Dynamic convex risk measures: a survey

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Outline

• Static risk measures: a recap

• Dynamic framework
  – Conditional risk measures
  – Time consistency
  – Asymptotics and bubbles
  – Risk measures for processes: capturing discounting ambiguity
History

- Starting point: Problem of quantifying the risk undertaken by a financial institution (bank, insurance company...)
- From the point of view of a supervising agency a specific monetary purpose comes into play
- Axiomatic analysis of capital requirements needed to cover the risk of some future liability was initiated by Artzner, Delbaen, Eber, Heath (1997, 1999)
- Convex risk measures: Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002)...)
Monetary measures of risk

The set of financial positions: $\mathcal{X}$. Usually $\mathcal{X} = L^\infty(\Omega, \mathcal{F}, P)$. More generally: $\mathcal{X}$ is an ordered locally convex topological vector space.

A mapping $\rho : \mathcal{X} \to \mathbb{R}$ is called a monetary convex risk measure if it satisfies the following conditions for all $X, Y \in \mathcal{X}$:

- **Cash (or Translation) Invariance**: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- **(Inverse) Monotonicity**: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- **Convexity**: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda \rho(X) + (1 - \lambda)\rho(Y)$ for $\lambda \in [0, 1]$.
- **(sometimes) Normalization**: $\rho(0) = 0$.

A convex risk measure with

- **Positive homogeneity**: If $\lambda \geq 0$, then $\rho(\lambda X) = \lambda \rho(X)$

is a coherent risk measure.
Acceptance sets

An important characterization of a monetary risk measure is the acceptance set

\[ \mathcal{A} := \{ X \in \mathcal{X} \mid \rho(X) \leq 0 \} . \]

- By cash invariance, \( \rho \) is uniquely determined through its acceptance set:

\[ \rho(X) = \inf \{ m \in \mathbb{R} \mid X + m \in \mathcal{A} \} \]  \hspace{1cm} (1)

\( \to \rho(X) \) is the minimal capital requirement that has to be added to a financial position \( X \) in order to make it acceptable.

- The acceptance set of a convex risk measure is convex, solid, and such that \( \text{ess inf} \{ m \in \mathbb{R} \mid m \in \mathcal{A} \} = 0 \) and \( 0 \in \mathcal{A} \).

- On the other hand, every set \( \mathcal{A} \) with the above properties defines a monetary convex risk measure via (1).
Robust representation

- A convex risk measure $\rho : \mathcal{X} \to \mathbb{R}$ has a robust representation if it can be written as

$$\rho(X) = \sup_{Y \in \mathcal{X}^*} (Y(X) - \alpha(Y)),$$

with some convex penalty function $\alpha : \mathcal{X}^* \to [0, \infty]$.

- Typically a penalty function is given by

$$\alpha_{\text{min}}(Y) = \sup_{X \in \mathcal{X}} (Y(X) - \rho(X)) = \sup_{X \in A} Y(X).$$

- If $\rho$ is coherent, then the robust representation is of the form

$$\rho(X) = \sup_{Y \in \mathcal{Q}} Y(X)$$

for some convex set $\mathcal{Q} \subseteq \mathcal{X}^*$. 

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Robust representation (continued)

For a convex risk measure \( \rho : L^\infty \to \mathbb{R} \) the following are equivalent:

- \( \rho \) has a robust representation in terms of probability measures, i.e.
  \[
  \rho(X) = \sup_{Q \in \mathcal{M}_1(P)} (E_Q[-X] - \alpha(Q))
  \]
  with some penalty function \( \alpha : \mathcal{M}_1(P) \to [0, \infty] \), where \( \mathcal{M}_1(P) = \{ Q \mid Q \text{ probability measure on } (\Omega, \mathcal{F}), Q \ll P \} \).

- \( \rho \) is lower semicontinuous in the weak* topology \( \sigma(L^\infty, L^1) \).

- The acceptance set \( \mathcal{A} \) is weak* closed in \( L^\infty \).

- \( \rho \) is continuous from above, i.e. \( \rho(X_n) \nearrow \rho(X) \) for any decreasing sequence \( (X_n) \) with \( X_n \searrow X \).

- \( \rho \) has the Fatou property, i.e. \( \rho(X) \leq \lim \inf \rho(X_n) \) for any bounded sequence \( (X_n) \) with \( X_n \to X \).
Important examples

- Law-invariant risk measures
- Shortfall risk
- Measures of risk in a financial market
Law-invariant risk measures

Law-invariant risk measures depend only on the law of $X$.

- A very common law-invariant monetary risk measure is Value at Risk at level $\lambda \in [0, 1]$:
  \[
  V@R_\lambda(X) = \inf \left\{ m \in \mathbb{R} \mid P[X + m < 0] \leq \lambda \right\}
  \]
  - unfortunately not convex.

- An alternative: Average Value at Risk (Tail Value at Risk, Conditional Value at Risk, Expected Shortfall) at level $\lambda \in [0, 1]$
  \[
  AV@R_\lambda(X) = \frac{1}{\lambda} \int_0^\lambda V@R_\gamma(X) d\gamma = \max_{Q \in \mathcal{Q}_\lambda} E_Q[-X],
  \]
  where $\mathcal{Q}_\lambda = \left\{ Q \ll P \mid \frac{dQ}{dP} \leq \frac{1}{\lambda} P\text{-a.s.} \right\}$.

- $AV@R_\lambda$ is the smallest law-invariant coherent risk measure that dominates $V@R_\lambda$. 
**Shortfall risk**

Let $l : \mathbb{R} \to \mathbb{R}$ be some convex and increasing *loss function* (or alternatively $u : \mathbb{R} \to \mathbb{R}$ some concave and increasing *utility function*).

- The set
  \[ \mathcal{A} := \{ X \in L^\infty \mid E[l(-X)] \leq l(0) \} = \{ X \in L^\infty \mid E[u(X)] \geq u(0) \} \]
  defines an acceptance set.

- The corresponding convex risk measure $\rho(X)$ is *continuous from below*.

- The *minimal penalty function* in the robust representation of $\rho$ is given by
  \[ \alpha_{\text{min}}(Q) = \inf_{\lambda > 0} \frac{1}{\lambda} E \left[ l^* \left( \lambda \frac{dQ}{dP} \right) \right] \]
  for all $Q \ll P$. 

Shortfall risk: entropic risk measure

• For the exponential loss function \( l(x) = e^{\gamma x} \) with the risk aversion \( \gamma > 0 \) we obtain the entropic risk measure

\[
\rho(X) = \frac{1}{\gamma} \log E[e^{-\gamma X}].
\]

• Entropic risk measure has the robust representation

\[
\rho(X) = \max_{Q \in M_1(P)} \left( E_Q[-X] - \frac{1}{\gamma} H(Q|P) \right),
\]

i.e., the minimal penalty function of the entropic risk measure is given by

\[
\alpha_{\text{min}}(Q) = \frac{1}{\gamma} H(Q|P) = E_Q[\log(dQ/dP)],
\]

where \( H(Q|P) \) denotes the relative entropy of \( Q \) w.r.t. \( P \).
Measures of risk in a financial market

Consider a financial market model with a discounted price process given by a semi-martingale $(S_t)_{t \in [0,T]}$. The set of admissible strategies is denoted by $S$.

- The set of all financial positions that can be hedged with some admissible strategy at no cost

$$\mathcal{A} := \left\{ X \in L^\infty \mid \exists \xi \in S : \int_0^T \xi_t dS_t \geq -X \right\}$$

defines an acceptance set.

- The corresponding risk measure $\rho$ is the superhedging price.

- $\rho$ is a convex risk measure if $S$ is convex and $\rho$ is coherent if $S$ is a cone.

- The existence of a robust representation is related to the no-arbitrage condition.
Outline

- Static risk measures: a recap
- Dynamic framework: Assume there is an information flow described by a filtered probability space

\[(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, P), \quad \mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F} = \mathcal{F}_T.\]

The time horizon \(T\) might be finite or infinite.

How can this information be used for risk assessment?
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How can this information be used for risk evaluation?

- Conditional risk measures
- Time consistency
- Asymptotics and bubbles
- Risk measures for processes: capturing discounting ambiguity
Conditional convex risk measure

A conditional convex risk measure $\rho_t$ is a map $L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ with the following properties for all $X, Y \in L^\infty$:

- **Conditional Cash Invariance**: $\forall m_t \in L^\infty(\mathcal{F}_t)$:
  \[
  \rho_t(X + m_t) = \rho_t(X) - m_t
  \]

- **(Inverse) Monotonicity**: $X \leq Y \implies \rho_t(X) \geq \rho_t(Y)$

- **Conditional Convexity**: $\forall \lambda \in L^\infty(\mathcal{F}_t), 0 \leq \lambda \leq 1$:
  \[
  \rho_t(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_t(X) + (1 - \lambda)\rho_t(Y)
  \]

- **Normalization**: $\rho_t(0) = 0$.

$\phi_t := -\rho_t$ is a conditional monetary utility function.
Conditional coherent risk measure

A conditional convex risk measure with

- **Conditional positive homogeneity**: \( \forall \lambda \in L^\infty(F_t), \lambda \geq 0:\)

\[
\rho_t(\lambda X) = \lambda \rho_t(X)
\]

is called a **conditional coherent risk measure**.
Acceptance sets

The conditional version of acceptance set is given by

$$A_t := \{ X \in L^\infty \mid \rho_t(X) \leq 0 \}.$$  

Again, $\rho_t$ is uniquely determined through its acceptance set:

$$\rho_t(X) = \text{ess inf} \{ m \in L^\infty(F_t) \mid X + m \in A_t \}$$

$\rho_t(X)$ is the minimal conditional capital requirement that has to be added to a financial position $X$ in order to make it acceptable at time $t$. 
Robust representation (cf. Detlefsen and Scandolo (2005))

For a conditional convex risk measure $\rho_t$ the following are equivalent:

- $\rho_t$ is continuous from above;
- $\rho_t$ has the robust representation

$$
\rho_t(X) = \operatorname{ess sup}_{Q \in Q_t} \left( E_Q[-X|\mathcal{F}_t] - \alpha_t^\text{min}(Q) \right),
$$

where the penalty function $\alpha_t^\text{min}$ is given by

$$
\alpha_t^\text{min}(Q) = \operatorname{ess sup}_{X \in L^\infty} \left( E_Q[-X|\mathcal{F}_t] - \rho_t(X) \right) = \operatorname{ess sup}_{X \in A_t} E_Q[-X|\mathcal{F}_t]
$$

for $Q \in Q_t := \{ Q \ll P \mid Q = P \text{ on } \mathcal{F}_t \}$. 
Example: conditional Average Value at Risk

The conditional version of Average Value at Risk (Tail Value at Risk, Conditional Value at Risk, Expected Shortfall) at level $\lambda_t$ can be defined as

$$AV@R_{\lambda_t}(X) = \max_{Q \in Q_{\lambda_t}} E_Q[-X | \mathcal{F}_t],$$

where $Q_{\lambda_t} = \left\{ Q \in \mathcal{Q}_t \mid \frac{dQ}{dP} \leq \frac{1}{\lambda_t} \text{ P-a.s.} \right\}$ for some $\lambda_t \in L^\infty(\mathcal{F}_t)$ s.t. $0 < \lambda_t \leq 1$ P-a.s.
Example: conditional entropic risk measure

- The exponential loss function \( l_t(x) = e^{\gamma_t x} \) with the risk aversion \( \gamma_t \in L^\infty(F_t), \gamma_t > 0 \) defines the acceptance set
  \[
  \mathcal{A}_t := \{ X \in L^\infty \mid E[e^{-\gamma_t X}|F_t] \leq 1 \}.
  \]

- The corresponding conditional risk measure
  \[
  \rho_t(X) = \frac{1}{\gamma_t} \log E[e^{-\gamma_t X}|F_t]
  \]
  is called a conditional entropic risk measure.

- It has the robust representation with the minimal penalty function
  \[
  \alpha_t^{\min}(Q) = \frac{1}{\gamma_t} \hat{H}_t(Q|P) = \frac{1}{\gamma_t} E_Q[\log(dQ/dP)|F_t],
  \]
  where \( \hat{H}_t(Q|P) \) denotes the conditional relative entropy of \( Q \in \mathcal{Q}_t \)
  w.r.t. \( P \).
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Time consistency

- Dynamic framework: a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\).
- A dynamic risk measure is a sequence \((\rho_t)_{t=0,1,...}\) such that each \(\rho_t\) is a conditional risk measure w.r.t. \(\mathcal{F}_t\).
- We obtain for each \(X \in L^\infty\) a sequence of risk assessments \((\rho_t(X))_{t=0,1,...}\), called risk process of \(X\).

The question arises: How are the risk assessments at different times interrelated?

\(\rightarrow\) Several notions of time consistency.

Several notions of time consistency

Assume that we have a sequence of benchmark sets \((\mathcal{Y}_t)_{t=0,1,...}\) in \(L^\infty\). Then a dynamic risk measure \((\rho_t)\) is called acceptance (resp. rejection) consistent with respect to \((\mathcal{Y}_t)\), if

\[
\rho_{t+1}(X) \leq \rho_{t+1}(Y) \quad (\text{resp.} \geq) \quad \Rightarrow \quad \rho_t(X) \leq \rho_t(Y) \quad (\text{resp.} \geq).
\]

for all \(t\) and for any \(X \in L^\infty\) and \(Y \in \mathcal{Y}_{t+1}\).

→ If a financial position is always preferable tomorrow to some element of the benchmark set, then it should also be preferable today. The bigger the benchmark set, the stronger is the resulting notion of time consistency.

\((\text{Tutsch (2006)})\)
Several notions of time consistency

We call a dynamic convex risk measure $\rho_t$

- (strongly) time consistent, if the sequence of benchmark sets is given by $\mathcal{Y}_t = L^\infty(\mathcal{F}_T)$ for all $t$. In this case acceptance and rejection consistency coincide.

- (middle) acceptance (resp. (middle) rejection) consistent, if the sequence of benchmark sets is given by $\mathcal{Y}_t = L^\infty(\mathcal{F}_t)$ for all $t$.

- weakly acceptance (resp. weakly rejection) consistent, if the sequence of benchmark sets is given by $\mathcal{Y}_t = \mathbb{R}$ for all $t$. 

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Time consistency

- Weak time consistency
- Acceptance and rejection consistency
- Strong time consistency
Weak time consistency

A dynamic convex risk measure \((\rho_t)\) is weakly acceptance (resp. weakly rejection) consistent, if and only if

\[
\rho_{t+1}(X) \leq 0 \ (\text{resp.} \geq) \implies \rho_t(X) \leq 0 \ (\text{resp.} \geq).
\]

for all \(t\) and all \(X \in L^\infty\).

→ If some position is accepted (or rejected) for any scenario tomorrow, it should be already accepted (or rejected) today.

Weak acceptance consistency

Let \((\rho_t)_{t=0,1,...}\) be a dynamic convex risk measure such that each \(\rho_t\) is continuous from above. Then the following properties are equivalent:

1. \((\rho_t)\) is weakly acceptance consistent.

2. \(\mathcal{A}_{t+1} \subseteq \mathcal{A}_t\) for all \(t\).

3. The inequality
   \[
   E_Q[\alpha_{t+1}^{\min}(Q)\,|\,F_t] \leq \alpha_t^{\min}(Q)
   \]
   holds for all \(Q \ll P\) and all \(t\).

In particular, the penalty process \((\alpha_t^{\min}(Q))\) is a \(Q\)-supermartingale for all \(Q \in \mathcal{Q}_0\), where \(\mathcal{Q}_0 = \{Q \ll P|\alpha_0^{\min}(Q) < \infty\}\).

\((\text{Tutsch (2006), Roorda and Schumacher (2007, 2008)})\)
Time consistency

- Weak time consistency
- Acceptance and rejection consistency
- Strong time consistency
Acceptance and rejection consistency

- A dynamic convex risk measure \((\rho_t)_{t=0,1,...}\) is rejection (resp. acceptance) consistent if and only if
  \[
  \rho_t(-\rho_{t+1}) \leq \rho_t \quad (\text{resp. } \geq) \quad \forall \ t.
  \]

- Another equivalent characterization of rejection consistency is
  \[
  \rho_t(\rho_t(X) - \rho_{t+1}(X)) \leq 0 \quad \forall \ t, \forall \ X,
  \]
  or, equivalently,
  \[
  X \in \mathcal{A}_t \quad \Rightarrow \quad -\rho_{t+1}(X) \in \mathcal{A}_t \quad \forall \ t, \forall \ X
  \]
  (→ “prudent”, “stay on the safe side”).
Step by step

Consider a conditional convex risk measure $\rho_t$ restricted to the space $L^\infty(\mathcal{F}_{t+1})$, i.e. just looking one step ahead.

The corresponding "one-step" acceptance set is given by

$$\mathcal{A}_{t,t+1} := \{ X \in L^\infty(\mathcal{F}_{t+1}) \mid \rho_t(X) \leq 0 \}$$

and the minimal "one-step" penalty function by

$$\alpha_{t,t+1}^{\min}(Q) := \text{ess sup}_{X \in \mathcal{A}_{t,t+1}} E_Q[-X|\mathcal{F}_t], \quad Q \ll P.$$

In the same way we define for $s \geq t$

$$\mathcal{A}_{t,s} := \{ X \in L^\infty(\mathcal{F}_s) \mid \rho_t(X) \leq 0 \},$$

and denote the corresponding penalty function by $\alpha_{t,s}^{\min}(Q)$. 

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Acceptance and rejection consistency

Let \((\rho_t)_{t=0,1,...}\) be a dynamic convex risk measure such that each \(\rho_t\) is continuous from above. Then the following properties are equivalent:

1. \((\rho_t)\) is rejection consistent (resp. acceptance consistent).
2. \(A_t \subseteq A_{t,t+1} + A_{t+1}\) resp. \(A_t \supseteq A_{t,t+1} + A_{t+1}\) for all \(t\).
3. The inequality
   \[
   \alpha_{t}^{\min}(Q) \leq \alpha_{t,t+1}^{\min}(Q) + E_Q[\alpha_{t+1}^{\min}(Q) | \mathcal{F}_t] \quad (\text{resp.} \geq)
   \]
   holds for all \(t\) and all \(Q \ll P\).
Acceptance and rejection consistency

- In case of acceptance consistency, the penalty process $(\alpha_t^{\min}(Q))$ is a $Q$-supermartingale for all $Q \in Q_0 = \{Q \ll P|\alpha_0^{\min}(Q) < \infty\}$.

- In case of rejection consistency, The process

$$\rho_t(X) - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q), \quad t = 0, 1, \ldots$$

is a $Q$-supermartingale for all $X \in L^\infty$ and all $Q \ll P$ for which it is integrable.
Sustainability

Let \((\rho_t)_{t=0,1,...}\) be any dynamic risk measure, and let \(X = (X_t)_{t=0,1,...}\) be a bounded adapted process. We interpret \(X\) as a cumulative investment process, and we call \(X\) sustainable with respect to the dynamic risk measure \((\rho_t)\), if

\[
\rho_t(X_t - X_{t+1}) \leq 0 \quad \text{for all } t = 0, 1, \ldots
\]

- The adjustment \(X_{t+1} - X_t\) at time \(t + 1\) is acceptable at time \(t\) with respect to the risk measure \(\rho_t\).

- A dynamic risk measure \((\rho_t)\) is rejection consistent iff for each \(X\) the risk process \((\rho_t(X))\) is sustainable with respect to \((\rho_t)\).
Sustainability (continued)

Let \((\rho_t)\) be a dynamic convex risk measure such that each \(\rho_t\) is continuous from above, and let \(X\) be any bounded adapted process. Then the following properties are equivalent:

1. The process \(X\) is sustainable with respect to the risk measure \((\rho_t)\).

2. \(E_Q[X_t|\mathcal{F}_{t-1}] \leq X_{t-1} + \alpha_{\min}^{t-1,t}(Q)\) \(Q\)-a.s. for all \(Q \ll P\) and all \(t\).

In particular, the process

\[
X_t - \sum_{k=0}^{t-1} \alpha_{\min}^{k,k+1}(Q), \quad t = 0, 1, \ldots
\]

is a \(Q\)-supermartingale for all \(Q \ll P\) under which it is integrable.
Time consistency

- Weak time consistency
- Acceptance and rejection consistency
- Strong time consistency
**Time consistency**

A dynamic convex risk measure \((\rho_t)_{t=0,1,...}\) is called *(strongly) time consistent*, if and only if for all \(X, Y \in L^\infty\) and \(t, s \geq 0\) one of the following four equivalent conditions holds:

1. \(\rho_{t+1}(X) \geq \rho_{t+1}(Y) \Rightarrow \rho_t(X) \geq \rho_t(Y)\)
2. \(\rho_{t+1}(X) = \rho_{t+1}(Y) \Rightarrow \rho_t(X) = \rho_t(Y)\)
3. \(\rho_{t+s}(X) = \rho_{t+s}(Y) \Rightarrow \rho_t(X) = \rho_t(Y)\)
4. **Recursiveness:** \(\rho_t = \rho_t(-\rho_{t+s})\)

Equivalent characterizations

Let \((\rho_t)_{t=0,1,...}\) be a dynamic convex risk measure such that each \(\rho_t\) is continuous from above. Then the following conditions are equivalent:

1. \((\rho_t)\) is (strongly) time consistent.

2. \(\mathcal{A}_t = \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \quad \forall t, s.\)

3. \(\alpha^{\min}_t(Q) = \alpha^{\min}_{t,t+s}(Q) + E_Q[\alpha^{\min}_{t+s}(Q)|\mathcal{F}_t] \quad \forall t, s, Q \ll P.\)

4. \(E_Q[\rho_{t+s}(X) + \alpha^{\min}_{t+s}(Q)|\mathcal{F}_t] \leq \rho_t(X) + \alpha^{\min}_t(Q) \quad \forall X, t, s, Q \ll P.\)
Equivalent characterizations: coherent case

Let \((\rho_t)_{t=0,1,...}\) be a dynamic convex risk measure such that each \(\rho_t\) is continuous from above. Assume that \(Q^* := \{Q \approx P | \alpha_0^\text{min}(Q) = 0\} \neq \emptyset\), and that the initial risk measure \(\rho_0\) is coherent. Then the following conditions are equivalent:

1. \((\rho_t)\) is time consistent.

2. The representation

\[
\rho_t(X) = \text{ess sup}_{Q \in Q^*} E_Q[-X|\mathcal{F}_t]
\]

holds for all \(X\) and all \(t\), and the set \(Q^*\) is stable.

3. The representation (2) holds for all \(X\) and all \(t\), and the process \((\rho_t(X))\) is a \(Q\)-supermartingale for all \(Q \in Q^*\).

In each case \((\rho_t)\) is a dynamic coherent risk measure.
Supermartingale properties

Let \((\rho_t)_{t=0,1,...}\) be a \textit{time consistent} dynamic convex risk measure such that each \(\rho_t\) is continuous from above. Then

- the process
  \[ V_t^Q(X) := \rho_t(X) + \alpha_t^\min(Q), \quad t = 0, 1, \ldots \]
  is a \(Q\)-supermartingale for all \(X \in L^\infty\) and all \(Q \in \mathcal{Q}_0\).

- \((V_t^Q(X))\) is a \(Q\)-martingale if \(Q\) is a “worst case” measure for \(X\) at time 0, i.e. if
  \[ \rho_0(X) = E_Q[-X] - \alpha_0^\min(Q). \]
  In this case \(Q\) is a “worst case” measure for \(X\) at any time \(t\).

- The converse holds if \(T < \infty\) or if \(\lim_{t \to \infty} \rho_t(X) = -X\) \(P\)-a.s.: If \((V_t^Q(X))\) is a \(Q\)-martingale, then \(Q\) is a “worst case” measure for \(X\) at any time \(t\).
Supermartingale properties

Let \((\rho_t)_{t=0,1,...}\) be a time consistent dynamic convex risk measure such that each \(\rho_t\) is continuous from above.

- For \(Q \ll P\) we can view the process

\[
F^Q_t(X) := E_Q[-X|\mathcal{F}_t] - \alpha_{t}^{\text{min}}(Q), \quad t = 0, 1, \ldots
\]

as a risk evaluation of the position \(X\) at time \(t\), using the specific model \(Q\).

- We have the following maximal inequality for the excess of the required capital \(\rho_t(X)\) over the risk evaluation \(F^Q_t(X)\): for \(c > 0\)

\[
Q \left( \sup_{t \geq 0} \left( \rho_t(X) - F^Q_t(X) \right) \geq c \right) \leq \frac{\rho_0(X) - F^Q_0(X)}{c}.
\]
Supermartingale properties

Let \((\rho_t)_{t=0,1,...}\) be a time consistent dynamic convex risk measure such that each \(\rho_t\) is continuous from above. Then the penalty process \((\alpha_t^{\min}(Q))\) is a \(Q\)-supermartingale for all \(Q \in Q_0\) with the Riesz decomposition

\[
\alpha_t^{\min}(Q) = E_Q \left[ \sum_{k=t}^{\infty} \alpha_{k,k+1}^{\min}(Q) \left| \mathcal{F}_t \right. \right] + \lim_{s \to \infty} E_Q \left[ \alpha_s^{\min}(Q) \left| \mathcal{F}_t \right. \right]
\]

\(Q\)-potential \(Q\)-martingale, “bubble"

and the Doob-decomposition

\[
\alpha_t^{\min}(Q) = E_Q \left[ \sum_{k=0}^{\infty} \alpha_{k,k+1}^{\min}(Q) \left| \mathcal{F}_t \right. \right] + M_t^Q - \sum_{k=0}^{t-1} \alpha_{k,k+1}^{\min}(Q).
\]
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Asymptotics

Let $(\rho_t)_{t=0,1,...}$ be a time consistent dynamic convex risk measure, and assume that $T = \infty$. Then for every $Q \in Q_0$ there exist

$$\rho_\infty(X) := \lim_{t \to \infty} \rho_t(X) \quad \text{and} \quad \alpha_\infty(Q) := \lim_{t \to \infty} \alpha_t^{\min}(Q) \quad Q\text{-a.s.}.$$ 

The functional $\rho_\infty : L^\infty \to L^\infty$ is normalized, monotone, conditionally convex and conditionally cash invariant with respect to $\mathcal{F}_t$ for any $t \geq 0$. We call $(\rho_t)_{t=0,1,...}$

- asymptotically safe under the model $Q$, if $\rho_\infty(X) \geq -X \quad Q\text{-a.s.}$ (→ covers the final loss)

- asymptotically precise under the model $Q$, if $\rho_\infty(X) = -X \quad Q\text{-a.s.}$ (→ in this case $\rho_\infty$ is a risk measure itself).

Not every time consistent dynamic risk measure is asymptotically precise or asymptotically safe.
Asymptotic safety

For a time consistent dynamic risk measure \((\rho_t)_{t=0,1,...}\) and a model \(Q \in Q_0\) the following properties are equivalent:

1. \((\rho_t)_{t=0,1,...}\) is asymptotically safe under \(Q\).

2. \(\alpha_\infty(Q) = \lim_{t \to \infty} \alpha^\min_t(Q) = 0\) \(Q\)-a.s. and in \(L^1(Q)\), i.e., \((\alpha^\min_t(Q))\) is a \(Q\)-potential.

3. The martingale \(M^Q\) in the Riesz decomposition of the process \((\alpha^\min_t(Q))\) vanishes, i.e., there is no bubble.

4. No model \(R \in Q_0\) with \(R \ll Q\) admits bubbles.

In particular, every time consistent coherent dynamic risk measure is asymptotically safe under each \(Q \in Q_0\).
A counterexample

- Let $\Omega := (0, 1]$, $P := \lambda$, and let
  
  $F_t := \sigma \left( (k 2^{-t}, (k+1) 2^{-t}) \mid k = 0, \ldots, 2^t - 1 \right), \ t = 0, 1, \ldots$

- Take $A \in F := \sigma(\bigcup_{t \geq 0} F_t)$ such that $P[A] > 0$ and
  $P[A^c \cap J_{t,k}] \neq 0$ for any dyadic interval $J_{t,k}$.

- For any $t \geq 0$ we fix the same acceptance set
  
  $A_t := \{ X \in L^\infty \mid X \geq -I_A \}$.

- The corresponding conditional convex risk measure $\rho_t$ is given by
  
  $\rho_t(X) = -\text{ess sup} \left\{ m \in L^\infty(F_t) \mid m \leq X + I_A \right\}$. 
A counterexample (continued)

- $\rho_t$ is normalized and continuous from above with the penalty function given by
  \[ \alpha_{t}^{\min}(Q) = E_Q[I_A | \mathcal{F}_t]. \]
- The dynamic risk measure $(\rho_t)_{t=0,1,...}$ is time consistent, since
  \[ A_t = A_{t+s} = A_{t+s} + L_{+}^{\infty}(\mathcal{F}_{t+s}) = A_{t+s} + A_{t,t+s} \]
- $\rho_{\infty}(X) = -\operatorname{ess} \sup \{ m \in \bigcup_{t=0}^{\infty} L^{\infty}(\mathcal{F}_t) \mid m \leq X + I_A \}$
- $(\rho_t)_{t=0,1,...}$ is not asymptotically safe:
  \[ \rho_{\infty}(-I_A) = 0 \not\geq I_A, \]
- The penalty function is a $Q$-martingale (bubble) with
  \[ \alpha_{\infty}^{\min}(Q) = \lim_{t \to \infty} \alpha_{t}^{\min}(Q) = I_A \neq 0. \]
Asymptotic precision

A simple sufficient condition for asymptotic precision is the following:

Let \((\rho_t)_{t=0,1,...}\) be a time consistent dynamic convex risk measure, and suppose that there exists a worst case measure \(Q^X \approx Q\) for each \(X \in L^\infty\), i.e.,

\[
\rho_0(X) = E_{Q^X}[-X] - \alpha_0^{\min}(Q^X)
\]

for some \(Q^X \approx Q\). Then \((\rho_t)_{t=0,1,...}\) is asymptotically precise under \(Q\).
Examples

- Dynamic entropic risk measure
- Hedging under constraints
- Dynamic average value at risk
- Recursive construction
Example: Dynamic entropic risk measure

- Let \((\gamma_t)_{t=0,1,...}\) denote the process of strictly positive adapted risk aversion parameters s.t. \(\gamma_t, \frac{1}{\gamma_t} \in L^\infty(\mathcal{F}_t)\) for each \(t\).

- The corresponding dynamic risk measure

\[
\rho_t(X) = \frac{1}{\gamma_t} \log E[e^{-\gamma_t X}|\mathcal{F}_t], \quad t = 0, 1, \ldots
\]

is a dynamic entropic risk measure.

- The penalty process of the entropic risk measure is given by

\[
\alpha^{\text{min}}_t(Q) = \frac{1}{\gamma_t} \tilde{H}_t(Q|P), \quad t = 0, 1, \ldots,
\]

where \(\tilde{H}_t(Q|P)\) the conditional relative entropy of \(Q \ll P\) w.r.t. \(P\).
Dynamic entropic risk measure (continued)

The time consistency properties of the dynamic entropic risk measure are completely determined by the adapted process of risk aversion (\(\gamma_t\)):

1. \((\rho_t)_{t=0,1,...}\) is time consistent if \(\gamma_t = \gamma \in \mathbb{R}\) for all \(t\);
2. \((\rho_t)_{t=0,1,...}\) is rejection consistent if \(\gamma_t \geq \gamma_{t+1}\) for all \(t\);
3. \((\rho_t)_{t=0,1,...}\) is acceptance consistent if \(\gamma_t \leq \gamma_{t+1}\) for all \(t\).

Moreover, the converse holds, if \(\gamma_t \in \mathbb{R}\) for all \(t\), or if the filtration \((\mathcal{F}_t)_{t=0,1,...}\) is rich enough in the sense that for all \(t\) and for all \(B \in \mathcal{F}_t\) such that \(P[B] > 0\) there exists \(A \subset B\) such that \(A \notin \mathcal{F}_t\) and \(P[A] > 0\).

In fact, the entropic risk measure with constant risk aversion \(\gamma \in [0, \infty]\) is the only time consistent dynamic risk measure such that \(\rho_0\) is law invariant. (Kupper and Schachermayer (2009))
Examples

- Dynamic entropic risk measure
- Hedging under constraints
- Dynamic average value at risk
- Recursive construction
Hedging under constraints

(cf. Föllmer and Kramkov (1997))

• Financial market with $d$ risky assets. Their discounted prices are given by the adapted process $(S_t)_{t=0,...,T}$.

• The set of allowed investment strategies $S$ is subject to some convex trading constraints.

• The set of all financial positions that can be hedged at time $t$ by means of some admissible strategy at no cost

$$A_t := \left\{ X \in L^\infty \mid \exists \xi \in S : \sum_{k=t+1}^{T} \xi_k(S_k - S_{k-1}) \geq -X \right\}$$

defines a convex acceptance set.

• The corresponding conditional convex risk measure $\rho_t(X)$ is the superhedging price of $X$ at time $t$. 
Hedging under constraints (continued)

Under the no arbitrage assumption the dynamic risk measure
\((\rho_t(X))_{t=0,\ldots,T}\) is

- normalized, i.e. \(\rho_t(0) = 0\) for all \(t\)
- continuous from above, i.e. each \(\rho_t\) has a robust representation. The minimal penalty functions can be identified in terms of the upper variation process of \(Q\) w.r.t. \(S\)
- (strongly) time consistent
- a minimal process hedging \(X\) under constraints.
Examples

- Dynamic entropic risk measure
- Hedging under constraints
- Dynamic average value at risk
- Recursive construction
Dynamic Average Value at Risk

- Let \((\lambda_t)_{t=0,1,...}\) denote the adapted process of level parameters s.t. \(0 < \lambda_t \leq 1\) \(P\)-a.s. for each \(t\).

- The corresponding dynamic Average Value at Risk is given by

\[
AV@R_{\lambda_t}(X) = \max_{Q \in Q_{\lambda_t}} E_Q[-X|\mathcal{F}_t], \quad t = 0, 1, \ldots,
\]

where \(Q_{\lambda_t} = \left\{ Q \in Q_t \mid \frac{dQ}{dP} \leq \frac{1}{\lambda_t} \right\} \) for each \(t\).

- The dynamic Average Value at Risk with constant parameter \(\lambda < 1\) is in general neither weakly acceptance nor weakly rejection consistent. Moreover, there exists no sequence of parameters \((\lambda_t)\) such that the corresponding dynamic Average Value at Risk is time consistent.
Dynamic Average Value at Risk

- Let \((\lambda_t)_{t=0,1,...}\) denote the adapted process of level parameters s.t.
  \[0 < \lambda_t \leq 1\ P\text{-a.s. for each } t.\]

- The corresponding dynamic Average Value at Risk is given by
  \[
  AV_{R_{\lambda_t}}(X) = \max_{Q \in Q_{\lambda_t}} E_Q[-X|\mathcal{F}_t], \quad t = 0, 1, \ldots,
  \]
  where \(Q_{\lambda_t} = \left\{ Q \in Q_t \mid \frac{dQ}{dP} \leq \frac{1}{\lambda_t} \ P\text{-a.s.} \right\}\) for each \(t\).

- The dynamic Average Value at Risk with constant parameter \(\lambda < 1\) is in general neither weakly acceptance nor weakly rejection consistent. Moreover, there exists no sequence of parameters \((\lambda_t)\) such that the corresponding dynamic Average Value at Risk is time consistent.

- What can one do?
Examples

• Dynamic entropic risk measure
• Hedging under constraints
• Dynamic average value at risk
• Recursive construction
Recursive construction

Suppose that $T < \infty$ and let $(\rho_t)_{t=0,...,T}$ be a dynamic convex risk measure. Consider a new risk measure $(\tilde{\rho}_t)_{t=0,...,T}$ defined recursively by

\[
\tilde{\rho}_T(X) := \rho_T(X) = -X \\
\tilde{\rho}_t(X) := \rho_t(-\tilde{\rho}_{t+1}(X)), \quad t = 0, \ldots, T-1, \quad X \in L^\infty.
\]

• Then $(\tilde{\rho}_t)$ is again a dynamic convex risk measure and it is (strongly) time consistent by definition. (cf. Cheridito et al. (2006), Cheridito and Kupper (2006), Drapeau (2006))

• If the original risk measure $(\rho_t)$ is rejection (resp. acceptance) consistent, then $(\tilde{\rho}_t)$ lies below (resp. above) $(\rho_t)$.
Recursive construction (continued)

The recursively constructed dynamic convex risk measure $\tilde{\rho}_t^{t=0,\ldots,T}$ has the following properties:

- **Sustainability w.r.t. $(\rho_t)$:**
  \[
  \rho_t(\tilde{\rho}_t(X) - \tilde{\rho}_{t+1}(X)) \leq 0 \quad \forall t, \forall X,
  \]
  (→ the adjustments $\tilde{\rho}_t(X) - \tilde{\rho}_{t+1}(X)$ are acceptable at any time)

- $\rho_T(X) = -X$ (→ the final loss is covered)

Moreover, for each $X \in L^\infty$ the risk process $(\tilde{\rho}_t(X))_{t=0,\ldots,T}$ is the smallest bounded adapted process with these both properties.
Outline

• Static risk measures: a recap

• Dynamic framework
  – Conditional risk measures
  – Time consistency
  – Asymptotics and bubbles
  – Risk measures for processes: capturing discounting ambiguity
Risk measures for processes

Dynamic framework: Multiperiod information structure:

\[(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,\ldots,T}, P), \quad T \leq \infty, \quad \mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_\infty = \sigma(\cup_{t\geq 0} \mathcal{F}_t).\]

- So far we have discussed risk assessment for discounted payoffs taking into account the new information but not the timing of the payment.

- Now we will consider cash flow processes that model the evolution of financial values. Risk assessment in this framework accounts not only the amounts but also the timing of a payment.
Risk measures for processes

- Setup and definition
- Transforming processes into random variables
- Translating the results from Part I:
  - Robust representation
  - Cash additivity and subadditivity
  - Time consistency, supermartingales, asymptotics and bubbles...
- Examples
Risk measures for processes: setup

- $\mathcal{R}^\infty$ denotes the set of all bounded adapted processes on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0, \ldots, T}, P)$. A process $X \in \mathcal{R}^\infty$ is understood as a cumulated cash-flow (value process).

- For $0 \leq t \leq s \leq T$, we define the projection $\pi_{t,s} : \mathcal{R}^\infty \to \mathcal{R}^\infty$ as
  \[
  \pi_{t,s}(X)_r = 1_{\{t \leq r\}} X_{r \wedge s}, \quad r \in \mathbb{T}
  \]
  and denote $\mathcal{R}_{t,s}^\infty := \pi_{t,s}(\mathcal{R}^\infty), \mathcal{R}_t^\infty := \mathcal{R}_{t,T}^\infty$.

Risk measures for processes: setup

We characterize risk measures for processes in the following 3 cases:

**Case 1:** $T < \infty$, $\mathbb{T} := \{0, \ldots, T\}$

**Case 2:** $T = \infty$, $\mathbb{T} := \mathbb{N}_0$

**Case 3:** $T = \infty$, $\mathbb{T} := \mathbb{N}_0 \cup \{\infty\}$

In case 2 and 3 we assume that each $(\Omega, \mathcal{F}_t)$ is a standard Borel space, and that for any sequence of atoms $A_0 \supseteq A_2 \supseteq \ldots$, where $A_t \in \mathcal{F}_t$ for each $t \in \mathbb{N}_0$, one has $\cap_{t \in \mathbb{N}_0} A_t \neq \emptyset$. (Parthasarathy)
Conditional convex risk measure: a recap

A map $\rho_t : L^\infty(\Omega, \mathcal{F}_T, P) \to L^\infty(\mathcal{F}_t)$ is called a conditional convex risk measure for random variables if it satisfies the following properties for all $X, Y \in L^\infty$:

- **Conditional cash invariance:** $\forall m \in L^\infty(\mathcal{F}_t)$:
  $$\rho_t(X + m) = \rho_t(X) - m$$

- **Monotonicity:** $X \leq Y \Rightarrow \rho_t(X) \geq \rho_t(Y)$

- **Conditional convexity:** $\forall \lambda \in L^\infty(\mathcal{F}_t), 0 \leq \lambda \leq 1$:
  $$\rho_t(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_t(X) + (1 - \lambda)\rho_t(Y)$$

- **Normalization:** $\rho_t(0) = 0$. 
Conditional risk measure for processes

A map $\rho_t : \mathcal{R}_t^\infty \to L^\infty(\mathcal{F}_t)$ is called a conditional convex risk measure for processes if it satisfies the following properties for all $X, Y \in \mathcal{R}_t^\infty$:

- **Conditional cash invariance:** $\forall m \in L^\infty(\mathcal{F}_t)$:
  $$\rho_t(X + m 1_{\{t, t+1, \ldots\}}) = \rho_t(X) - m$$

- **Monotonicity:** $X \leq Y \implies \rho_t(X) \geq \rho_t(Y)$

- **Conditional convexity:** $\forall \lambda \in L^\infty(\mathcal{F}_t), 0 \leq \lambda \leq 1$:
  $$\rho_t(\lambda X + (1 - \lambda)Y) \leq \lambda \rho_t(X) + (1 - \lambda)\rho_t(Y)$$

- **Normalization:** $\rho_t(0) = 0$. 
Acceptance set

- In the same way as in case of random variables we define the acceptance set of a conditional convex risk measure for processes:

\[ \mathcal{A}_t = \{ X \in \mathcal{R}^\infty \mid \rho_t(X) \leq 0 \} . \]

- Again, \( \rho_t \) is uniquely determined through its acceptance set:

\[ \rho_t(X) = \text{ess inf} \{ m \in L^\infty(\mathcal{F}_t) \mid X + m1_{\{t,t+1,...\}} \in \mathcal{A}_t \} . \]

\( \rightarrow \rho_t(X) \) is the minimal conditional capital requirement that has to be added to the cash flow \( X \) at time \( t \) in order to make it acceptable.
Risk measures for processes

• Setup

• Definition

• Transforming processes into random variables

• Translating the results from Part I:
  – Robust representation
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  – Time consistency, supermartingales, asymptotics and bubbles...

• Examples
Transforming processes into random variables

We define

- the product space \((\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})\) as: \(\bar{\Omega} = \Omega \times \mathbb{T}\)

\[
\bar{\mathcal{F}} = \sigma(A_t \times \{t\} \mid A_t \in \mathcal{F}_t, t \in \mathbb{T}), \quad \bar{P} = P \otimes \mu,
\]

where \(\mu = (\mu_t)_{t=0,\ldots,T}\) is any adapted sequence with \(\mu_t > 0\), \(\sum_{t \in \mathbb{T}} \mu_t = 1\). Then

\[
\mathcal{R}^\infty = L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}).
\]

- the filtration \((\bar{\mathcal{F}}_t)_{t=0,\ldots,T}\) on \((\bar{\Omega}, \bar{\mathcal{F}})\) by

\[
\bar{\mathcal{F}}_t = \sigma \left( \{A_j \times \{j\}, A_t \times \{t, t+1, \ldots\} \mid A_j \in \mathcal{F}_j, j \leq t\} \right).
\]

Then \(X\) is \(\bar{\mathcal{F}}_t\)-measurable iff \(X = (X_0, \ldots, X_{t-1}, X_t, X_t, \ldots)\) with \(X_s \in \mathcal{F}_s\)-measurable, \(s = 0, \ldots, t\).
Risk measures on the product space

There is a one-to-one correspondence between

- conditional convex risk measures for processes

\[ \rho_t : \mathcal{R}_t^\infty \to L^\infty(\mathcal{F}_t) \]

and

- conditional convex risk measures for random variables on the product space

\[ \bar{\rho}_t : L^\infty(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P}) \to L^\infty(\bar{\Omega}, \bar{\mathcal{F}}_t, \bar{P}) \]

The relation is given by

\[ \bar{\rho}_t(X) = -X_01_{\{0\}} - \ldots - X_{t-1}1_{\{t-1\}} + \rho_t(X)1_{\{t,t+1,\ldots\}}. \]
Optional random measures

Let $Q$ be a probability measure on $(\Omega, (\mathcal{F}_t), \mathcal{F})$ with $Q \ll_{loc} P$ and consider the set $\Gamma(Q)$ of optional random measures on $\mathbb{T}$ under $Q$:

$$\gamma = (\gamma_t)_{t \in \mathbb{T}} \text{ s.t. } \gamma_t \in L^\infty(\mathcal{F}_t), \quad \gamma_t \geq 0 \text{ P-a.s. } \forall t \in \mathbb{T}$$

and

- $T < \infty$: $\sum_{t=0}^{T} \gamma_t = 1$ P-a.s. ($\Gamma(Q) := \Gamma$ independent of $Q$).
- $\mathbb{T} = \mathbb{N}_0$: $\sum_{t=0}^{\infty} \gamma_t = 1$ Q-a.s.
- $\mathbb{T} = \mathbb{N}_0 \cup \{\infty\}$: $\sum_{t=0}^{\infty} \gamma_t + \gamma_\infty = 1$ Q-a.s. and $\gamma_\infty = 0$ Q-a.s. on the support of the singular part of $Q$ w.r.t. $P$. 
Discounting processes

Let $Q$ be a probability measure on $(\Omega, (\mathcal{F}_t), \mathcal{F})$ with $Q \ll_{\text{loc}} P$ and consider the set $D(Q)$ of predictable discounting processes on $\mathbb{T}$ under $Q$:

$$D = (D_t)_{t \in \mathbb{T}} \text{ s.t. } D \text{ decreasing, predictable, } D_0 = 1$$

and

- $T < \infty$: $D_{T+1} := 0$ ($D(Q) := \mathcal{D}$ independent of $Q$).

- $T = \mathbb{N}_0$: $D_t \downarrow 0$ $Q$-a.s.

- $T = \mathbb{N}_0 \cup \{\infty\}$: $D_t \downarrow D_\infty \geq 0$ $Q$-a.s. and $D_\infty = 0$ $Q$-a.s. on the support of the singular part of $Q$ w.r.t. $P$. 
Correspondence

There is a one-to-one correspondence between optional measures \( \gamma \in \Gamma(Q) \) and predictable discounting processes \( D \in D(Q) \).

The relation is given by

\[
D_t = 1 - \sum_{s=0}^{t-1} \gamma_s = \sum_{s \in T, s \geq t} \gamma_s, \quad t = 0, 1, \ldots, \quad D_\infty = \gamma_\infty \quad Q\text{-a.s.}
\]

and

\[
\gamma_t = D_t - D_{t+1}, \quad t = 0, 1, \ldots, \quad \gamma_\infty = D_\infty \quad Q\text{-a.s.}
\]
Probability measures on the product space

For any probability measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ the following holds: $\bar{Q} \ll \bar{P}$ if and only if there exists

- a probability measure $Q$ on $(\Omega, \mathcal{F})$ s.t. $Q \ll P$ if $T < \infty$ and $Q \ll_{\text{loc}} P$ if $T = \infty$
- a random measure $\gamma \in \Gamma(Q)$ (resp. a discounting factor $D \in \mathcal{D}(Q)$)

such that

$$E_{\bar{Q}}[X] = E_{Q} \left[ \sum_{t \in T} \gamma_t X_t \right] = E_{Q} \left[ \sum_{t=0}^{T} D_t (X_t - X_{t-1}) \right]$$

for all $X \in \mathcal{R}^\infty$. In this case we write

$$\bar{Q} = Q \otimes \gamma = Q \otimes D.$$
Idea of the proof

We use the Itô-Watanabe decomposition for nonnegative supermartingales:

**Proposition 1** Let \((U_t)\) be a nonnegative \(P\)-supermartingale on some probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\). Then there exist a nonnegative \(P\)-martingale \((M_t)\) and a predictable decreasing process \((D_t)\) such that \(M_0 = U_0, D_0 = 1\) and

\[ U_t = M_t D_t \quad \forall t. \]

Moreover such a decomposition is unique on \(\{t < \tau_0\}\), where \(\tau_0 := \inf\{t > 0 \mid U_t = 0\}\).
Idea of the proof (continued)

Let $\bar{Q} \ll \bar{P}$ with the density $\bar{Z} = (Z_t)_{t \in \mathbb{T}}$.

- We apply the Itô-Watanabe decomposition to the nonnegative $P$-supermartingale

$$U_t := E_P \left[ \sum_{s \geq t} \mu_s Z_s | \mathcal{F}_t \right] = M_t D_t, \quad t \in \mathbb{T}.$$  

- The martingale $(M_t)$ defines a consistent sequence of probability measures $Q_t$ on $\mathcal{F}_t$ via $\frac{dQ_t}{dP} = M_t$, $t \in \mathbb{T} \cap \mathbb{N}_0$.

- By Parthasarathy's extension theorem, the family $(Q_t)$ admits a unique extension to a probability measure $Q$ on $\mathcal{F}_\infty = \sigma(\bigcup_t \mathcal{F}_t)$, such that $Q \ll_{\text{loc}} P$ with

$$M_t = \frac{dQ}{dP} |_{\mathcal{F}_t}, \quad t \in \mathbb{T} \cap \mathbb{N}_0.$$
Idea of the proof (continued)

- $(D_t)$ defines a discounting factor with the corresponding random measure $(\gamma_t)$.

- We have

$$E_{\tilde{Q}}[X] = E_P \left[ \sum_{t \in T} X_t \mu_t Z_t \right] = \sum_{t=0}^{\infty} E_P \left[ X_t E_P [U_t - U_{t+1} | F_t] \right]$$

$$= \sum_{t=0}^{\infty} E_P \left[ X_t (M_tD_t - M_{t+1}D_{t+1}) \right]$$

$$= \sum_{t=0}^{\infty} E_Q [X_t \gamma_t] = E_Q \left[ \sum_{t=0}^{\infty} X_t \gamma_t \right].$$
Probability measures on the product space

For $\bar{Q} \ll \bar{P}$ with the decomposition $\bar{Q} = Q \otimes \gamma = Q \otimes D$, the conditional expectation given $\bar{F}_t$ takes the form

$$E_{\bar{Q}}[X | \bar{F}_t] = X_0 1_{\{0\}} + \ldots + X_{t-1} 1_{\{t-1\}} + E_Q \left[ \sum_{s \geq t} \frac{\gamma_s}{D_t} X_s | \mathcal{F}_t \right] 1_{\{t, t+1, \ldots\}},$$

where the last term on the right-hand-side is well defined $Q$-a.s. on $\{D_t > 0\}$.
Risk measures for processes

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- Examples
Robust representation: a recap

A conditional convex risk measure $\rho_t$ for random variables is continuous from above if and only if it has the robust representation

$$\rho_t(X) = \underset{Q \in Q_t}{\text{ess sup}} (E_Q[-X] - \alpha_t^{\text{min}}(Q)),$$

where

$$Q_t = \{ Q \ll P \mid Q = P|_{\mathcal{F}_t} \},$$

and the minimal penalty function $\alpha_t$ is given by

$$\alpha_t^{\text{min}}(Q) = \underset{X \in \mathcal{A}_t}{\text{ess sup}} E_Q[-X|\mathcal{F}_t] = \underset{X \in L^\infty}{\text{ess sup}} (E_Q[-X|\mathcal{F}_t] - \rho_t(X)).$$
Robust representation

A conditional convex risk measure \( \rho_t \) for processes is continuous from above if and only if it has the robust representation

\[
\rho_t(X) = \operatorname{ess sup}_{Q \in \mathcal{Q}_{loc}^t} \operatorname{ess sup}_{\gamma \in \Gamma_t(Q)} \left( E_Q \left[ - \sum_{s \in T, s \geq t} \gamma_s X_s \mid \mathcal{F}_t \right] - \alpha_t(Q \otimes \gamma) \right)
\]

\[
= \operatorname{ess sup}_{Q \in \mathcal{Q}_{loc}^t} \operatorname{ess sup}_{D \in \mathcal{D}_t(Q)} \left( E_Q \left[ - \sum_{s = t}^T D_s \Delta X_s \mid \mathcal{F}_t \right] - \alpha_t(Q \otimes D) \right)
\]

where

\[
\mathcal{Q}_{loc}^t = \{ Q \ll_{loc} P \mid Q = P|_{\mathcal{F}_t} \}
\]

\[
\Gamma_t(Q) = \{ \gamma \in \Gamma(Q) \mid \gamma_0 = \ldots = \gamma_{t-1} = 0 \},
\]

\[
\mathcal{D}_t(Q) = \{ D \in \mathcal{D}(Q) \mid D_0 = \ldots = D_t = 1 \},
\]
Robust representation

A conditional convex risk measure $\rho_t$ for processes is continuous from above if and only if it has the robust representation

$$\rho_t(X) = \text{ess sup}_{Q \in \mathcal{Q}_t^{\text{loc}}} \text{ess sup}_{\gamma \in \Gamma_t(Q)} \left( E_Q \left[ - \sum_{s \in \mathcal{T}, s \geq t} \gamma_s X_s \mid \mathcal{F}_t \right] - \alpha_t(Q \otimes \gamma) \right)$$

$$= \text{ess sup}_{Q \in \mathcal{Q}_t^{\text{loc}}} \text{ess sup}_{D \in \mathcal{D}_t(Q)} \left( E_Q \left[ - \sum_{s = t}^{T} D_s \Delta X_s \mid \mathcal{F}_t \right] - \alpha_t(Q \otimes D) \right)$$

and where

$$\alpha_t(Q \otimes \gamma) = \text{ess sup}_{X \in A_t} E_Q \left[ - \sum_{s \in \mathcal{T}, s \geq t} \gamma_s X_s \mid \mathcal{F}_t \right]$$

for $Q \in \mathcal{Q}_t^{\text{loc}}$ and $\gamma \in \Gamma_t(Q)$. 
Risk measures for processes

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Cash additivity and sub-additivity

A conditional convex risk measure for processes $\rho_t$ is called

- **cash sub-additive** if for all $t, s \geq 0$ and $m \in L^\infty(\mathcal{F}_t)$
  \[ \rho_t(X + m1_{\{t+s,\ldots\}}) \geq \rho_t(X) - m \quad \text{for} \quad m \geq 0 \]
  \[ \text{(resp.} \quad \leq \quad \text{for} \quad m \leq 0) \]

  *(El Karoui & Ravanelli (2008));*

- **cash additive at** $t + s$ for some $s \geq 1$ if
  \[ \rho_t(X + m1_{\{t+s,\ldots\}}) = \rho_t(X) - m, \quad \forall \ m \in L^\infty(\mathcal{F}_t); \]

- **cash additive** if it is cash additive at all times.

**Remark.** By monotonicity and cash invariance every conditional convex risk measure on processes is cash sub-additive.
Cash additivity and sub-additivity

Let $\rho_t$ be a conditional convex risk measure on processes, continuous from above. Then

- $\rho_t$ is cash additive at time $t + s$ if and only if there is no discounting up to time $t + s$. In this case $\rho_t$ is cash additive up to time $t + s$.

- $T < \infty$ or $T = \mathbb{N}_0 \cup \{\infty\}$: $\rho_t$ is cash additive if and only if it reduces to a risk measure for random variables, i.e.

$$
\rho_t(X) = \operatorname{ess sup}_{Q \in \mathcal{Q}_t} \left( E_Q[-X_T \mid \mathcal{F}_t] - \alpha_t(Q) \right).
$$

- $T = \mathbb{N}_0$: $\rho_t$ cannot be cash additive.
Risk measures for processes

- Setup
- Definition
- Transforming processes into random variables
- Translating the results from Part I:
  - Robust representation
  - Cash additivity and subadditivity
  - Time consistency, supermartingales, asymptotics and bubbles...
- Examples
Time consistency

- As in the case of risk measures for random variables, in the dynamic setting we obtain for each cash flow process \( X \) a sequence of risk assessments \( (\rho_t(X))_{t=0,1,...} \).

  (Notation: \( \rho_t(X) := \rho_t(\pi_{t,T}(X)) \) for \( X \in \mathcal{R}^\infty \).)

- The issue of time consistency arises.

- Using the one-to-one correspondence on the product space, we can translate all the notions of time consistency and their characterizations from the first part to the present context.

- Here we concentrate only on the strong time consistency.
(Strong) Time consistency

A dynamic convex risk measure for random variables $(\rho_t)_{t=0,1,...}$ is called time consistent if

$$\rho_t(X) = \rho_t(-\rho_{t+1}(X)) \quad \forall \ t \geq 0, \ \forall \ X \in L^\infty(\mathcal{F}_T).$$

A dynamic convex risk measure for processes $(\rho_t)_{t=0,1,...}$ is called time consistent if

$$\rho_t(X) = \rho_t(X_t1_{\{t\}} - \rho_{t+1}(X)1_{\{t+1,...\}}) \quad \forall \ t \geq 0, \forall \ X \in \mathcal{R}^\infty.$$

(Cheridito, Delbaen & Kupper (2006), Cheridito & Kupper (2006))
Step by step

- **"One-step" acceptance set** for processes:

  \[ A_{t,t+1} := \{ X \in \mathcal{R}_{t,t+1}^\infty \mid \rho_t(X) \leq 0 \} . \]

  \( X \in \mathcal{R}_{t,t+1}^\infty \) iff \( X = (0, \ldots, 0, X_t, X_{t+1}, X_{t+1}, \ldots) \)

- **"One-step" penalty function** for processes:

  \[ \alpha_{t,t+1}(Q \otimes D) = \frac{1}{D_t} \text{ess sup}_{X \in A_{t,t+1}} \mathbb{E}_Q [-\gamma_t X_t - D_{t+1} X_{t+1} \mid \mathcal{F}_t] . \]
Equivalent characterizations: a recap

Let \((\rho_t)_{t=0,\ldots,T}\) be a dynamic convex risk measure for random variables such that each \(\rho_t\) is continuous from above. Then the following conditions are equivalent:

1. \((\rho_t)\) is time consistent
2. \(A_t = A_{t,t+1} + A_{t+1}\)
3. \(\alpha_{t}^{\text{min}}(Q) = \alpha_{t,t+1}^{\text{min}}(Q) + E_Q[\alpha_{t+1}^{\text{min}}(Q) | \mathcal{F}_t]\)
4. \(E_Q[\rho_{t+1}(X) + \alpha_{t+1}^{\text{min}}(Q) | \mathcal{F}_t] \leq \rho_t(X) + \alpha_t^{\text{min}}(Q).\)
Equivalent characterizations

Let \((\rho_t)_{t=0,...,T}\) be a dynamic convex risk measure for processes such that each \(\rho_t\) is continuous from above. Then the following conditions are equivalent:

1. \((\rho_t)\) is time consistent

2. \(A_t = A_{t,t+1} + A_{t+1} \ \forall t\)

3. \(D_t\alpha_t(Q \otimes D) = D_t\alpha_{t,t+1}(Q \otimes D) + E_Q[D_{t+1}\alpha_{t+1}(Q \otimes D)|\mathcal{F}_t]\)
   for all \(t \geq 0\), all \(Q \ll_{\text{loc}} P\) and \(D \in \mathcal{D}(Q)\).

4. \(E_Q[D_{t+1}(X_t + \rho_{t+1}(X) + \alpha_{t+1}(Q \otimes D))|\mathcal{F}_t] \leq D_t(X_t + \rho_t(X) + \alpha_t(Q \otimes D))\)
   for all \(Q \ll_{\text{loc}} P\) and \(D \in \mathcal{D}(Q)\).
Supermartingale properties

Let \((\rho_t)_{t=0,\ldots,T}\) be a time consistent dynamic convex risk measure

- for random variables. Then
  - the penalty process \((\alpha_t^{\min}(Q))_{t \geq 0}\)
  - the process
    \[ \rho_t(X) + \alpha_t^{\min}(Q), \quad t \geq 0 \]
    are \(Q\)-supermartingales for all \(Q \ll P\) s.t. \(\alpha_0^{\min}(Q) < \infty\).

- for processes. Then
  - the discounted penalty function process \((D_t\alpha_t^{\min}(Q)))_{t \geq 0}\)
  - the process
    \[ D_t \rho_t(X - X_{t1}\{t,\ldots\}) + \sum_{s=0}^{t} D_s(-\Delta X_s) + D_t\alpha_t(Q \otimes D), \quad t \geq 0 \]
    are \(Q\)-supermartingales for all \(Q\) and \(D\) s.t. \(\alpha_0(Q \otimes D) < \infty\).
Bubbles

The discounted penalty function process \((D_t \alpha_t^{\text{min}}(Q))_{t=0,1,\ldots}\) has the Riesz decomposition

\[
D_t \alpha_t(Q \otimes D) = E_Q \left[ \sum_{k=t}^T D_k \alpha_{k,k+1}(Q \otimes D) \mid \mathcal{F}_t \right] + N_q^{Q,D} \quad Q\text{-a.s.},
\]

where

\[
N_q^{Q,D} := \begin{cases} 
0 & \text{if } T < \infty, \\
\lim_{s \to \infty} E_Q [D_s \alpha_s(Q \otimes D) \mid \mathcal{F}_t] & \text{if } T = \infty 
\end{cases} \quad Q\text{-a.s.}
\]

is a non-negative \(Q\)-martingale \(\rightarrow \) "bubble".

In case \(T = \mathbb{N}_0 \cup \{\infty\}: \) no "bubble" \(\Leftrightarrow \) "asymptotic safety":

\[
\rho_\infty(X) := \lim_{t \to \infty} \rho_t(X) \geq -X_\infty \quad Q\text{-a.s. on } \{D_\infty > 0\}.
\]
Risk measures for processes

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Separation of model- and discounting ambiguity

- A simple class of risk measures for processes can be obtained using the representation

\[ \rho_t(X) = \text{ess sup}_{\gamma \in G_t} \psi_t \left( \sum_{s \geq t} X_s \gamma_s \right), \]

where \( \psi_t \) is a fixed conditional convex risk measure for random variables and \( G_t \subseteq \Gamma_t \) is a fixed collection of discounting factors.
Separation of model- and discounting ambiguity

Some examples of a risk measure for processes with the representation

$$\rho_t(X) = \text{ess sup}_{\gamma \in G_t} \psi_t \left( \sum_{s \geq t} X_s \gamma_s \right):$$

- $G_t = \{ \delta_{\{s\}} \}$ for some $s \in \{t, t + 1, \ldots \}$. In this case

$$\rho_t(X) = \psi_t(X_s)$$

is a conditional convex risk measure on $L^\infty(\Omega, \mathcal{F}_s, P)$. 
Separation of model- and discounting ambiguity

Some examples of a risk measure for processes with the representation

$$\rho_t(X) = \text{ess sup}_{\gamma \in G_t} \psi_t \left( \sum_{s \geq t} X_s \gamma_s \right) :$$

- $G_t = \{(\gamma_t, \ldots, \gamma_T)\}$, i.e. a single discounting factor. In this case

$$\rho_t(X) = \psi_t \left( \sum_{s=t}^{T} D_s \Delta X_s \right) = \psi_t \left( \sum_{s=t}^{T} \Delta Y_s \right) = \psi_t(Y_T)$$

i.e. on discounted cash-flows $\rho_t$ reduces to a risk measure for random variables.
Separation of model- and discounting ambiguity

Some examples of a risk measure for processes with the representation

\[ \rho_t(X) = \text{ess sup} \psi_t \left( \sum_{s \geq t} X_s \gamma_s \right) : \]

- \( G_t = \{(1 \{ \tau = s \})_{s=t,t+1,...} | \tau \in \Theta_t \} \), where \( \Theta_t \) denotes the set of all stopping times valued in \( \{t, t + 1, \ldots \} \). Here we obtain

\[ \rho_t(X) = \text{ess sup} \psi_t \left( \sum_{s=t}^{T} X_s \gamma_s \right) = \text{ess sup} \psi_t(X_{\tau}) \]

i.e. \( \rho_t \) is defined by the worst stopping of \( (\psi_t(X_s))_{s=t,t+1,...} \).

(Cheridito & Kupper (2006))
Entropic risk measure for processes

On the product space we define the conditional entropic risk measure
\( \tilde{\rho}_t : L^\infty(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}) \to L^\infty(\tilde{\Omega}, \tilde{\mathcal{F}}_t, \tilde{P}) \) by

\[
\tilde{\rho}_t(X) = \frac{1}{R_t} \cdot \log E_{\tilde{P}} \left[ e^{-R_t \cdot X} \big| \tilde{\mathcal{F}}_t \right]
\]

with the risk aversion parameter \( R_t = (r_0, \ldots, r_{t-1}, r_t, \ldots, r_T), \ r_s > 0, \ \mathcal{F}_s\text{-mb. for all } s = 0, \ldots, t. \)

The corresponding conditional entropic risk measure for processes
\( \rho_t : \mathcal{R}_t^\infty \to L^\infty(\mathcal{F}_t) \) takes the form

\[
\rho_t(X) = \rho^P_t \left( -\frac{1}{r_t} \log \left( \sum_{s=t}^{T} e^{-r_t X_s \mu^t_s} \right) \right) = \rho^P_t \left( -\rho^\mu_t \left( X(\omega, \cdot) \right) \right),
\]

where \( \rho^P_t : L_T^\infty \to L_t^\infty \) is the usual conditional entropic risk measure for random variables and \( \rho^\mu_t : \mathbb{R}^{T-t+1} \to \mathbb{R} \) is its analogous “with respect to time”.

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Average Value at Risk for processes

- On the product space the conditional AVaR at level

\[ \Lambda_t = (\lambda_0, \ldots, \lambda_{t-1}, \lambda_t, \ldots, \lambda_t), \ 0 < \lambda_s \leq 1, \ \lambda_s \in L^\infty(\mathcal{F}_s) \ \forall \ s \]

is defined by

\[ \bar{\rho}_t(X) = \text{ess sup} \{ E_{\bar{Q}}[-X|\bar{F}_t] | \bar{Q} \in \bar{Q}_t, d\bar{Q}/d\bar{P} \leq \Lambda_t^{-1} \} \]

- The corresponding conditional Average Value at Risk for processes

\[ \rho_t : \mathcal{R}_t^\infty \rightarrow L^\infty(\mathcal{F}_t) \] is given by

\[ \rho_t(X) = \text{ess sup} \left\{ E_Q \left[ - \sum_{s=t}^{T} X_s \gamma_s |\mathcal{F}_t \right] | Q \in Q_t, \gamma \in \Gamma_t, \frac{\gamma_s}{\mu_s^{t}} \left( \frac{dQ}{dP} \right)_s \leq \frac{1}{\lambda_t} \ \forall \ s \right\}. \]
Average Value at Risk for processes

- One can also define a “decoupled version” of conditional Average Value at Risk for processes as

\[
\rho_t^{\lambda_1,\lambda_2}(X) := \text{ess sup}_{\gamma \in \Gamma_t^{\lambda_1}} AV @ R_t^\lambda \left( \sum_{s \in \mathcal{T}_t} X_s \gamma_s \right), \quad X \in \mathcal{R}_t^\infty.
\]

Here, \( \lambda_1, \lambda_2 \in L^\infty(\mathcal{F}_t) \) with \( 0 < \lambda_1, \lambda_2 \leq 1 \), \( AV @ R_t^\lambda \) is the usual Average Value at Risk for random variables, and

\[
\Gamma_t^{\lambda_1} = \left\{ \gamma \in \Gamma_t(P) \mid \frac{\gamma_s}{\mu_s} \leq \frac{1}{\lambda_1}, \ s \geq t \right\}.
\]

- The decoupled version is less conservative than the Average Value at Risk with \( \lambda_t = \lambda_1 \lambda_2 \) defined the previous slide, i.e.,

\[
\rho_t^{\lambda_1,\lambda_2}(X) \leq \rho_t^{\lambda_1\lambda_2}(X) \quad \forall \ X \in \mathcal{R}_t^\infty.
\]
Thank you for your attention!