A NOTE ON THE INFINITE DIVISIBILITY OF SKEW-SYMMETRIC DISTRIBUTIONS

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A Note on the Infinite Divisibility of Skew-Symmetric Distributions

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Abstract

Infinite divisibility of some of the most important symmetric distributions skewed by an additive component is investigated. We find in particular that the skew-normal distribution of Azzalini (1985) and the multivariate skew-normal distribution of Azzalini and Dalla Valle (1996) are not infinitely divisible.

1. Introduction

Skew-symmetric distributions have been developed as natural extensions of the skew-normal distribution introduced by Azzalini (1985). The aim of this note is to determine the infinite divisibility of skew-symmetric distributions.

There are several ways of skewing a symmetric distribution; see Arnold and Beaver (2002). Here we only consider those symmetric distributions skewed by an additive component.

Definition 1. Let \( c \geq 0 \). A random variable \( Y \) is said to have skew-symmetric distribution if there exist constants \( a \) and \( b \neq 0 \); and independent random variables \( X \) and \( X_c \) such that

\[
Y \overset{d}{=} aX + bX_c,
\]

where \( X \) is symmetric and \( X_c \) is a copy of \( X \) truncated below at \( c \). For the special case \( c = 0 \) is said that \( X_0 \) has a half-distribution of \( X \).
We discuss the infinite divisibility of a skew-symmetric distribution of the form (1) only when \( X \) is infinitely divisible. This leads to the problem of determining the infinite divisibility of \( X_c \).

First we put attention to the case \( c = 0 \).

The skew-normal distribution of Azzalini has representation (1), see Henze (1986), with \( X \) standard normal, \( a = 1/\sqrt{1+\delta^2} \), \( b = \delta/\sqrt{1+\delta^2} \) and \( \delta \) is a real skewness parameter. It is important to remark that it is not infinitely divisible due to the half-normal distribution is not infinitely divisible as is noted in Steutel and Van Harn (2003, p. 126).

Immediate examples of infinitely divisible skew-symmetric distributions are skew-Laplace and skew-Cauchy, since the half-Laplace is the exponential distribution and the half-Cauchy is infinitely divisible as is shown in Steutel and Van Harn (2003, p. 411).

2. The case \( c > 0 \)

The skew-\( t \) distribution with \( \nu \) degree of freedom and the skew-double Pareto are infinitely divisible since, in both cases, their corresponding \( X_c \) in (1) is infinitely divisible by a log-convexity argument.

**Proposition 2.** The skew-Student \( t \) with \( \nu \) degree of freedom is infinitely divisible if \( c > \sqrt{\nu} \).

**Proof.** We prove that \( X_c \) in (1) is infinitely divisible by showing that its density is log-convex, see Sato (1999, Th. 51.4). Let us consider the Student-\( t \) density \( f \) with \( \nu \) degree of freedom truncated below at \( c \)

\[
f(x) = K \left( 1 + \frac{1}{\nu} x^2 \right)^{-\frac{\nu+1}{2}} 1_{[c,\infty)}(x).
\]

where \( K \) is the normalizing constant. Let \( x > c \). Differentiating twice, we have

\[
f'(x) = -K \frac{1+\nu}{\nu} x \left( 1 + \frac{1}{\nu} x^2 \right)^{-\frac{\nu+3}{2}},
\]

\[
f''(x) = -K \frac{1+\nu}{\nu} \left( \frac{x^2}{\nu} + 1 \right)^{-\frac{\nu+3}{2}} + K \frac{3+4\nu+\nu^2}{\nu^2} x^2 \left( \frac{x^2}{\nu} + 1 \right)^{-\frac{\nu+3}{2}}
\]

and

\[
[f'f'' - f'^2](x) = K^2 \frac{(\nu + 1)}{\nu^2} (x^2 - \nu) \left( 1 + \frac{1}{\nu} x^2 \right)^{-3-\nu}.
\]
Thus \( f f'' - (f')^2 > 0 \) if \( x > \sqrt{\nu} \). Hence \( f \) is log-convex.

In particular the skew-Cauchy distribution is infinitely divisible if \( c > 1 \) since the Student-t is Cauchy when \( \nu = 1 \). We are not able to give an answer in the truncation range \((0, 1]\) for Cauchy and \([0, \sqrt{\nu}]\) for Student-t.

**Proposition 3.** Let \( c > 0 \). The skew-double Pareto distribution is infinitely divisible.

**Proof.** We proceed similarly as the former proof. Consider the double-Pareto density truncated below at \( c \)
\[
f(x) = K \frac{1}{(1+x)^r} 1_{[c,\infty)}(x),
\]
where \( r > 1 \) and \( K \) is the normalizing constant. We obtain \( [f'' f'' - f'^2] (x) = K^2 \frac{r}{(1+x)^{2(r+1)}} > 0 \) for any \( x > c \).

Using a tail behavior criterion for infinite divisibility we show that skew-normal distribution is not infinitely divisible.

**Proposition 4.** Let \( c > 0 \). The skew normal distribution is not infinitely divisible.

**Proof.** Let us consider \( X \) in model (1) be a standard normal random variable. We prove that \( X_c \) is not infinitely divisible by proving that it does not fulfill the necessary condition \( - \log P (X_c > x) \leq \alpha x \log x \) for some \( \alpha > 0 \) and \( x \) sufficiently large, cf. Steutel (1979). Consider the standard normal density truncated below at \( c \)
\[
f(x) = K \phi(x) 1_{[c,\infty)}(x),
\]
where \( \phi \) is the standard normal density \( K \) is the normalizing constant. Let \( \Phi(x) \) denote the standard normal distribution. Observe that
\[
\lim_{x \to \infty} \frac{- \log P (X_c > x)}{x \log x} = \lim_{x \to \infty} \frac{- \log K [1 - \Phi(x)]}{x \log x}
\]
and apply L'Hôpital Rule twice to lead the limit to
\[
\lim_{x \to \infty} \left[ \frac{x}{1 + \log x} + \frac{1}{x (1 + \log x)^2} \right] = \infty.
\]
We finally conclude that the multivariate skew-normal distribution of Azzalini and Dalla Valle (1996) is not infinitely divisible.

Let \( Y = (Y_1, ..., Y_p)^T \) be a random vector with coordinates

\[
Y_i = a_i X_i + b_i X_0, \quad i = 1, ..., p, \tag{2}
\]

where \((X_1, ..., X_p)^T\) is a jointly normal vector independent of the half-normal random variable \(X_0\). Any linear combination \(\alpha^T Y\) is of the form

\[
\sum_{i=1}^{p} \alpha_i a_i X_i + \left( \sum_{i=1}^{p} \alpha_i b_i \right) X_0,
\]

which is not infinitely divisible by Proposition 4 and hence \(Y\) so is not.

The representation of the skew-normal random vector in Section 2.1 of Azzalini and Dalla Valle (1996) is a special case of the model (2).

References


