Central limit theorem for multiple Skorohod integrals

David Nualart

Department of Mathematics
Kansas University

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\[ H \text{ is a real separable Hilbert space} \]

\[ X = \{ X(h), h \in \mathcal{H} \} \text{ is a Gaussian family of random variables with zero mean and covariance} \]

\[ E(X(h)X(g)) = \langle h, g \rangle_{\mathcal{H}} \]

**Example:** If \( W = \{ W_t, t \geq 0 \} \) is a Brownian motion, then \( \mathcal{H} = L^2([0, \infty)) \) and for any \( h \in \mathcal{H}, \)

\[ W(h) = \int_0^\infty h_t dW_t \]
Derivative operator:

- For $F \in S$ of the form $F = f(X(h_1), \ldots, X(h_n))$, where $n \geq 1$, $f \in C^\infty_b(\mathbb{R}^n)$, $h_i \in \mathcal{H}$, we define

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(h_1), \ldots, X(h_n))h_i$$

- $D$ is closable from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$
- $\mathbb{D}^{k,p}$ is the closure of $S$ with respect to the norm

$$\|F\|_{k,p}^p = E(|F|^p) + \sum_{j=1}^{k} E\left(\|D^j F\|_{\mathcal{H} \otimes j}^p\right)$$
**Divergence operator:**

- $\delta$ is the adjoint of the derivative operator:

$$\mathbb{E}(F\delta(u)) = \mathbb{E}(\langle DF, u \rangle_{\mathcal{H}}),$$

for any $F \in \mathbb{D}^{1,2}$, where $u \in \text{Dom}\delta \subset L^2(\Omega; \mathcal{H})$.

- If $X = W$ is Brownian motion, $\delta$ is an extension of the Itô stochastic integral called **Skorohod integral**

- $\delta^k$ is the adjoint of $D^k$ and it coincides with the multiple stochastic integral $I_k$ on deterministic elements of $\mathcal{H}^\otimes k$

- $\mathbb{D}^{k,p}(\mathcal{H}^\otimes k) \subset \text{Dom}\delta^k$
Non-Gaussian Central Limit Theorem

Theorem

Fix an integer $q \geq 1$. Let $F_n$ be a sequence of random variables of the form $F_n = \delta^q(u_n)$, where $u_n$ is a symmetric element in $D^{2q, 2q}(\mathcal{H} \otimes q)$. Suppose that $\sup_n \mathbb{E}(|F_n|) < \infty$ and

$$\langle u_n, D^{k_1}F_n \otimes \cdots \otimes (D^{q-1}F_n)^{k_{q-1}} \otimes h \rangle_{\mathcal{H} \otimes q} \xrightarrow{L^1} 0$$

for all $h \in \mathcal{H} \otimes r$, $r, k_1, \ldots, k_{q-1} \geq 0$, with $k_1 + 2k_2 + \cdots + (q-1)k_{q-1} + r = q$,

$$\langle u_n, D^qF_n \rangle_{\mathcal{H} \otimes q} \xrightarrow{L^1} S^2$$

Then, $F_n$ converges stably in law to $N(0, S^2)$
The sequence $F_n$ converges stably to $N(0, S^2)$ if for any bounded random variable $Z$ measurable with respect to $X$ and for any $\lambda \in \mathbb{R}$

$$\lim_{n} \mathbb{E} \left( e^{i\lambda F_n Z} \right) = \mathbb{E} \left( e^{-\frac{\lambda^2}{2} S^2 Z} \right)$$
Sketch of the proof

- Fix $\Phi_m = (X(e_1), \ldots, X(e_m))$, where $e_i$ is a basis of $\mathcal{H}$. It suffices to show that for each $m$

  $$(F_n, \Phi_m) \xrightarrow{\text{Law}} (F_\infty, \Phi_m),$$

  where

  $$\mathbb{E} \left( e^{i\lambda F_\infty} | X \right) = e^{-\frac{\lambda^2}{2} S^2} \quad (3)$$

- Because the laws of $(F_n, \Phi)$ are tight, we can assume that the above convergence in law holds, and it suffices to show that the limit satisfies (3)

- Set $Y = g(\Phi_m)$, where $g \in C_b^\infty(\mathbb{R}^m)$ and define

  $$\phi_n(\lambda) = \mathbb{E}(e^{i\lambda F_n Y})$$
We compute the limit of $\phi'(\lambda)$ in two ways:

1. Using weak convergence:

$$\phi'(\lambda) = i\mathbb{E}(e^{i\lambda F_n} F_n Y) \rightarrow i\mathbb{E}(e^{i\lambda F_\infty} F_\infty Y)$$

2. Using Malliavin calculus and our assumptions:

$$\phi'(\lambda) = i\mathbb{E}(e^{i\lambda F_n} F_n Y) = i\mathbb{E}(e^{i\lambda F_n} \delta^q(u_n) Y)$$

$$= i\mathbb{E} \left( \left\langle D^q \left( e^{i\lambda F_n} Y \right) \right| u_n \right\rangle_{\mathcal{H} \otimes q}$$

$$= -\lambda \mathbb{E} \left( e^{i\lambda F_n} \left\langle u_n, D^q F_n \right\rangle_{\mathcal{H} \otimes q} Y \right) + R_n$$

$$\rightarrow -\lambda \mathbb{E}(e^{i\lambda F_\infty} S^2 Y)$$
Hence,

$$i\mathbb{E}(e^{i\lambda F_\infty} F_\infty Y) = -\lambda \mathbb{E}(e^{i\lambda F_\infty} S^2 Y)$$

This leads to a linear differential equation satisfied by the conditional characteristic function of $F_\infty$:

$$\frac{\partial}{\partial \lambda} \mathbb{E}(e^{i\lambda F_\infty} | X) = -S^2 \lambda \mathbb{E}(e^{i\lambda F_\infty} | X),$$

and we obtain

$$\mathbb{E}(e^{i\lambda F_\infty} | X) = e^{-\frac{\lambda^2}{2} S^2}$$
Corollary

Fix an integer $q \geq 1$. Let $F_n$ be a sequence of random variables of the form $F_n = \delta^q(u_n)$, where $u_n$ is a symmetric element in $\mathbb{H}^{2q,2q}(\mathcal{H}^\otimes q)$. Suppose that $\sup_n \|F_n\|_{q,p} < \infty$ for any $p \geq 2$ and

(i) $\langle u_n, h \rangle_{\mathcal{H}^\otimes q} \to 0$ in $L^1$ for any $h \in \mathcal{H}^\otimes q$, and $u_n \otimes_j D^j F_n \to 0$ in $L^2$ for all $j = 1, \ldots, q - 1$,

(ii) $\langle u_n, D^q F_n \rangle_{\mathcal{H}^\otimes q} \xrightarrow{L^1} S^2$

Then, $F_n$ converges stably in law to $N(0, S^2)$
Consider a sequence of multiple integrals $F_n = I_m(g_n)$, where $m \geq 2$. We can take $q = 1$ and $u_n = I_{m-1}(g_n)$. Under the conditions

(i) For any $h \in \mathcal{H}$,

$$
\mathbb{E}\langle u_n, h \rangle^2_{\mathcal{H}} = \mathbb{E}\langle I_{m-1}(g_n), h \rangle_{\mathcal{H}} = (m - 1)! \langle g_n \otimes_{m-1} g_n, h \otimes^2 \rangle_{\mathcal{H} \otimes^2} \to 0
$$

(ii) $\langle u_n, DF_n \rangle_{\mathcal{H}} = \langle I_{m-1}(g_n), mI_{m-1}(g_n) \rangle_{\mathcal{H}} = \frac{1}{m} \| DF_n \|^2_{\mathcal{H}} \xrightarrow{\text{Law}} S^2$, we have $F_n \xrightarrow{\text{stably}} N(0, S^2)$

- If $S^2 = \sigma^2$ is constant and we assume that $E(F_n^2) \to \sigma^2$, then (ii) is equivalent to the convergence to a normal law $N(0, \sigma^2)$ (Nualart and Ortiz-Latorre 08)
The fractional Brownian motion (fBm) \( B^H = \{ B^H_t, t \geq 0 \} \) with Hurst parameter \( H \in (0, 1) \) is a zero mean Gaussian process with covariance

\[
\mathbb{E}(B^H_s B^H_t) = R_H(s, t) = \frac{1}{2} \left( s^{2H} + t^{2H} - |t - s|^{2H} \right)
\]

Some basic properties:

- **Stationary increments:** \( \mathbb{E}(B^H_t - B^H_s)^2 = |t - s|^{2H} \)
- By Kolmogorov’s continuity criterion, with probability one, the trajectories \( t \rightarrow B^H_t(\omega) \) are Hölder continuous of order \( \gamma \), for all \( \gamma < H \):
  \[
  |B^H_t(\omega) - B^H_s(\omega)| \leq G_{\gamma, T}(\omega)|t - s|^{\gamma},
  \]
  if \( s, t \in [0, T] \)
- For \( H = \frac{1}{2} \), \( B = B^{1/2} \) is a Brownian motion
**Self-similarity:** For any \( a > 0 \), the processes

\[
\{ a^{-H} B_{at}^H, t \geq 0 \}
\]

and

\[
\{ B_t^H, t \geq 0 \}
\]

have the same probability distribution (they are fractional Brownian motions with Hurst parameter \( H \)).

- This is a fractal property in probability
- The fBm is characterized by being a self-similar continuous Gaussian centered process with stationary increments
For $H = \frac{1}{2}$ the fBm is a Brownian motion and it has *independent increments*

For $H \neq \frac{1}{2}$, the fBm $B^H$ has correlated increments:

$$
\rho_H(n) = \mathbb{E}(B_1^H(B_{n+1}^H - B_n^H)) = \frac{1}{2} \left( (n + 1)^{2H} + (n - 1)^{2H} - 2n^{2H} \right) \sim H(2H - 1)n^{2H-2},
$$

as $n \to \infty$

If $H > \frac{1}{2}$, then $\rho_H(n) > 0$ and $\sum_n \rho_H(n) = \infty$ (*long memory*)

If $H < \frac{1}{2}$, then $\rho_H(n) < 0$ and $\sum_n |\rho_H(n)| < \infty$
Define $\Delta B^H_k = B^H_{(k+1)/n} - B^H_{k/n}$

- For any $p \geq 1$, using self-similarity and the Ergodic Theorem we obtain
  \[ n^{pH-1} \sum_{k=0}^{n-1} |\Delta B^H_k|^p \xrightarrow{\text{a.s., } L^1} c_p = \mathbb{E}(|B^H_1|^p) \]

- For any $p \geq 1$ and $H < \frac{3}{4}$ (Breuer and Major 83)
  \[ \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left( n^{pH} |\Delta B^H_k|^p - c_p \right) \xrightarrow{\text{Law}} N(0, \sigma_{p,H}^2) \]
There has been some recent interest in the study of weighted variations of the form

$$G_n = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f(B_{k/n}^H) H_q(n^H \Delta B_k^H),$$

where $f$ is a smooth function with polynomial growth and $H_q$ is the $q$th Hermite polynomial.

Motivation: Itô’s formula in the case $H < \frac{1}{2}$ where the stochastic integral is the limit of Riemann sums defined using the midpoint rule of the trapezoid rule (Burdzy and Swanson 08 and Nourdin and Reveillac 08).
Fix an integer \( q \geq 2 \) and a function \( f \in C^{2q}(\mathbb{R}) \) such that for all \( p \geq 1 \), \( \sup_{t \in [0,1]} \mathbb{E}(|f^{(i)}(B^H_t)|^p) < \infty \) for all \( i \leq 2q \).

(i) If \( \frac{1}{2q} < H < 1 - \frac{1}{2q} \), then

\[
G_n \xrightarrow{\text{Stably}} \sigma_{H,q} \int_0^1 f(B^H_s) dW_s,
\]

where \( W \) is a Brownian motion independent of \( B^H \)

(ii) If \( H < \frac{1}{2q} \), then

\[
n^{qH - \frac{1}{2}} G_n \xrightarrow{L^2} \frac{(-1)^q}{2^q q!} \int_0^1 f^{(q)}(B^H_s) ds
\]
These convergences were proved by Nourdin, Nualart and Tudor 08.

The convergence (5) in the case $q = 2$ was obtained by Nourdin 08, using Malliavin calculus.

The critical case $q = 2, H = \frac{1}{4}$ was solved by Nourdin and Réveillac. The convergence is in law and the limit is a linear combination of the two above limits.
Theorem

Suppose that \( H \in \left( \frac{1}{4q}, \frac{1}{2} \right) \). Then

\[
G_n - n^{1 - qH} \left( \frac{-1}{2q} \right)^q \int_0^1 f^{(q)}(B_s^H) \, ds \xrightarrow{\text{Stably}} \sigma_{H,q} \int_0^1 f(B_s^H) \, dW_s
\]

where \( W \) is a Brownian motion independent of \( B^H \)

- This implies the convergence (4) if \( H \in \left( \frac{1}{2q}, \frac{1}{2} \right) \) and the convergence (5) if \( H \in \left( 0, \frac{1}{2q} \right) \).
- In the critical case \( H = \frac{1}{2q} \) we obtain

\[
G_n \xrightarrow{\text{Stably}} \left( \frac{-1}{2q} \right)^q \int_0^1 f^{(q)}(B_s^H) \, ds + \sigma_{H,q} \int_0^1 f(B_s^H) \, dW_s
\]
Sketch of the proof

Define

\[ u_n = n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} f(B^H_{k/n}) 1_{[\frac{k}{n}, \frac{k+1}{n}]} \otimes q_{[\frac{k}{n}, \frac{k+1}{n}]} \].

Notice that \( H_q(n^H \Delta B^H_k) = \frac{1}{q!} n^{qH} \delta_q \left( 1_{[\frac{k}{n}, \frac{k+1}{n}]} \right) \). Therefore

\[ G_n = \frac{1}{q!} \sum_{k=0}^{n-1} f(B^H_{k/n}) \delta_q \left( 1_{[\frac{k}{n}, \frac{k+1}{n}]} \right) \]

We need the following general formula

\[ F \delta^q(u) = \sum_{r=0}^{q} \binom{q}{r} \delta^{q-r} (D^r F, u) \langle H \otimes r \rangle \]

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We obtain the decomposition

\[ G_n = \frac{1}{q!} \delta^q(u_n) + \sum_{r=1}^{q-1} \delta^{q-r}(v_{n,r}) + R_n, \]

where

\[ v_{n,r} = \frac{1}{q!} \binom{q}{r} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_k^r f^{(r)}(B_{k/n}^H) 1_{[k/n, k+1/n]}^{\otimes(q-r)} \]

\[ R_n = \frac{1}{q!} n^{qH-\frac{1}{2}} \sum_{k=0}^{n-1} \alpha_k^q f^{(q)}(B_{k/n}^H), \]

and \( \alpha_k = \langle 1_{[0, k/n]}, 1_{[k/n, k+1/n]} \rangle_H \).
By the Non-Gaussian Central Limit Theorem for multiple Skorohod integrals we obtain that for any $H \in \left( \frac{1}{4q}, \frac{1}{2} \right)$

$$
\frac{1}{q!} \delta^q (u_n) G_n \xrightarrow{\text{Stably}} \sigma_{H,q} \int_0^1 f(B^H_s) dW_s
$$

For any $H \in (0, \frac{1}{2})$, and $r = 1, \ldots, q - 1$, $\delta^{q-r}(v_{n,r})$ converges to zero in $L^2$

$$
R_n - n^{1-qH}(-1)^{qH/2q} \int_0^1 f(q)(B^H_s) ds
$$

converges to zero in $L^2$
Further developments

Obtain new change-of-variable formulas in law. We conjecture that

$$\int_0^1 f'(B_{s}^{1/6}) d^\circ B_s^{1/6} \overset{\mathcal{L}}{=} f(B_1^{1/6}) - f(0) - \frac{\tilde{\kappa}}{6} \int_0^1 f'''(B_s^{1/6}) dW_s,$$

where $W$ is a Brownian motion independent of $B^{1/6}$, and $d^\circ$ denotes the Stratonovich (trapezoid rule) integral