ is just an ordinary dg algebra.

Given a filtered quadratic algebra A, choose a subspace $V \subset F^1 A$ splitting the map $F^1 A \to F^1 A/F^0 A$. Let R denote the space $J_A \bmod \boldsymbol{T}_1(1_A \in F^1 A)$ viewed as a subset in $V \otimes V$. In terms of V and R we have

- $A^{(0)} = \{V, R\}$, $\boldsymbol{T}_i(1_A \in F^1 A) = \oplus_{k=0}^{i} V^{\otimes i}$, and
- $J_A \subset \boldsymbol{T}_2(1_A \in F^1 A) = \mathbb{C} \oplus V \oplus (V \otimes V)$ is the graph of some linear map

$$\psi = (\varphi, h) : R \to V \oplus \mathbb{C}$$

Moreover the map ψ satisfies

$$(*) \qquad (\psi^{12} - \psi^{23})(V \otimes R \cap R \otimes V) \subset \Gamma_\psi$$

Let $B := (A^{(0)})^!$. By definition $B_2 = R^\vee$ and so φ and h dualize to $\varphi^\vee : B_1 \to B_2$ and $h_B \in B_2$. The condition (*) translates into the fact that φ^* can be extended to an odd derivation d_B of degree 1 and that (B, d_B, h_B) is a curved dg algebra.

Note: The above construction assigns a curved dga $\boldsymbol{B} = (B, d_B, h_B)$ to every filtered quadratic algebra $\{e \in W, J\}$ equipped with a splitting $W/\mathbb{C} \cdot e \widetilde{\to} V \subset W$ of the map $W \twoheadrightarrow W/\mathbb{C} \cdot e$. We will call such algebras *almost-split*.

Choosing a different splitting V' results in an isomorphic curved dga $\boldsymbol{B}' = (B, d'_B, h'_B)$. Indeed, given $V' \subset W$ we can find a linear map $\alpha : V \to \mathbb{C}$ such that

$$V' = \{x - \alpha(x) | x \in V\}.$$

In terms of α the isomorphism $\boldsymbol{f} : \boldsymbol{B} \widetilde{\to} \boldsymbol{B}'$ is given by the pair $\boldsymbol{f} = (\mathrm{id}_B, \alpha^\vee)$.

The assignment $A \mapsto ((A^{(0)})^!, \varphi^\vee, h^\vee)$ gives rise to a filtered quadratic duallity functor

$$! : (\text{filt-quadAlg}_{\mathbb{C}}^{as}) \to (\text{c-dgAlg}_{\mathbb{C}})^{op}$$

which is fully faithful.

Under the Koszulity assumption one can say more:

Theorem [L.Positselski,[Pos93, PP98]] *The filtered quadratic duality functor establishes an anti-equivalence between the category of almost split filtered Koszul algebras (respectively augmented filtered Koszul algebras) and the category of Koszul curved dg algebras (respectively Koszul dg algebras).*

Similarly one can show:

Theorem [L.Positselski,[Pos93, PP98] *For any almost split filtered quadratic algebra A there is a fully faithful functor between the category of quadratic A-modules and the category of curved dg modules over $A^!$. For Koszul algebras this functor induces an anti-equivalence of the respective categories of Koszul modules.*

Remark 3.11 The special case of quadratic duality for augmented filtered algebras is due to S.Priddy [Pri70]. It was rediscovered later and generalized to the relative context by C.Simpson under the name 'duality for split almost polynomial rings of differential operators' [Sim92].

3.3 Higgs bundles revisited

We are now ready to recast the spectral construction for Higgs bundles on a smooth complex space S as a filtered Koszul duality for families of algebras over S.

Let as before K be a fixed coefficient bundle and let $X = \operatorname{tot}(K)$ be its total space. Consider the sheaf of algebras $A = S^{\bullet}K^{\vee}$ on S with the natural filtration induced from the grading. The quadratic dual algebra is the trivial dg algebra $(\wedge^{\bullet}K, 0, 0)$:

$$\mathcal{O}_S \xrightarrow{0} K \xrightarrow{0} \wedge^2 K \xrightarrow{0} \ldots \xrightarrow{0} \wedge^n K.$$

Every quasi-coherent sheaf $\mathscr{E}$ on X can be viewed as a filtered module over A and so corresponds by quadratic duality to a dg module $\mathscr{E}^!$ over $(\wedge^{\bullet}K, 0, 0)$. Explicitly we have

$$\mathscr{E}^! = \left(E \xrightarrow{\wedge\phi} E \otimes K \xrightarrow{\wedge\phi} \ldots \xrightarrow{\wedge\phi} E \otimes \bigwedge^n K \right),$$

where (E, ϕ) is the corresponding Higgs sheaf.

Note that even though the differential in the quadratic dual algebra $A^!$ is trivial, we still can have a non-trivial differential for the module.

The Koszul reinterpretation of the spectral construction is useful because it gives us a way to deform the correspondence. Next we explore various commutative and non- commutative deformations of the spectral construction.

4 Deformations of the spectral construction

Fix a smooth complex variety S and a (coefficient) vector bundle K of rank n on S.

Let $p : X = \mathrm{tot}(K) \to S$ be the total space of K. The spectral correspondence establishes a bijection between the following types of geometric data

(Spectral data) Coherent sheaves $\mathscr{E} \to X$ which are finite over S (B-branes on X).

(K-valued Higgs data) Coherent sheaves $E \to S$ equipped with a Higgs field ϕ, i.e. an $\mathcal{O}_S$-linear K-valued endomorphism

$$\phi : E \to E \otimes K$$

satisfying $\phi \wedge \phi = 0$.

We interpreted the correspondence as a special case of filtered Koszul duality as follows:

- View the spectral sheaf $\mathscr{E} \to X$ as a module over the sheaf of algebras $S^\bullet K^\vee$ over S, i.e. replace $\mathscr{E}$ with the equivalent data

$$(E := p_* \mathscr{E} \to S) + (S^\bullet K^\vee - \text{action on } E).$$

- View the Higgs sheaf (E, ϕ) as a dg module

$$E \stackrel{\wedge\phi}{\to} E \otimes K \stackrel{\wedge\phi}{\to} \ldots \stackrel{\wedge\phi}{\to} E \otimes \wedge^n K$$

over the dga $\mathcal{O}_S \stackrel{0}{\to} K \stackrel{0}{\to} \ldots \stackrel{0}{\to} \wedge^n K$.

- Use filtered Koszul duality to convert modules over the filtered quadratic algebra $S^\bullet K^\vee$ and dg modules over the dg algebra $(\wedge^\bullet K, 0)$.

Remark 4.1 • The Koszul reformulation of the spectral correspondence has the adavntage of exhibiting both the Higgs and the spectral data in a manifestly deformable form. Indeed, by deforming the structures on $S^\bullet K^\vee$ and $(\wedge^\bullet K, 0)$ so that the Koszul duality still holds, we can obtain a new kind of spectral duality between the deformed module structures.

• Note that there are three possible ways in which we can perturb the structure of $S^\bullet K^\vee$ so that the resulting algebra will still be filtered quadratic. Clearly $S^\bullet K^\vee$ is a filtered quadratic algebra of the most trivial type: it is commutative, augmented and the filtration is completely split.

Thus when we start deforming the product structure on $S^\bullet K^\vee$ we can perform the deformation so that:

◊ the product becomes non-commutative;

◊ the augmentation ceases to be an algebra morphism;

◊ the filtration is not split anymore.

Similarly we can deform the curved dg algebra structure on $(\wedge^\bullet K, 0, 0)$ so that:

◊ the product becomes non-commutative;

◊ the differential becomes non-zero;

◊ the curvature becomes non-zero.

Note: An interesting feature of Koszul duality is that the duality transformation mixes the different types of deformations. Here are specific examples of this phenomenon.

Examples: (i) There are natural deformations of $S^\bullet K^\vee$ as a filtered commutative algebra which is filtered quadratic and has associated graded isomorphic to $S^\bullet K^\vee$. (In particular the deformation will be filtered Koszul.)

Indeed, let $\omega \in H^1(S, K)$. Then ω determines a deformation of the variety $p : X \to S$, namely the total space $X_\omega := \mathrm{tot}(K_\omega)$ of the affine bundle $K_\omega \to S$ corresponding to the class ω.

Concretely ω corresponds to an extension

$$(\omega) \qquad 0 \to K \to F_\omega \overset{\pi_\omega}{\to} \mathcal{O}_S \to 0,$$

and $K_\omega = \pi_\omega^{-1}(1)$ (note that $K = \pi_\omega^{-1}(0)$). Let $p_\omega : X_\omega \to S$ be the natural projection and let

$$S_\omega^\bullet K^\vee := p_{\omega *}\mathcal{O}_{X_\omega}$$

be the sheaf of commutative algebras of functions along the fibers of p_ω.

Note: The fact that X_ω has no section means that $S_\omega^\bullet K^\vee$ is filtered but not graded. Geometrically $X_\omega = \mathbb{P}(F_\omega) - \mathbb{P}(K)$, and in fact

$$p_{\omega *}\mathcal{O}_{X_\omega} = p_{\omega *}\mathcal{O}_{\mathbb{P}(F_\omega)}(\infty \cdot \mathbb{P}(K)).$$

Thus $S_\omega^\bullet K^\vee$ is filtered by "order of poles along the divisor $\mathbb{P}(K) \subset \mathbb{P}(F_\omega)$ at infinity".

Algebraically we have

$$S_\omega^\bullet K^\vee = S^\bullet F_\omega^\vee / \langle 1_{S^\bullet F_\omega^\vee} - 1\rangle,$$

and so

$$F^i S_\omega^\bullet K^\vee \cong S^i F_\omega^\vee$$

and the filtration is given by the natural maps

$$\mathcal{O}_S \overset{\cdot\pi_\omega^\vee}{\hookrightarrow} F_\omega^\vee \overset{\cdot\pi_\omega^\vee}{\hookrightarrow} S^2 F_\omega^\vee \overset{\cdot\pi_\omega^\vee}{\hookrightarrow} S^3 F_\omega^\vee \overset{\cdot\pi_\omega^\vee}{\hookrightarrow} \dots$$

This implies that $S^\bullet_\omega K^\vee$ is filtered quadratic and that $\mathrm{gr}_F(S^\bullet_\omega K^\vee) = S^\bullet K^\vee$.

In particular the non-homogeneous quadratic dual of $S^\bullet_\omega K^\vee$ will be a sheaf $\boldsymbol{\wedge^\bullet_\omega K}$ of curved dga whose underlying sheaf of graded algebras is $\wedge^\bullet K = (S^\bullet K^\vee)^!$.

Note that the modules over the filtered algebra $S^\bullet_\omega K^\vee$ are just the quasi-coherent sheaves (= B-branes) on the deformed space X_ω. So the filtered quadratic duality will convert the B-branes on X_ω into Higgs-like objects on S.

It is not hard to describe the curved dga $\boldsymbol{\wedge^\bullet_\omega K}$ and the corresponding modules explicitly.

Let $\{U_i\}$ be a Čech cover of S w.r.t. which $\omega \in H^1(S, K)$ is represented by a Čech cocycle

$$\{\omega_{ij}\} \in Z^1(\{U_i\}, K).$$

Now for each i the restricted fibration

$$X_{\omega|U_i} \to U_i$$

has a section which gives rise to a natural splitting of $F_\omega^\vee \twoheadrightarrow K^\vee$ over U_i. Thus we can repeat the previous construction over each U_i. In this way we get the sheaf of curved dga

$$\boldsymbol{\wedge^\bullet_\omega K} := \coprod_i (\wedge^\bullet K, 0, 0)_{|U_i} / \sim,$$

where $(\wedge^\bullet K_{|U_i}, 0, 0)$ and $(\wedge^\bullet K_{|U_j}, 0, 0)$ are glued to each other via the isomorphism of curved dga given by the pair $(\mathrm{id}, \omega_{ij})$.

Note that this pair indeed gives a well defined automorphism of the curved dga $(\wedge^\bullet K_{|U_{ij}}, 0, 0)$:

- $[\omega_{ij}, x] = 0$ for all $x \in \wedge^\bullet K$ since $\wedge^\bullet K$ is supercommutative;
- $\omega_{ij} \wedge \omega_{ij} = 0$ since $\omega_{ij} \in K$.

Thus $\boldsymbol{\wedge^\bullet_\omega K}$ is a sheaf of curved dga which is a twisted form of (i.e. is locally isomorphic to) $(\wedge^\bullet K, 0, 0)$.

Filtered quadratic duality converts $S^\bullet_\omega K^\vee$ modules into $\boldsymbol{\wedge^\bullet_\omega K}$ curved dg modules or ω-*twisted K-valued Higgs sheaves* on S.

Explicitly an ω-twisted Higgs sheaf is a quasi-coherent sheaf $E \to S$ equipped with a collection of local K-valued Higgs fields

$$\phi_i : E_{|U_i} \to E_{|U_i} \otimes K_{|U_i}, \quad \phi_i \wedge \phi_i = 0,$$

so that

$$\phi_i - \phi_j = \omega_{ij} \cdot \mathrm{id}_E \ \text{ on } U_{ij}.$$

Remark: If $(E, \{\phi_i\})$ is an ω-twisted Higgs bundle of rank r, then

$$\frac{1}{r}\operatorname{tr}(\phi_i) - \frac{1}{r}\operatorname{tr}(\phi_j) = \omega_{ij},$$

i.e. $\omega = 0 \in H^1(S, K)$.

Conclusion: There are *no* classical spectral sheaves in X_ω, i.e. sheaves $\mathscr{E} \to X_\omega$ which are coherent and finite over S, as long as $\omega \neq 0$.

In fact if $\mathscr{E}$ is a classical spectral sheaf on X_ω, then

$$\omega_{|\operatorname{supp}(p_{\omega *}\mathscr{E})} = 0.$$

Thus a B-brane on X can deform to X_ω only if it is quasi-coherent or if the deformation direction vanishes on its support.

Digression on gerbes: The notion of a twisted Higgs sheaf on S resembles a lot the notion of a sheaf twisted by a class $\alpha \in H^2(S, \mathcal{O}_S^\times)$.

Recall: If $\{U_i\}$ is a Čech cover of S w.r.t. which $\alpha \in H^2(S, \mathcal{O}_S^\times)$ is represented by a cocycle

$$\{\alpha_{ijk}\} \in Z^2(\{U_i\}, \mathcal{O}^\times),$$

then an α-twisted sheaf on S is a collection (F_i, g_{ij}), where:

- F_i is a coherent sheaf on U_i;
- $g_{ij} : F_{i|U_{ij}} \widetilde{\to} F_{j|U_{ij}}$ are sheaf isomorphisms satisfying the twisted cocycle condition

$$g_{ij} \circ g_{jk} \circ g_{ki} = \alpha_{ijk} \cdot \mathrm{id}\,.$$

From the viewpoint of the geometry of stacks, twisted sheaves appear as ordinary sheaves. More precisely: the element $\alpha \in H^2(S, \mathcal{O}_S^\times)$ classifies an algebraic (or analytic) $\mathcal{O}^\times$-gerbe ${}_\alpha S$ on S and an α-twisted sheaf is simply a (special kind of) sheaf on ${}_\alpha S$.

The ω-twisted Higgs sheaves admit a similar interpretation as sheaves on a gerbe. Before we spell this out we need to recast the ordinary (untwisted) Higgs sheaves as sheaves on some geometric object. Such an incarnation of Higgs sheaves was proposed and studied by C. Simpson [Sim97, Sim02]:

For a vector bundle $N \to S$, let $\widehat{N} \to S$ denote the formal completion of N along the zero section. Then $\widehat{N}$ is a formal group scheme and we can consider the formal stack

$$S_N := [S/\widehat{N}] = B\widehat{N}.$$

Now a sheaf on S_N is simply a $N^\vee$-valued Higgs sheaf on S .Similarly one may consider the algebraic stack $BN = [S/N]$. Sheaves on BN correspond to nilpotent $N^\vee$-valued Higgs sheaves on S.

Going back to the ω-twisted Higgs sheaves, note that the Leray spectral sequence applied to the map $S_{K^\vee} \to S$ allows us to view $\omega \in H^1(S,K)$ as an element in $H^2(S_{K^\vee}, \mathcal{O})$. In particular, ω gives rise to an $\mathcal{O}$-gerbe ${}_\omega S_{K^\vee}$ on the formal stack $S_{K^\vee}$.

In these terms, the ω-twisted Higgs sheaves on S become simply sheaves on the gerbe ${}_\omega S_{K^\vee}$. This gives yet another (stacky) interpretation of the spectral data living on the variety X_ω.

Comments:

- The correspondence

$$\left\{\begin{array}{l}\text{quasi-coherent sheaves on}\\ \text{the affine bundle } X_\omega\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{quasi-coherent sheaves on}\\ \text{the } \mathcal{O}\text{-gerbe } {}_\omega S_{K^\vee}\end{array}\right\}$$

  is another instance of a mathematical duality similar to the spectral correspondence. In this case it is the Fourier-Mukai duality for commutative group stacks.

  Recently a different occurence of this duality, dealing with gerbes over elliptic fibrations, was worked out in [DP03].

- It is not hard to describe the gerbe ${}_\omega S_{K^\vee}$ explicitly. Indeed, the short exact sequence of vector bundles

$$0 \to K \to F_\omega \to \mathcal{O}_S \to 0,$$

  gives rise to a short exact sequence of commutative group stacks

$$0 \to B\widehat{\mathcal{O}_S} \to B\widehat{F_\omega^\vee} \to S_{K^\vee} \to 0,$$

  which in turn can be viewed as an $\mathcal{O}$-gerbe on $S_{K^\vee}$. This is precisely the gerbe ${}_\omega S_{K^\vee}$.

  It is also instructive to note here that this gerbe can be naturally identified with the stack of homomorphisms $\underline{\mathrm{Hom}}_{\mathrm{cgs}}(F_\omega, B\mathcal{O}_S)$, where both F_ω and $B\mathcal{O}_S$ are both viewed as commutative group stacks over S.

(ii) Let $h \in H^0(S, \wedge^2 K)$ be any element. Again, since $\wedge^\bullet K$ is supercommutative we have

$$[h, x] = 0$$

for all $x \in \wedge^\bullet K$. Thus $(\wedge^\bullet, 0, h)$ is again a curved dg algebra over S. The dg modules over this algebra are the h-curved Higgs bundles, i.e. the pairs

$$(E, \phi : E \to E \otimes K), \quad \text{such that } \phi \wedge \phi = h \cdot \mathrm{id}_E\,.$$

It is not hard to describe the Koszul dual objects.

The section $h \in H^0(S, \wedge^2 K)$ gives an extension of $K^\vee$ as a sheaf of Lie algebras. Explicitly consider the vector bundle

$$L_h := \mathcal{O}_S \cdot c \oplus K^\vee$$

with c being a dummy variable. Now h defines a Lie bracket on L_h given by

$$\begin{aligned}
&[c, c] = 0 \\
&[c, a] = 0, \text{ for all } a \in K^\vee \\
&[a, b] = \langle h, a \wedge b \rangle \cdot c, \text{ for all } a, b \in K^\vee.
\end{aligned}$$

By construction $(L_h, [\ ,\ ])$ is a sheaf of nilpotent Lie algebras on S with an $\mathcal{O}_S$-linear Lie bracket.

Moreover $\mathcal{O} \cdot c \subset L_h$ is a central ideal and we have a short exact sequence of Lie algebra sheaves

$$0 \to \mathcal{O} \cdot c \to L_h \to K^\vee \to 0$$

where both $\mathcal{O} \cdot c$ and $K^\vee$ are taken to be commutative.

With this notation we can identify the Koszul dual

$$(\wedge^\bullet K, 0, h)^! = U(L_h)/\langle 1_{U(L_h)} - c \rangle =: U_h$$

with the enveloping algebra of the central extension L_h taken with its natural filtration.

Comment: Note that again

$$\mathrm{gr}_F U_h = S^\bullet K^\vee$$

and so the h-curved Higgs bundles have a dual interpretation as *non-commutative branes*, i.e. modules over U_h.

Here we think of U_h as "$p_{h*}\mathcal{O}_{X_h}$" of a non-commutative deformation

$$p_h : X_h \to S$$

of the map $p : X \to S$.

(iii) More generally one can check that if

$$(\wedge^\bullet K, d, h)$$

is an arbitrary curved dg algebra having $\wedge^\bullet K$ as the underlying graded algebra, then

$$d^\vee : \wedge^2 K^\vee \to K^\vee$$

will be a Lie bracket on $K^\vee$ and $h \in H^0(S, \wedge^2 K)$ is a two cocycle for the Lie algebra $(K^\vee, d^\vee)$. Let

$$0 \to \mathcal{O} \cdot c \to L_{h,d} \to K^\vee \to 0$$

be the corresponding central extension as sheaves of Lie algebras.

Then

$$(\wedge^\bullet K, d, h)^! = U(L_{h,d})/\langle 1_{U(L_{h,d})} - c\rangle =: U_{h,d}$$

as a filtered algebra and so can be thought of as a non-commutative deformation of $p : X \to S$ again.

Observe: If $h = 0$ then $(\wedge^\bullet K, d, 0)$ as the Cartan-Eilenberg complex of $L_{0,d}$ with trivial coefficients. That is:

$$(\wedge^\bullet K, d, 0) = C^\bullet(K^\vee, \mathcal{O}_S).$$

Similarly, for any module M over the Lie algebra $(K^\vee, d^\vee)$ we have that M viewed as a module over $U_{0,d}$ is a filtered quadratic module and that

$$M^! = C^\bullet(K^\vee, M)$$

is the Cartan-Eilenberg complex computing the cohomology of $(K^\vee, d^\vee)$ with coefficients in M.

(iv) An important special case is when $K = \Omega^1_S$. Then we have a natural dg algebra

$$(\Omega^\bullet_S, d) = (\wedge^\bullet \Omega^1_S, d, 0)$$

- the *holomorphic de Rham complex* on S.

Recall that the modules over $(\Omega^\bullet_S, d)$ are precisely the quasi-coherent sheaves equipped with a flat connection, i.e. are the analytic (algebraic) $\mathscr{D}_S$-modules. This fact is a manifestation of filtered Koszul duality since

$$\mathscr{D}_S = (\wedge^\bullet \Omega^1_S, d, 0)^!$$

is exactly the Koszul dual filtered algebra.

Conclusion: The local systems on S can be interpretted dually as non-commutative spectral covers, i.e. as $\mathscr{D}_S$-modules. The latter should be thought of as sheaves on a non-commutative deformation of $p : X \to S$ whose structure sheaf pushes forward to $\mathscr{D}_S$.

Variant: Taking a deormation of Ω^1_S to a ω-twisted cotangent bundle and introducing a curvature $h \in \Omega^2_S$ one gets (similarly to **(i)** and **(ii)**) a ω-twisted curved dg algebra

$$(\Omega^\bullet_S, d, h)_\omega$$

which is Koszul dual to a sheaf $(\mathscr{D}_S)_{h,\omega}$ of twisted differential opertors on S. This gives yet another non-commutative deformation of $p : X \to S$.

5 Framed sheaves on ruled Poisson surfaces

Heuristics: Non-commutative deformations of a scheme X should be thought of as deformations of the abelian category $\mathrm{Coh}(X)$ or more generally of the triangulated category $D^b(X)$.

Bondal observed that the typical infinitesimal deformations of $D^b(X)$ come from deformations of the identity functor on $D^b(X)$ which in turn can be computed as the second Hoschchild cohomology of X, i.e.

$$\begin{aligned}\mathrm{Ext}^2_{D^b(X)}(\mathbb{I},\mathbb{I}) &= \mathrm{Hom}_{D^b(X)}(\mathbb{I},\mathbb{I}[2])\\ &= HH^2(X).\end{aligned}$$

Assume for simplicty that X is smooth. Then it is known by the Gerstenhaber-Shack theorem that $HH^2(X)$ has a Hodge type decomposition:

$$HH^2(X) = H^2(\mathcal{O}_X) \oplus H^1(T_X) \oplus H^0(\wedge^2 T_X).$$

The different pieces in this decomposition have different meaning:

$H^1(T_X)$ parameterizes ordinary geometric deformations of X.

$H^0(\wedge^2 T_X)$ parameterizes deformations of the product on the algebra $\mathcal{O}_X$ to some associative product.

parameterizes deformations of X in the stacky direction, i.e. deformations as an $\mathcal{O}_X^{\times}$-gerbe.

Important observation: The passage to moduli can mix different types of deformations. In particular if we have a moduli problem on X for which the discrete data is fixed so that it deforms unobstructedly in any of the three directions above, then the moduli space may have commutative deformations which are interprettable as moduli spaces of the same type of data on a non-commutative or stacky deformation of X.

A large class of examples of this phenomenon are provided by moduli spaces of framed sheaves.

General setup: Let S be a complex surface which has some non-commutative deformations (at least infinitesimally). This means that S must be a Poisson surface.

Let $\lambda \in H^0(S, \wedge^2 T_S)$ be a fixed Poisson structure. We would like to say that the moduli space M of sheaves on S with framing along $\lambda = 0$ will have some additional deformations M_t (not corresponding to deformations of S) and we would like to identify M_t with the moduli space of framed sheaves on the non-commutative deformation of S in the direction of λ.

(a) (Nekrasov-Schwartz, Kapustin-Kuznetsov-Orlov) Let $S = \mathbb{P}^2$ with the Poisson structure $\lambda = f^3$ where $f \in H^0(\mathbb{P}^2, \mathcal{O}(1))$ is the equation of some line $L \subset S$.

Consider the moduli space

$$M = \left\{ \left(E, \varphi : E_{|L} \to \mathcal{O}_L^{\oplus r}\right) \middle| \begin{array}{l} E \text{ is a rank } r \text{ vector bundle on } S \\ \text{with } c_1(E) = 0 \text{ and } c_2(E) = k. \end{array} \right\}$$

of framed sheaves on $\mathbb{P}^2$. Using the ADHM construction Donaldson showed that M is isomorphic to the moduli space of $SU(r)$ instantons of instanton numbers k on S^4.

In particular M is hyper-Kähler and has a natural twistor deformation

$$\begin{array}{ccccc} M & = & \mathscr{M}_0 & \subset & \mathscr{M} \\ & & \downarrow & & \downarrow \\ & & 0 & \in & \mathbb{P}^1 \end{array}$$

Moreover the restriction of the twistor family $\mathscr{M}$ on the open subset $\mathbb{C}^\times = \mathbb{P}^1 - \{0, \infty\}$ is a holomorphically trivial family, i.e.

$$\mathscr{M}_{|\mathbb{C}^\times} \cong \mathscr{M}_1 \times \mathbb{C}^\times,$$

where $\mathscr{M}_1$ is the fiber of $\mathscr{M} \to \mathbb{P}^1$ over $1 \in \mathbb{P}^1$.

Now Nekrasov-Shwartz and Kapustin-Kuznetsov-Orlov prove [NS98, KKO01]

Theorem *$\mathscr{M}_1$ is isomorphic to the moduli space of pairs (E, φ), where:*

- *E is a rank r vector bundle on the non-commutative $\mathbb{P}^2_\lambda$ defined by λ, having $c_1(E) = 0$ and $c_2(E) = k$;*
- *φ is a framing of E along the commutative line $L \subset \mathbb{P}^2_\lambda$.*

(b) (Baranovsky-Ginzburg-Kuznetsov) Let X be a projective variety and let A be a graded algebra defining X, i.e. $X = \mathrm{Proj}(A)$. Recall that by Serre's theorem the category $\mathrm{Coh}(X)$ of coherent sheaves on X is equivalent to the quotient category

$$\mathrm{qgr}(A) := \frac{(\text{finitely generated graded } A\text{-modules})}{(\text{finite dimensional graded } A\text{-modules})}.$$

Consider now a finite group Γ acting on X and let $A\sharp\mathbb{C}\Gamma$ be the smash product of algebras with the grading in which $\mathbb{C}\Gamma$ is placed in degree zero.

The algebra $A\sharp\mathbb{C}\Gamma$ is a graded non-commutative algebra and when it happens to be regular of dimension d, then we can view it as the algebra of homogeneous functions on some non-commutative d-dimensional projective scheme $\mathrm{Proj}(A\sharp\mathbb{C}\Gamma)$.

Remark 5.1 Recall that a locally finite dimensional graded algebra $A = \oplus_{i \geq 0} A_i$ is called *regular of dimension* d if:

- A_0 is a semisimple $\mathbb{C}$-algebra;
- A global homological dimension d, i.e. d is the minimal integer such that

$$\mathrm{Ext}_{A-\mathrm{Mod}}(M, N) = 0, \text{ for all } M, N \in A - \mathrm{Mod};$$

- A is Noetherian of polynomial growth, i.e. we can find integers $m, n > 0$ so that

$$\dim_{\mathbb{C}} A_i \leq m i^n, \text{ for } i \gg 0.$$

- A is Gorenstein of type (d, ℓ), i.e.

$$\mathrm{Ext}^i_{A-\mathrm{Mod}}(A_0, A) = \begin{cases} A_0(\ell), & \text{if } i = d \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly we can make sense of the category $\mathrm{Coh}(\mathrm{Proj}(A\sharp\mathbb{C}\Gamma))$ of coherent sheaves on the non-commutative scheme $\mathrm{Proj}(A\sharp\mathbb{C}\Gamma)$ as the quotient category

$$\mathrm{qgr}(A\sharp\mathbb{C}\Gamma) := \frac{(\text{finitely generated graded } A\sharp\mathbb{C}\Gamma\text{-modules })}{(\text{finite dimensional graded } A\sharp\mathbb{C}\Gamma\text{-modules})}.$$

The next step is consider deformations of $A\sharp\mathbb{C}\Gamma$ as a graded associative algebra. These deformations will give rise to new non-commutative projective varieties.

Digression: The non-commutative projective variety $\mathrm{Proj}(A\sharp\mathbb{C}\Gamma)$ defined above is a non-commutative variety of a very special kind - its non-commutativity is coming from twisting the multiplication in the homogeneous algebra of functions on the ordinary projective variety $X = \mathrm{Proj}(A)$ by the action of the group of authomorphisms $\Gamma \subset \mathrm{Aut}(X)$. Such non-commutative varieties are called *twisted projective varieties.* An interesting feature of these varieties is that their geometry (e.g. their sheaf theory) can also be interpreted by using quotient stacks rather than non-commutative spaces. Indeed, it is not hard to see that the category $\mathrm{qgr}(A\sharp\mathbb{C}\Gamma)$ of coherent sheaves on $\mathrm{Proj}(A\sharp\mathbb{C}\Gamma)$ is equivalent to the category of coherent sheaves on the stack quotient $[X/\Gamma]$ or equivalently to the category of Γ-equivariant coherent sheaves on X.

In particular, if Γ acts freely, then $[X/\Gamma]$ is an ordinary variety and so the sheaf theory on $\mathrm{Proj}(A\sharp\mathbb{C}\Gamma)$ is undistinguishable from the sheaf theory on a commutative variety. This setup also shows that we can deform the category of coherent sheaves on a projective variety to a category which admits two different interpretations: as the category of sheaves on a stack and as the category of sheaves of a twisted projective variety.

(c) (Nevins-Stafford) Let again $S = \mathbb{P}^2$ but now take $\lambda \in H^0(\mathbb{P}^2, \mathcal{O}(3))$ to be the equation of a smooth cubic $\Sigma \subset S$.

Consider the Hilbert schemes

$$\mathrm{Hilb}^k(S), \qquad \mathrm{Hilb}^k(S - \Sigma)$$

of k-points on S and $S - \Sigma$ respectively. As usual we can identify $\mathrm{Hilb}^k(S)$ with the moduli space of stable torsion free rank one sheaves on S having $c_1 = 0$ and $c_2 = k$. Similarly $\mathrm{Hilb}^k(S - \Sigma)$ admits an interpretation as the moduli of framed rank one torsion free sheaves on S with framing $\mathcal{O}_\Sigma$ along Σ.

From the work of Artin-Tate-Van den Bergh it is known that the infinitesimal non-commutative deformation of S given by λ can be integrated to an actual non-commutative deformation (given by a Sklyanin algebra) as long as we choose an automorphism $s : \Sigma \to \Sigma$ of the elliptic curve Σ.

(d) In the remainder of the section we focus on the case of Poisson ruled surfaces. To set things up we fix the following notation:

C - a smooth compact complex curve of genus $g > 1$.

X - the projective bundle $X := \mathbb{P}(\mathcal{O}_C \oplus \omega_C) \xrightarrow{\pi} C$.

$\mathcal{O}_X(1)$ - the relative hyperplane bundle of X over C, normalized by the condition $\pi_* \mathcal{O}_C(1) \cong \mathcal{O}_C \oplus \omega_C^{-1}$.

D - the divisor $D \subset X$ at infinity corresponding to the line $\mathcal{O}_C \subset \mathcal{O}_C \oplus \omega_C$.

Note that the surface X has a canonical Poisson structure. Indeed, observe that

- The map $\pi_{|D} : D \to C$ is an isomorphism and $X = D \coprod \mathrm{tot}(\omega_C)$.
- We have $\mathcal{O}_X(1) = \mathcal{O}_X(D)$ and $\mathcal{O}_X(D)_{|D} \cong \omega_C^{-1}$ under the identification $\pi : D \widetilde{\to} C$.
- From the Euler sequence on $\pi : X \to C$ we get that the canonical class ω_X satisfies $\omega_X \cong \mathcal{O}X(-2D)$. Thus $\omega_X^{-1} = \wedge^2 T_X$ has a unique (up to scale) section λ with divisor $\mathrm{div}(\lambda) = 2D$. Thus (X, λ) is a Poisson surface.

Let now $F \to C$ be a fixed vector bundle of rank r. We claim that there is an equivalence of the following two types of data associated with X and F:

F**-framed bundles on** X**:** These are pairs (E, φ) where

- $E \to X$ is a vector bundle of rank r with the property that $E_{|X_t}$ is generated by global sections for all $t \in C$.
- $\varphi : E_{|D} \widetilde{\to} F$ is an isomorphism.

F**-prolonged Higgs bundles on** C**:** These are triples $(W, W \overset{i}{\hookrightarrow} V, \theta)$, where

- W and V are vector bundles on C.
- $0 \to W \overset{i}{\to} V \to F \to 0$ is a short exact sequence of vector bundles.
- $\theta : W \to V \otimes \omega_C$ is a sheaf homomorphism.

Similarly a F-framed pair on the non-commutative deformation of X in the direction of λ will correspond to an F-prolonged differential operator, i.e. a triple $(A, i : A \hookrightarrow B, \delta)$, where again A and B are vector bundles on C, $B/A \cong F$ and $\delta : A \to B \otimes \omega_C$ is a $\mathbb{C}$-linear map satisfying the Leibnitz rule w.r.t. multiplication by functions.

The identification between (E, φ) and $(W, W \subset V, \theta)$ goes as follows.

Start with (E, φ). since E is assumed to be globally generated along the fibers of π it follows that the natural evaluation map

$$\mathrm{ev} : \pi^* \pi_* E \to E$$

is surjective. In particular we have a short exact sequence of vector bundles on X:

$$0 \to \ker(\mathrm{ev}) \to \pi^* \pi_* E \overset{\mathrm{ev}}{E} \to 0.$$

Pushing this sequence forward to C we get a long exact sequence of sheaves:

$$0 \to \pi_*(\ker(\mathrm{ev})) \to \pi_* E \overset{\mathrm{id}}{\to} \pi_* E \to R^1 \pi_*(\ker(\mathrm{ev})) \to 0.$$

In particular, we must have $\pi_*(\ker(\mathrm{ev})) = R^1 \pi_*(\ker(\mathrm{ev})) = 0$ and so $\ker(\mathrm{ev})$ must be of the form

$$\ker(\mathrm{ev}) \cong \pi^* S \otimes \mathcal{O}_X(-D)$$

by the see-saw theorem.

Now twist by $\mathcal{O}(D)$ to get the sequence

$$0 \to \pi^* S \to (\pi^* \pi_* E) \otimes \mathcal{O}_X(D) \to E(D) \to 0.$$

Pushing this sequence forward we get

$$0 \to S \xrightarrow{g} \pi_* E \otimes (\mathcal{O}_C \oplus \omega_C^{-1}) \to \pi_*(E(D)) \to 0.$$

Let

$$S \to \pi_* E \tag{1}$$

$$S \to \pi_* E \otimes \omega_C^{-1} \tag{2}$$

be the two components of the map g. Define $V := \pi_* E$ and $W := S \otimes \omega_C$. Let $i : W \to V$ and $\theta : W \to V \otimes \omega_C$ be the two maps corresponding to (1) and (2) respectively. Note that by construction the map i can be identified with the restriction to D of the natural inclusion

$$(\pi^* S)(-D) \hookrightarrow \pi^* V.$$

Since this inclusion fits in the short exact sequence

$$0 \to (\pi^* S)(-D) \to \pi^* V \to E \to 0,$$

it follows that i fits in the sequence

$$0 \to W \xrightarrow{i} V \to E_{|D} \to 0,$$

which via φ can be rewritten as the sequence

$$0 \to W \xrightarrow{i} V \to F \to 0.$$

Conversely, given an F-prolonged Higgs bundle $(W, W \subset V, \theta)$ we can construct a F-framed pair (E, φ). Indeed, consider the maps

$$i : W \to V$$
$$\theta : W \to V \otimes \omega_C$$

and interpret them as maps

$$j : W \otimes \omega_C^{-1} \to V \otimes \omega_C^{-1}$$
$$\kappa : W \otimes \omega_C^{-1} \to V.$$

Thus we get a natural inclusion

$$\kappa \oplus j : W \otimes \omega_C^{-1} \hookrightarrow V \oplus (V \otimes \omega_C^{-1}),$$

which by adjunction can be rewritten as a map

$$\alpha : \pi^*(W \otimes \omega_C^{-1}) \hookrightarrow (\pi^* V)(D).$$

Define E as

$$E := \operatorname{coker}(\pi^*(W \otimes \omega_C^{-1})(-D) \to \pi^* V).$$

Then $E_{|D}$ is naturally identified with the cokernel of the map $W \otimes \omega_C^{-1} \otimes \omega_C \to V$, i.e. with the vector bundle F.

References

[AD02] P.Aspinwall and M. Douglas. D-brane stability and monodromy. *J. High Energy Phys.*, 5:no. 31, 35, 2002.

[Aetall09] P.Aspinwall et al. Dirichlet Branes and Mirror Symmetry, Clay Math Monographs, volume 4, 2009, 681 pp..

[Bri07] T. Bridgeland. Stability conditions on triangulated categories, Ann. of Math. (2) 166 (2007), no. 2, 317345. arXiv:math.AG/0212237.

[Bri08] T. Bridgeland. Stability conditions on K3 surfaces, Duke Math. J. 141 (2008), no. 2, 241291. arXiv:math.AG/0307164.

[Cor88] K. Corlette. Flat G-bundles with canonical metrics. J. Differential Geom. 28 (1988), no. 3, 361382.

[DG02] R. Donagi and D. Gaitsgory. The gerbe of Higgs bundles. *Transform. Groups*, 7(2):109–153, 2002.

[Do95] R. Donagi. Spectral covers. In *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, volume 28 of *Math. Sci. Res. Inst. Publ.*, pages 65–86. Cambridge Univ. Press, Cambridge, 1995.

[Do98] R. Donagi. Taniguchi lectures on principal bundles on elliptic fibrations. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 33–46. World Sci. Publishing, River Edge, NJ, 1998.

[DP03] R. Donagi, T. Pantev. Torus fibrations, gerbes, and duality, With an appendix by Dmitry Arinkin. Mem. Amer. Math. Soc. 193 (2008), no. 901, vi+90 pp. 2003, arXiv:math.AG/0306213.

[Don83] . Donaldson. A new proof of a theorem of Narasimhan and Seshadri. J. Differential Geom. 18 (1983), no. 2, 269277.

[Dou02] M. Douglas. Dirichlet branes, homological mirror symmetry, and stability. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 395–408. Higher Ed. Press, 2002.

[Fei01] B. Feix. Hyperkähler metrics on cotangent bundles. *J. Reine Angew. Math.*, 532:33–46, 2001.

[GC98] R. Gopakumar and C.Vafa. M-theory and topological strings. ii, 1998, hep-th/9812127.

[Kal99] D. Kaledin. A canonical hyperkähler metric on the total space of a cotangent bundle. In *Quaternionic structures in mathematics and physics (Rome, 1999)*, pages 195–230 (electronic). Univ. Studi Roma "La Sapienza", Rome, 1999.

[KKO01] A. Kapustin, A. Kuznetsov, and D. Orlov. Noncommutative instantons and twistor transform. *Comm. Math. Phys.*, 221(2):385–432, 2001.

[KO01] A. Kapustin and D. Orlov. Remarks on A-branes, Mirror Symmetry, and the Fukaya category. CALT-68-2348, 2001, arXiv:hep-th/0109098.

[NS98] N. Nekrasov and A. Schwarz. Instantons on noncommutative $\mathbf{R}^4$, and $(2,0)$ superconformal six-dimensional theory. *Comm. Math. Phys.*, 198(3):689–703, 1998.

[PW11] T.Pantev, M.Wijnholt Hitchin's equations and M-theory phenomenology. Journal of Geometry and Physics, 61 (2011), pp. 1223-1247.

[Pos93] L. Positselski. Nonhomogeneous quadratic duality and curvature. *Funktsional. Anal. i Prilozhen.*, 27(3):57–66, 96, 1993.

[PP98] A. Polishchuk and L. Positselski. Quadratic algebras, 1998. preprint.

[Pri70] S. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.

[Sim92] C. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, 75:5–95, 1992.

[Sim97] C. Simpson. The Hodge filtration on nonabelian cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 217–281. Amer. Math. Soc., Providence, RI, 1997.

[Sim02] C. Simpson. Algebraic aspects of higher nonabelian Hodge theory. In *Motives, polylogarithms and Hodge theory, Part II (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 417–604. Int. Press, Somerville, MA, 2002.

[SYZ96] A. Strominger, S.-T. Yau, E. Zaslow. Mirror symmetry is T-duality. Nuclear Phys. B 479 (1996), no. 1-2, 243259.