Twisted Borel K-theory and isomorphisms between differential models of K-theory

Ph.D. Thesis

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Resumo

Nesta tese discutimos alguns tópicos sobre cálculos da K-teoría torcida e equivalências entre dois modelos de extensão diferencial. Começamos com uma revisão matemática de modelos para a K-teoria torcida, extensões diferenciais para o caso não torcido e sequências espectrais de Serre e Atiyah-Hirzebruch, a fim de fornecer uma ligação explícita entre a extensão diferencial dos casos no torcidos e torcidos, além dar ferramentas para a exploração posterior dos torcimentos que serão utilizados. Na primeira parte desta tese, calculamos uma fórmula salvo extensões de grupo para a K-teoria torcida para um fibrado baseado no círculo S^1 com fibra uma variedade compacta e um torcimento dado em função de uma classe do segundo grupo de cohomología da fibra, posteriormente este caso é generalizado estendendo a base para o espaço classificatório de um grupo livre finitamente gerado e o torcimento será dado por derivações de feixes lineares associados ao grupo e à fibra, isto é acompanhado de exemplos para finalmente desenvolver um sequência espectral onde as fórmulas anteriores são enquadradas. Na segunda parte da tese, desenvolve-se uma equivalência topológica para os modelos de extensão diferencial de Freed-Lott e Carey-Mickelsson-Wang, além de indicar uma forma de alcançar a equivalência diferencial.

Palavras-chave: K-Teoria Topológica; Operadores de Fredholm; K-Teoria Torcida; K-Teoria Torcida de Borel; K-Teoria Diferencial.

Abstract

In this thesis we discuss some topics about twisted K-theory calculations and equivalences between a couple of differential extension models. We start with a mathematical review of models for twisted K-theory, differential extensions for the untwisted case, and Serre spectral sequences, in order to provide an explicit link between the differential extension of the untwisted and twisted cases, in addition to giving tools for the subsequent exploration of the twists that will be used. In the first part of this thesis we determine a formula up to group extensions for the twisted K-theory for a fiber bundle over the circle S^1 with fiber a compact manifold with respect to certain twists constructed from elements of the second cohomology group of the fiber. Later this case is generalized by allowing the base to be the classifying space of a finitely generated free group and the twisting will be given by a derivation of line bundles associated to the group and the fiber. This is accompanied by examples and we finally develop a spectral sequence where the previous formulas are framed. In the second part of the thesis, a topological equivalence is developed for the Freed-Lott and Carey-Mickelsson-Wang differential extension models. Additionally, we indicate a way to achieve the differential equivalence.

Keywords: Topological K-Theory; Fredholm operators; Twisted K-Theory; Twisted Borel K-Theory; Differential Twisted K-Theory.

Contents

Chapter 1

Introduction

Topological K-theory is a generalized cohomology theory that was introduced around 1960 by Atiyah and Hirzebruch [3], based on the Periodicity Theorem of Bott proved just a few years earlier and motivated by a more general K-theory introduced by Grothendieck in his formulation of the Riemann-Roch theorem [7]. Atiyah and Hirzebruch considered a topological analogue defined for any compact and Hausdorff space X, a group $K(X)$ constructed from the category of complex vector bundles on X. It is this "topological K-theory" that we are going to deal with in this document.

Topological K-theory has become an important tool in algebraic topology. Some of the best-known applications of algebraic topology are the non-existence theorem of division algebras after the Cayley octonions by Bott and Milnor, Adams' theorem determining the maximum number of linearly independent tangent vector fields on a sphere of arbitrary dimension or the proof that the only spheres which can be provided with H-space structures are S^0, S^1, S^3 and S^7. These facts have relatively elementary proofs using K–theory. Further applications to analysis, algebra and physics are found in the work of Atiyah-Singer [6], Quillen [39], Minasian and Moore [35] and others.

K-theory has different but equivalent descriptions. For a compact manifold, the group $K^0(M)$ can be described in terms of:

- Vector bundles on M [2].
- Maps from M to a certain space of Fredholm operators ([2], Appendix).
- Maps $p\colon Z \to M$ where Z is compact and p is K-oriented [14].

This also gave rise to twisted K-theory first introduced by Donovan and Karoubi [15] from a local coefficients approach, and later by Atiyah and Segal [4] (we refer to [30] for a history of its development). More recently, twisted K-theory has received much attention because of its applications classifying D-brane charges in string theory [36] and its connections with Verlinde algebras [17], [19], [18] and topological insulators [21].

Complementing the previous cohomological theories, their differential versions emerged where essentially given a cohomological theory K a refinement is made by adding differential information. This refinement is accompanied by three natural transformations, namely, we have I that forgets the differential structure, R that we call the curvature and finally a that inserts differential information in a refinement class. Additionally there are different compatibility conditions such as $R \circ a = d$, where d denotes the differential of forms, or that I and R complete a certain commutative diagram involving cohomology with real coefficients.

In the same way as in topological K-theory, there are multiple models that represent the differential extensions. For the three descriptions given above of K-theory, there are corresponding models for differential K-theory:

- The Freed-Lott model, as in [20] and [41].
- The Hopkins-Singer model [28].
- The geometric families of Bunke-Schick [12].

In the case of twisted differential K-theory, there are also different models, among them the model of Freed-Lott given by Park [38] in which for a twist representing a torsion class, a suitable connection is added to a twisted vector bundle. A more general case with general twist is given by Gorokhovsky and Lott [23]. A model based on sections to $Gr^p(\mathcal{P}_\sigma)$, a bundle of Grassmannians asociated to a principal $PU(\mathcal{H})$–bundle P_σ, was given by Carey, Mickelsson and Wang [13].

In this thesis we have two principal goals, the first is related to contributing to the challenge of calculating twisted K-theory and the second is focused on constructing an equivalence between the models for twisted differential K-theory given by Park and by Carey, Mickelsson and Wang.

Now let us get into the first objective. K-theory on a topological space X can be twisted by an integral cohomology class σ of degree 3. For different choices of spaces X and twists σ, the calculation of ${}^{\sigma}K(X)$ requires the search for multiple strategies, including spectral sequences [5], [40]. Since it was introduced, different calculations have been made, among them for Lie groups [9], stacks [43] and others. Some recently studied spaces and twists are product spaces $X = S^1 \times M$ with σ a decomposable twist. In this case σ is decomposable if σ is the cup product of the basic integral one-form on S^1 and an integral class in $H^2(M, \mathbb{Z})$.

Our work is motivated by the formula given by Harju and Mickelsson in [24] for $X = S^1 \times M$ where M is a compact manifold and σ a decomposable class associated to a line bundle over M. This case was recently studied by V. Mathai, R. Melrose, and I.M. Singer [33].

The first main goal here is obtain a formula to calculate up to extensions the twisted K-group ${}^{\sigma}K(X)$, when X is the total space in a fiber bundle $M \hookrightarrow X \xrightarrow{\pi} S^1$ with M a compact manifold. The twist σ corresponds to a special class in $H^3(X)$, which is obtained from elements in $H^2(M)$ and $H^1(S^1)$, namely, through the construction of a projective bundle on X using the property of decomposing a principal bundle on S^1 as a mapping torus. The following theorem provides a calculation up to extensions for this case.

Theorem. *Let $M \hookrightarrow X \xrightarrow{\pi} S^1$ be a fiber bundle where M is a compact manifold and X is obtained as the mapping torus of a homeomorphism $\varsigma\colon M \to M$. Given a class $[\lambda] \in H^2(M)$ represented by a complex line bundle λ, let $\sigma \in H^3(X)$ be the image of the class of $[\lambda]$ under the inclusion $H^2(M)_{\mathbb{Z}} \cong H^1(S^1; H^2(M)) \to H^3(X)$. Then the twisted K-theory group ${}^{\sigma}K^*(X)$, for $* = 0, 1$, is isomorphic to an extension of*

$$\{x \in K^*(M) \mid x = \lambda \cdot \varsigma^* x\} \ \textit{by} \ \frac{K^{*+1}(M)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^{*+1}(M)\}}. \tag{1.1}$$

The second main objective here resulted from regarding S^1 as the classifying space of $\mathbb{Z}$ and looking for a formula that generalizes the theorem above, but in the case where the base of the bundle X was the classifying space of a finitely generated free group G. The following theorem provides a calculation up to extensions for this case.

Theorem. *Let G be a finitely generated free group with generators $\{g_i\}_{i\in J}$ which acts on a compact manifold M and let $M \hookrightarrow EG \times_G M \xrightarrow{\pi} BG$ be the fiber bundle associated to the Borel construction. Given a derivation of line bundles $\kappa\colon G \to MAP(M, PU(\mathcal{H}))$, if σ is the associated $PU(\mathcal{H})$–principal bundle, then the twisted K-theory group ${}^{\sigma}K^*(X)$, for $* = 0, 1$, is isomorphic to an extension group of*

$$\{x \in K^*(M) \mid x = \kappa(g_i) \cdot g_i^* x \ \textit{for all} \ i \in J\} \ \textit{by} \ \bigoplus_{i\in J} K_i^{*+1}(M)/N. \tag{1.2}$$

where N is the subgroup of tuples indexed by J with ith coordinate equal to $\kappa(g_i) \cdot g_i^* x - x \in K^{*+1}(M)$ for a certain $x \in K^{*+1}(M)$.

Unlike the previous case, for the construction of the projective bundle we must consider that G may have more than one generator. The total space that was previously obtained as a mapping torus will now be a twisted Borel construction. The fibers will be glued using a derivation of line bundles, which is a map $\kappa\colon G \to MAP(M, PU(\mathcal{H}))$ satisfying the following compatibility conditions.

$$\begin{aligned} \kappa_{gh} &= \kappa_h \circ L_{g^{-1}} \cdot \kappa_g \\ \kappa_e &= id \end{aligned}$$

where e is the identity of G and id is the constant function equal to the identity of $PU(\mathcal{H})$. In the case of $G = \mathbb{Z}$, such a map assigned the generator to a representative of a highlighted class in $H^2(M) = [M, PU(\mathcal{H})]$.

The approach that was taken after these results was to search for a spectral sequence that encompasses these two cases. Since our space X in these two cases is a Borel construction, this corresponds to computing twisted Borel equivariant K-theory. Actually there is a spectral sequence to compute the twisted K-theory of total spaces of fibrations given in Theorem 20.4.1 of [34], but this is not useful for our intention because the twist considered in this theorem comes from the base space. In our case, our twists do not come from the base space, but from a mixture of homotopical information of the base and the fiber.

For this purpose, we consider a compact manifold M with the action of a discrete group G. We consider the spectral sequence induced by the filtration $H^{(k)} = p^{-1}(BG^{(k)})$ to determine the twisted K-theory of $H = EG \times_G F$, where $BG^{(k)}$ denotes the k-skeleton of a CW-structure on the classifying space BG of G. The first page of the spectral sequence has the form

$$E_1^{k,mk} =^P K^m(H^{(k)}, H^{(k-1)}) \cong \prod_{\alpha \in J_k} {}^{P'_\alpha}K^{m-k}(p^{-1}(o_\alpha))$$

where J_k is the set of k-dimensional cells in a CW-structure of BG and P'_α are certain twists. The homogenization of the twists P'_α is achieved thanks to the fact that we have an explicit description of the twist through the projective bundle.

Theorem. *Let G be a discrete group and let M be a compact manifold with a G–action. Given a derivation of line bundles $\kappa\colon G \to MAP(M, PU(\mathcal{H}))$, there is a spectral sequence*

$$E_2^{p,q} \cong H^p(BG; K^q(M)) \Longrightarrow {}^P K^{p+q}(EG \times_G M)$$

where P is the principal $PU(\mathcal{H})$–bundle associated to κ and the cohomology of BG has local coefficients for the action

$$g \cdot z = \kappa_g \cdot L^*_{g^{-1}}(z).$$

on $K^q(M)$.

Along the way, other formulas for ${}^\sigma K(X)$ were found to calculate special cases such as when X is a compact, path connected and orientable 3-dimensional manifold with a twist $n\omega \in H^3(M;\mathbb{Z})$, where ω is the generator class and $n \in \mathbb{Z}$ or when X is a path connected and non-orientable 3-dimensional manifold with a twist $\omega \in H^3(M;\mathbb{Z})$ given by the non-trivial class. Finally, the previous results were validated with the different particular calculations:

- $X = S^1 \times S^2$
- $X = S^1 \times \mathbb{T}^2$
- $X = \mathbb{R} \times_\rho S^2$, with $\rho\colon \pi_1(S^1) \times S^2 \to S^2$ is the action given by $\rho(z, x) = x$ if z is even and $\rho(z, x) = -x$ if z is odd.
- $X = \mathbb{R} \times_\rho \mathbb{T}^2$, with $\rho\colon \pi_1(S^1) \times \mathbb{T}^2 \to \mathbb{T}^2$ the action over $\mathbb{T}^2 = S^1 \times S^1$ induced by an element of $GL_2(\mathbb{Z})$.

Now let us get into the second objective. Differential K-theory on a manifold can be twisted by an integral cohomological differential torsion class of degree 3. For different models this twist can be represent by a gerbe with connection and curvature [13]. This goal was motivated by the models given by Carey-Mickelsson-Wang [13] and Park [38]. We wish to obtain an explicit equivalence between these different models of twisted differential K-theory.

The steps that will be followed to achieve this equivalence will be first to give an explicit description of each model, pointing out the differential information added and describing how the twisting is characterized in each case. Eventually topological equivalence will be addressed initially and the idea to extend differential equivalence will be established.

Chapter 2

Preliminaries

In this chapter we introduce some basic tools and terminology that will be used in the rest of the thesis. Most of the results in this chapter are well-known and the interested reader can check the references [2], [4], [16], [26] and [41] for further information.

2.1 K-theory

The most intuitive definition of the degree 0 K-theory group is given through a relation on the set of vector bundles, although this definition is only appropriate for compact Hausdorff spaces. All vector bundles will be complex vector bundles unless mentioned otherwise.

Definition 2.1.1. Two vector bundles E_1 and E_2 over X are stably isomorphic if there is a trivial n-dimensional vector bundle ϵ^n over X such that

$$E_1 \oplus \epsilon^n \cong E_2 \oplus \epsilon^n$$

We denote this by $E_1 \sim E_2$.

Definition 2.1.2. Let X be a compact Hausdorff topological space. The complex K-theory group $K(X)$ is defined as the Grothendieck group completion of the monoid $\mathrm{Vect}_{\mathbb{C}}(X)$ of isomorphism classes of vector bundles over X.

In other words, $K(X)$ consists of formal differences $E - E'$ of isomorphism classes of vector bundles over X, with the equivalence relation

$$E_1 - E_1' = E_2 - E_2' \text{ iff } E_1 \oplus E_2' \sim E_2 \oplus E_1'. \tag{2.1}$$

In fact, $K(X)$ has a ring structure with the product

$$(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 - E_1 \otimes E_2' - E_1' \otimes E_2 + E_1' \otimes E_2' \tag{2.2}$$

Some examples of such rings are

$$K(pt) \cong \mathbb{Z}, \qquad K(S^2) \cong \frac{\mathbb{Z}[t]}{\langle (t-1)^2 \rangle}$$

Definition 2.1.3. (Reduced K-theory) Consider the inclusion of a basepoint $x_0 \hookrightarrow X$, it induces a map of rings $\phi\colon K(X) \to K(x_0) \cong \mathbb{Z}$. Then the reduced K-theory group of X is defined as

$$\widetilde{K}(X) = \mathrm{Ker}\{\phi\colon K(X) \to \mathbb{Z}\}$$

When X is path connected, it consists of classes $E_1 - E_2 \in K(X)$ such that $\dim(E_1) = \dim(E_2)$.

Some properties of K-theory are:

- There are natural splittings of rings $K(X) \cong \widetilde{K}(X) \oplus \mathbb{Z}$.
- A continuous map $f\colon X \to Y$ induces a ring homomorphism $f^*\colon K(Y) \to K(X)$.
- Let $f\colon X \to Y$ and $g\colon X \to Y$ be homotopic maps. Then the pullback homomorphisms f^* and g^* are equal.

The extension of the K-theory groups to all degrees is done via the following definition.

Definition 2.1.4. Let X be a compact Hausdorff space and Y a closed subspace. For $n \geq 0$, we define

- $K(X,Y) = \widetilde{K}(X/Y)$, in particular $K(X,\emptyset) \cong K(X)$
- $\widetilde{K}^{-n}(X) = \widetilde{K}(S^n X)$
- $K^{-n}(X,Y) = \widetilde{K}^{-n}(X/Y) = \widetilde{K}(S^n(X/Y))$
- $K^{-n}(X) = K^{-n}(X,\emptyset) = \widetilde{K}(S^n(X^+))$

where S denotes reduced suspension and X^+ is the disjoint union of X with the one-point space.

Since K-theory defines a cohomology theory, it is accompanied by its sequence of the pair.

Proposition 2.1.5. *If Y is a closed subspace of a compact Hausdorff space X, there is a natural exact sequence*

$$\cdots K^{-2}(Y) \xrightarrow{\partial} K^{-1}(X,Y) \xrightarrow{j^*} K^{-1}(X) \xrightarrow{i^*} K^{-1}(Y)$$
$$\xrightarrow{\partial} K^0(X,Y) \xrightarrow{j^*} K^0(X) \xrightarrow{i^*} K^0(Y).$$

A property that greatly facilitates the use of K-theory is that it is a periodic cohomology theory of periodicity 2, this is given by the following result.

Theorem 2.1.6. *(Bott's periodicity theorem) For any compact Hausdorff space X and any $n \leq 0$ there is a natural isomorphism*

$$\beta\colon K^{-n}(X) \to K^{-n-2}(X)$$

If we define $K^n(X,Y)$ for $n > 0$ inductively by $K^n = K^{n-2}$, the sequence of the pair becomes

$$\begin{tikzcd} K^0(X,Y) \arrow[r] & K^0(X) \arrow[r] & K^0(Y) \arrow[d] \\ K^1(Y) \arrow[u] & K^1(X) \arrow[l] & K^1(X,Y) \arrow[l] \end{tikzcd}$$

Another important property when doing calculations is:

Definition 2.1.7. (Mayer-Vietoris exact sequence) Let U_1 and U_2 be two open subspaces of a compact Hausdorff space X such that $U_1 \cup U_2 = X$. Then we have an exact sequence

$$\begin{tikzcd} K^0(X) \arrow[r] & K^0(U_1) \oplus K^0(U_2) \arrow[r] & K^0(U_1 \cap U_2) \arrow[d] \\ K^1(U_1 \cap U_2) \arrow[u] & K^1(U_1) \oplus K^1(U_2) \arrow[l] & K^1(X) \arrow[l] \end{tikzcd}$$

Now we are going to give a definition of K-theory for more general spaces, for this we will use the space of Fredholm operators.

Definition 2.1.8. (Fredholm Operator) Let $\mathcal{H}$ be an infinite-dimensional separable complex Hilbert space. A bounded operator $T\colon \mathcal{H} \to \mathcal{H}$ is a Fredholm operator if $\mathrm{Ker}(T)$ and $\mathrm{CoKer}(T)$ are finite-dimensional. We denote by $\mathrm{Fred}(\mathcal{H})$ the space of Fredholm operators in $\mathcal{H}$.

The justification for this definition can be seen in the following theorem.

Theorem 2.1.9. *(Index theorem) For any compact Hausdorff space X, we have a natural isomorphism*

$$\textit{index}\colon [X, \textit{Fred}(\mathcal{H})] \to K(X)$$

Finally the general definition can also be seen as a representation of K-theory. For $n \geq 0$ we define

$$K^{-n}(X) = [X, \Omega^n \mathrm{Fred}(\mathcal{H})] \tag{2.3}$$

Since $\Omega^2\mathrm{Fred}(\mathcal{H}) \simeq \mathrm{Fred}(\mathcal{H})$, we can extend this definition to positive degrees. Similarly, it can be extended to pairs of spaces in the same way as in Definition 2.1.4.

2.2 Brief review on Deligne cohomology

Given a smooth manifold X, we consider the complex of sheaves:

$$S^p_X := \underline{\mathrm{U}}(1) \xrightarrow{\tilde{d}} \Omega^1_{\mathbb{R}} \xrightarrow{d} \cdots \xrightarrow{d} \Omega^p_{\mathbb{R}}, \tag{2.4}$$

where $\underline{\mathrm{U}}(1)$ is the sheaf of smooth U(1)–valued functions, $\Omega^k_{\mathbb{R}}$ is the sheaf of real k-forms, d is the exterior differential and $\tilde{d}f := \frac{1}{2\pi i} f^{-1} df$. The Deligne cohomology group of degree p on X is by definition the sheaf hypercohomology group of the complex (2.4), i.e., $\check{H}^p(X; S^p_X)$. It can be concretely described through a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of X as follows: we consider the double complex whose rows are the Čech complexes of the sheaves involved in (2.4), and we consider the cohomology of the associated total complex. This means that a p-cocycle consists of a sequence $(\{g_{i_0\cdots i_p}\}, \{(C_1)_{i_0\cdots i_{p-1}}\}, \ldots, \{(C_{p-1})_{i_0 i_1}\}, \{(C_p)_{i_0}\})$, where C_k is a k-form, satisfying the conditions:

$$\begin{aligned}
&(C_p)_{i_1} - (C_p)_{i_0} = d(C_{p-1})_{i_0 i_1} \\
&(C_{p-1})_{i_1 i_2} - (C_{p-1})_{i_0 i_2} + (C_{p-1})_{i_0 i_1} = -d(C_{p-2})_{i_0 i_1 i_2} \\
&\vdots \\
&\check{\delta}^{p-1}(C_1)_{i_0\ldots i_{p-1}} = \tfrac{(-1)^{p+1}}{2\pi i} g^{-1}_{i_0\ldots i_p} dg_{i_0\ldots i_p} \\
&\check{\delta}^p g_{i_0\ldots i_p} = 1.
\end{aligned} \tag{2.5}$$

We call $\mathcal{G} := [g, C_1, \ldots, C_p]$ the corresponding cohomology class. The local forms C_p correspond to the Ramond-Ramond potentials in string theory (if we consider this model) and their differentials dC_p glue to a global gauge-invariant closed form G_{p+1} (the Ramond-Ramond field strength), which is called *curvature*. Moreover, from the underlying class $[\{g_{i_0\cdots i_p}\}] \in \check{H}^p(\mathfrak{U}; \underline{\mathrm{U}}(1))$, applying the isomorphism $\check{H}^p(\mathfrak{U}; \underline{\mathrm{U}}(1)) \cong H^{p+1}(X; \mathbb{Z})$, we get the *first Chern class* $c_1(\mathcal{G}) \in H^{p+1}(X; \mathbb{Z})$. The de-Rham cohomology class, represented by the curvature, is the real image of the first Chern class (Dirac quantization condition), therefore the curvature has integral periods. Of course we are free to add any coboundary to the cocycle $(g, C_1, \ldots, C_p)$, the meaningful datum being the corresponding cohomology class, since it is determined by the two real physical observables: the field strength G_{p+1} (corresponding to the field F in electromagnetism) and the holonomy or Wess-Zumino action (providing the additional piece of information that, in electromagnetism, is measured by the phase difference in the Aranhov-Bohm effect). The latter can be computed on a singular p-cycle of X, integrating the local k-forms C_k on the k-simplices and summing the results in a suitable way [22]. It is related to the curvature by a Stokes-type formula, hence, if the class is flat (i.e. the curvature vanishes), then its holonomy on a cycle only depends on the underlying homology class.

Differential cohomology diagram. We set $\hat{H}^p(X) := \check{H}^{p-1}(S_X^{p-1})$ and we get the following commutative diagram [28]:

$$\begin{tikzcd} \hat{H}^\bullet(X) \arrow[r, two heads, "c_1"] \arrow[d, "curv"'] & H^\bullet(X;\mathbb{Z}) \arrow[d, "\otimes_{\mathbb{Z}}\mathbb{R}"] \\ \Omega^\bullet_{int}(X) \arrow[r, "dR"] & H^\bullet_{dR}(X). \end{tikzcd} \tag{2.6}$$

Here c_1 is the first Chern class, $curv$ is the curvature, dR is the de-Rham cohomology class and $\Omega^\bullet_{int}(X)$ is the group of closed real forms with integral periods. The surjective map c_1 in Diagram (2.6) shows that $\hat{H}^\bullet(X)$ is a differential refinement of $H^\bullet(X;\mathbb{Z})$.

Relative Deligne cohomology. Given a smooth closed embedding $\rho\colon Y \hookrightarrow X$, we consider the complexes of sheaves S_X^p and $\rho_* S_Y^{p-1}$ on X. We recall that, for any open subset $U \subset X$, by definition $(\rho_* S_Y^{p-1})(U) := S_Y^{p-1}(\rho^{-1}U)$. We have the natural morphism

$$\rho^!\colon S_X^p \to \rho_* S_Y^{p-1}, \tag{2.7}$$

defined as follows:

$$\begin{tikzcd} \underline{\mathrm{U}}(1)_X \arrow[r, "\tilde{d}"] \arrow[d, "\rho^{!,0}"] & \Omega^1_{X,\mathbb{R}} \arrow[r, "d"] \arrow[d, "\rho^{!,1}"] & \cdots \arrow[r, "d"] & \Omega^{p-1}_{X,\mathbb{R}} \arrow[r, "d"] \arrow[d, "\rho^{!,p-1}"] & \Omega^p_{X,\mathbb{R}} \arrow[d] \\ \rho_*\underline{\mathrm{U}}(1)_Y \arrow[r, "\tilde{d}"] & \rho_*\Omega^1_{Y,\mathbb{R}} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \rho_*\Omega^{p-1}_{Y,\mathbb{R}} \arrow[r] & 0, \end{tikzcd} \tag{2.8}$$

where $\rho^{!,q}(\omega) = \rho^*\omega$ for any $q \geq 1$ and $\rho^{!,0}(f) = f \circ \rho$. The corresponding cone complex is the following one:

$$\check{\boldsymbol{C}}^\bullet(\rho) := \check{C}^\bullet(S_X^p) \oplus \check{C}^{\bullet-1}(\rho_* S_Y^{p-1}) \qquad \check{\boldsymbol{D}}^\bullet(\alpha,\beta) := \big(\check{D}^\bullet(\alpha), \rho^!(\alpha) - \check{D}^{\bullet-1}(\beta)\big). \tag{2.9}$$

The cohomology groups of $(\check{\boldsymbol{C}}^\bullet(\rho), \check{\boldsymbol{D}}^\bullet)$ are by definition the relative Deligne cohomology groups of ρ.

Remark 2.2.1. *The case $p = 1$ is quite clear geometrically. In fact, a relative Deligne class is represented by a cocycle of the form (α, β), where $\alpha = (\{g_{ij}\}, \{A_i\}) \in \check{Z}^1(S_X^1)$ represents a line bundle with connection (L, ∇) and $\beta = \{h_i\} \in \check{C}^0(\underline{\mathrm{U}}(1)_Y)$ represents a trivialization (i.e. a global non-vanishing section) of $\rho^* L$. Let us see why such a construction is natural. Fixing a piecewise smooth curve $\gamma\colon I \to X$, such that $\gamma(\partial I) \subset Y$, the parallel transport of ∇ along γ is not well-defined as a complex number, since γ is not a closed curve. Nevertheless, since $\gamma(\partial I) \subset Y$, the fixed trivialization of $L|_Y$ provides a canonical way to identify the fibres $L_{\gamma(0)}$ and $L_{\gamma(1)}$ with $\mathbb{C}$, hence the parallel transport, as a unitary linear map from $L_{\gamma(0)}$ to $L_{\gamma(1)}$, becomes a well-defined number belonging to* $\mathrm{U}(1)$. *Therefore, the 1-Deligne cohomology group of ρ naturally leads to the notion of* relative *holonomy.*

In general, if (α, β) represents a relative cocycle of degree p, then α represents a Deligne p-cohomology class on X such that $\rho^*\alpha$ has trivial first Chern class. This means that $\rho^*\alpha$ is topologically trivial, but not necessarily trivial as a Deligne class. It follows that $\rho^*\alpha$ is cohomologous to a cochain of the form $(1, 0, \ldots, 0, G)$, where G is a global potential. The cochain β provides a suitable reparametrization, i.e. $\rho^*\alpha - \check{D}\beta = (1, 0, \ldots, 0, G)$. By definition, the *curvature* of $[(\alpha, \beta)]$ is the relative form $(F, G) \in \Omega^p(\rho)$, where F is the curvature of α in X and G is the global potential on Y. Now we can understand why, in Diagram (2.8), the complex on Y has been truncated at degree $p-1$, not p. This is because, if we reach p even in the lower row, then the cocycle condition also imposes $G = 0$, hence $\rho^*\alpha$ must be trivial (not only topologically). Therefore, the morphism $\rho^! \colon S^p_X \to \rho_* S^p_Y$ (with p on both sides) leads to a proper subgroup of relative p-Deligne cohomology, whose elements are called *parallel classes*:

$$\begin{tikzcd}
\underline{U}(1)_X \arrow[r, "\tilde{d}"] \arrow[d, "\rho^{!,0}"] & \Omega^1_{X,\mathbb{R}} \arrow[r, "d"] \arrow[d, "\rho^{!,1}"] & \cdots \arrow[r, "d"] & \Omega^{p-1}_{X,\mathbb{R}} \arrow[r, "d"] \arrow[d, "\rho^{!,p-1}"] & \Omega^p_{X,\mathbb{R}} \arrow[d, "\rho^{!,p}"] \\
\rho_*\underline{U}(1)_Y \arrow[r, "\tilde{d}"] & \rho_*\Omega^1_{Y,\mathbb{R}} \arrow[r, "d"] & \cdots \arrow[r, "d"] & \rho_*\Omega^{p-1}_{Y,\mathbb{R}} \arrow[r, "d"] & \rho_*\Omega^p_{Y,\mathbb{R}}.
\end{tikzcd} \tag{2.10}$$

We set:

$$\hat{H}^{p+1}(\rho) := \check{H}^p(\rho^! \colon S^p_X \to \rho_* S^{p-1}_Y) \qquad \hat{H}^{p+1}_{\text{par}}(\rho) := \check{H}^p(\rho^! \colon S^p_X \to \rho_* S^p_Y). \tag{2.11}$$

We get the following diagram, generalizing (2.6) to the relative framework:

$$\begin{tikzcd}
\hat{H}^\bullet(\rho) \arrow[r, two heads, "c_1"] \arrow[d, two heads, "curv"'] & H^\bullet(\rho; \mathbb{Z}) \arrow[d, "\otimes_{\mathbb{Z}}\mathbb{R}"] \\
\Omega^\bullet_{int}(\rho) \arrow[r, "dR"] & H^\bullet_{dR}(\rho).
\end{tikzcd} \tag{2.12}$$

Moreover, $\hat{H}^{p+1}_{\text{par}}(\rho)$ is the subgroup of $\hat{H}^{p+1}(\rho)$ formed by classes with curvature of the form $(F, 0)$. It is possible to define the holonomy of a class $(\alpha, \beta) \in \hat{H}^{p+1}(\rho)$ on relative p-cycles, such cycles being defined through the homological version of the cone complex.

Remark 2.2.2. *In the case $p = 1$, using the notations of Remark 2.2.1, a class is parallel when the fixed trivialization of $L|_Y$ is a global* parallel *section with respect to ∇. It follows that $\nabla|_Y = 0$.*

2.3 Twisted K-theory

In this section we give two definitions of twisted K-theory, in the first part we give a definition based on twisted vector bundles for special twists and then we give a definition for generic twists using sections of a suitable bundle.

2.3.1 Torsion twisting class

We begin discussing twists which are classified by a torsion class.

Twisted vector bundles

We fix a Hausdorff paracompact topological space X and a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$, whose existence we assume by hypothesis. Every smooth manifold admits a good cover [8]. We denote by $\underline{\mathrm{U}}(r)$ the sheaf of $\mathrm{U}(r)$–valued continuous functions on X and, when $r = 1$, we denote by $\check{C}^\bullet(\mathfrak{U}; \underline{\mathrm{U}}(1))$, $\check{Z}^\bullet(\mathfrak{U}; \underline{\mathrm{U}}(1))$ and $\check{H}^\bullet(\mathfrak{U}; \underline{\mathrm{U}}(1))$ the corresponding Čech cochains, cocycles and cohomology classes, with respect to the fixed good cover $\mathfrak{U}$.

Definition 2.3.1. Given a cochain $\zeta := \{\zeta_{ijk}\} \in \check{C}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$, a ζ*-twisted vector bundle* of rank r on X is a collection of trivial Hermitian vector bundles $\pi_i \colon E_i \to U_i$ of rank r and unitary vector bundle isomorphisms $\varphi_{ij} \colon E_i|_{U_{ij}} \to E_j|_{U_{ij}}$, such that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \zeta_{ijk} \cdot \mathrm{id}$.

Of course, when $\zeta_{ijk} = 1$, we get an ordinary vector bundle by identifying v with $\varphi_{ij}(v)$ for every $v \in E_i|_{U_{ij}}$. It is easy to prove by direct computation that, if there exists a ζ-twisted vector bundle, then ζ is necessarily a cocycle, hence the cohomology class $[\zeta] \in \check{H}^2(\mathfrak{U}; \underline{\mathrm{U}}(1)) \cong H^3(X; \mathbb{Z})$ is well-defined. We will show in Remark 2.3.5 that it is necessarily a torsion class.

Definition 2.3.2. Given two ζ-twisted vector bundles $E := (\{E_i\}, \{\varphi_{ij}\})$ and $F := (\{F_i\}, \{\psi_{ij}\})$, a *morphism* from E to F is a collection of vector bundle morphisms $f_i \colon E_i \to F_i$ such that $f_j \circ \varphi_{ij} = \psi_{ij} \circ f_i$ for every $i, j \in I$. The morphism is called *unitary* if each f_i is unitary.

Of course an *isomorphism* is an invertible morphism, and this is equivalent to requiring that each f_i is a vector bundle isomorphism. The following definition easily generalizes to the non-abelian setting the basic tools of Čech cohomology in low degree.

Definition 2.3.3. A Čech cochain of degree 1 of the sheaf $\underline{\mathrm{U}}(r)$ is a collection of continuous functions $\{g_{ij} \colon U_{ij} \to \mathrm{U}(r)\}$. Similarly, a Čech cochain of degree 0 is a collection of continuous functions $\{g_i \colon U_i \to \mathrm{U}(r)\}$. We denote the set of p-cochains, for $p \in \{0, 1\}$, by $\check{C}^p(\mathfrak{U}; \underline{\mathrm{U}}(r))$. Given $\zeta := \{\zeta_{ijk}\} \in \check{C}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$, a cochain $\{g_{ij}\} \in \check{C}^1(\mathfrak{U}; \underline{\mathrm{U}}(r))$ is called a ζ*-cocycle* if $g_{ki}g_{jk}g_{ij} = \zeta_{ijk} \cdot I_r$. We denote by $\check{Z}^1_\zeta(\mathfrak{U}; \underline{\mathrm{U}}(r))$ the set of ζ-cocycles.

There is a natural action of 0-cochains on 1-cochains, defined by $\{h_i\} \cdot \{g_{ij}\} := \{h_i g_{ij} h_j^{-1}\}$. It is easy to prove that such an action determines an equivalence relation in $\check{C}^1(\mathfrak{U}; \underline{\mathrm{U}}(r))$, that restricts to an equivalence relation in $\check{Z}^1_\zeta(\mathfrak{U}; \underline{\mathrm{U}}(r))$.

Definition 2.3.4. The ζ*-twisted cohomology set* of degree 1 and rank r, that we denote by $\check{H}^1_\zeta(\mathfrak{U}; \underline{\mathrm{U}}(r))$, is the quotient of $\check{Z}^1_\zeta(\mathfrak{U}; \underline{\mathrm{U}}(r))$ by the action of the 0-cochains.

Clearly, when $\zeta = 1$, we get ordinary non-abelian cohomology of degree 1, which classifies the isomorphism classes of rank-r vector bundles on X. We can easily show that the same happens for any ζ. In fact, given a twisted vector bundle $E := (\{E_i\}, \{\varphi_{ij}\})$ of rank r, for each $i \in I$ we can fix a set of r pointwise-independent local sections $s_{1,i}, \dots, s_{r,i} \colon U_i \to E_i$ of unit norm, determining vector bundle isomorphisms $\xi_i \colon E_i \to U_i \times \mathbb{C}^r$ that send $\sum \lambda^k s_{i,k}(x)$ to $(x, (\lambda^1, \dots, \lambda^r))$. The isomorphisms φ_{ij} determine local transition functions $g_{ij} \colon U_{ij} \to \mathrm{U}(r)$ such that $\varphi_{ij}(\xi_i^{-1}(x,\lambda)) = \xi_j^{-1}(x, g_{ij}(x) \cdot \lambda)$. Equivalently, $g_{ij}(x)$ is the change of basis in $(E_j)_x$ from $\{s_{j,1}(x), \dots, s_{j,r}(x)\}$ to $\{\varphi_{ij}(s_{i,1}(x)), \dots, \varphi_{ij}(s_{i,r}(x))\}$. The condition $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \zeta_{ijk} \cdot \mathrm{id}$ is equivalent to $g_{ki}g_{jk}g_{ij} = \zeta_{ijk} \cdot I_n$, hence $\{g_{ij}\} \in \check{Z}^1_\zeta(\mathfrak{U}; \underline{\mathrm{U}}(r))$. Finally, it is straightforward to verify, as for ordinary vector bundles, that the cohomology class $[\{g_{ij}\}] \in \check{H}^1_\zeta(\mathfrak{U}; \underline{\mathrm{U}}(r))$ only depends on the isomorphism class of $E := (\{E_i\}, \{\varphi_{ij}\})$, in such a way that we get the natural bijection $[E] \mapsto [\{g_{ij}\}]$.

Remark 2.3.5. *The class* $[\zeta]$ *is necessarily torsion. In fact, computing the determinants, we get* $\det(g_{ki})\det(g_{jk})\det(g_{ij}) = \zeta^r_{ijk}$*; since* $\det(g_{ij})$ *is a* $\mathrm{U}(1)$*–valued function, this shows that* $\{\zeta^r_{ijk}\}$ *is a trivial cocycle, hence* $[\zeta]^r = 1$ *(or* $r[\zeta] = 0$*, thinking of* $H^3(X;\mathbb{Z})$*). In particular, the order of* $[\zeta]$ *divides* r*. One can prove that, for any cocycle representing a torsion class, there exist a corresponding twisted bundle [4].*

2.3.2 Generic twisting class

Before giving a definition of twisted K-theory we will recall some facts about the space of unitary projective operators $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$, where $U(\mathcal{H})$ is the space of unitary operators on an infinite-dimensional separable complex Hilbert space $\mathcal{H}$. Let $U(\mathcal{H})$ act on $\mathrm{Fred}(\mathcal{H})$ by

$$\begin{aligned} U(\mathcal{H}) \times \mathrm{Fred}(\mathcal{H}) &\to \mathrm{Fred}(\mathcal{H}) \\ (g, f) &\mapsto g^{-1} \circ f \circ g \end{aligned}$$

Since g is an isomorphism this is well defined. Now if $\alpha \in U(1)$, then

$$(\alpha g, f) \mapsto (\alpha g)^{-1} \cdot f \cdot \alpha g = g^{-1} \cdot f \cdot g = (g, f)$$

so indeed $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$ acts on $\mathrm{Fred}(\mathcal{H})$.

K-theory on a topological space X is twisted by an integral cohomology class H of degree 3, this means that the twists in twisted K-theory are classified by elements of $H^3(X;\mathbb{Z})$. Other convenient interpretations for this group of isomorphism classes of twists are:

$$H^3(X;\mathbb{Z}) \cong [X, K(\mathbb{Z},3)] \cong [X, B^2K(Z,1)] \cong [X, BPU(\mathcal{H})]$$

Remark 2.3.6. There is a free action of $U(1)$ on $U(\mathcal{H})$ and by Kuiper's Theorem [32] we have that $U(\mathcal{H})$ is contractible, so $U(\mathcal{H})/U(1)$ is a model for $BU(1)$.

Now we can construct two new bundles with the associated bundle construction. Let us denote by E_H the principal $PU(\mathcal{H})$–bundle classified by the map H.

$$\begin{tikzcd}
E_H \times_{PU(\mathcal{H})} \Omega^n \mathrm{Fred}(\mathcal{H}) \arrow[d, dashed] & EPU(\mathcal{H}) \times_{PU(\mathcal{H})} \Omega^n \mathrm{Fred}(\mathcal{H}) \arrow[d] \\
X \arrow[r, "H"'] & BPU(\mathcal{H})
\end{tikzcd}$$

Let $F^0_H = E_H \times_{PU(\mathcal{H})} \mathrm{Fred}(\mathcal{H}) \to X$ and $F^1_H = E_H \times_{PU(\mathcal{H})} \Omega\mathrm{Fred}(\mathcal{H}) \to X$. In the next definition, $\overline{\Gamma}(P)$ denotes homotopy classes of sections of P.

Definition 2.3.7. (Twisted K-theory) The twisted K-theory of X with twist $H \in H^3(X;\mathbb{Z})$ is given by

$$K^0_H(X) = \overline{\Gamma}(F^0_H)$$
$$K^1_H(X) = \overline{\Gamma}(F^1_H)$$

The higher twisted K-theory groups are defined by Bott periodicity, i.e. $K^n_H(X) = \overline{\Gamma}(E_H \times_{PU(\mathcal{H})} \Omega^n \mathrm{Fred}(\mathcal{H})) \cong \overline{\Gamma}(E_H \times_{PU(\mathcal{H})} \Omega^k \mathrm{Fred}(\mathcal{H}))$, where $k = 0$ if n is even and 1 otherwise.

Remark 2.3.8. We will also use the notation ${}^H K^*(X)$ in Chapters 4 and 5, or ${}^P K^*(X)$, when P is a $PU(\mathcal{H})$–principal bundle classified by H.

Some noteworthy properties satisfied by twisted K-theory are:

- If $H = 0$ then $K^*_H(X) \cong K^*(X)$.
- $K^*_H(X)$ is a module over $K^*(X)$.
- There is a product homomorphism $K^p_H(X) \otimes K^q_{H'}(X) \to K^{p+q}_{H+H'}(X)$.
- If $f\colon X \to Y$ is a continuous map, then there is a homomorphism $f^*\colon K^*_H(Y) \to K^*_{f^*H}(X)$

Proposition 2.3.9. *(Mayer-Vietoris exact sequence) Let U_1 and U_2 be two open subspaces of a space X such that $U_1 \cup U_2 = X$. Then we have an exact sequence:*

$$\begin{tikzcd}
K^0_H(X) \arrow[r] & K^0_H(U_1) \oplus K^0_H(U_2) \arrow[r] & K^0_H(U_1 \cap U_2) \arrow[d] \\
K^1_H(U_1 \cap U_2) \arrow[u] & K^1_H(U_1) \oplus K^1_H(U_2) \arrow[l] & K^1_H(X) \arrow[l]
\end{tikzcd}$$

In the case of a twist which represents a torsion class, a geometric representation through twisted vector bundles can be given.

Definition 2.3.10. (ζ-twisted vector bundle) For a Hausdorff paracompact topological space X and a good cover $\mathfrak{U} = \{U_i\}_{i\in I}$, given a cochain $\zeta := \{\zeta_{ijk} \in \check{C}^2(\mathfrak{U}, U(1))\}$, a ζ-twisted vector bundle of rank r on X is a collection of trivial Hermitian vector bundles $\pi_i\colon E_i \to U_i$ of rank r and unitary vector bundle isomorphisms $\phi_{ij}\colon E_i \restriction_{U_{ij}} \to E_j \restriction_{U_{ij}}$, such that $\phi_{ki}\phi_{jk}\phi_{ij} = \zeta_{ijk} \cdot id$.

We denote by $VB_\zeta(X)$ the set of isomorphism classes of ζ-twisted vector bundles.

Definition 2.3.11. The direct sum of ζ-twisted vector bundles is defined as $(\{E_i\}, \{\phi_{ij}\}) \oplus (\{F_i\}, \{\psi_{ij}\}) := (\{E_i \oplus F_i\}, \{\phi_{ij} \oplus \psi_{ij}\})$.

Definition 2.3.12. (ζ-twisted K-theory group) The set $VB_\zeta(X)$, endowed with this operation, is a commutative semi-group, hence we can define the corresponding Grothendieck group, that we call the ζ-twisted K-theory group of X and denote by $K_\zeta(X)$.

We fix an infinite-dimensional separable complex Hilbert space $\mathcal{H}$. We can easily generalize definition 6.1.7 as follows.

Definition 2.3.13. Given a cocycle $\zeta := \{\zeta_{ijk}\} \in \check{Z}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$, a *$\zeta$-twisted Hilbert bundle* with fibre $\mathcal{H}$ on X is a collection of trivial Hilbert bundles $\pi_i\colon E_i \to U_i$ with fibre $\mathcal{H}$ and of Hilbert bundle isomorphisms $\varphi_{ij}\colon E_i|_{U_{ij}} \to E_j|_{U_{ij}}$, such that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \zeta_{ijk} \cdot \mathrm{id}$.

The corresponding definition of (iso)morphism coincides with Definition 2.3.2. For every $\zeta \in \check{Z}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$, not necessarily of finite order in cohomology, there exists a ζ-twisted Hilbert bundle [4], the main difference with respect to the finite-dimensional setting being that any two ζ-twisted Hilbert bundles (for a fixed ζ) are isomorphic [31].

Projective Hilbert bundles. Given a twisted bundle $E = (\{E_i\}, \{\varphi_{ij}\})$, projecting each fibre $(E_i)_x \setminus \{0\}$ to the corresponding projective space, we get a well-defined (non-twisted) projective bundle, that we denote by $\mathbb{P}(E)$. It follows from local triviality that every projective bundle can be obtained in this way up to isomorphism, therefore we get a surjective map from isomorphism classes of twisted bundles to isomorphism classes of projective bundles. In the finite-dimensional case such a map is not injective for a fixed ζ (for example, every line bundle projects to the trivial one). On the contrary, in the infinite-dimensional case, the unique isomorphism class of ζ-twisted Hilbert bundles induces a unique isomorphism class of projective bundles. Moreover, fixing ζ and $\zeta' := \zeta \cdot \check{\delta}^1\eta$, let us consider the bijection

$$\begin{aligned} \Phi_\eta\colon \widetilde{\mathrm{VB}}_\zeta(X) &\xrightarrow{\cong} \widetilde{\mathrm{VB}}_{\zeta'}(X) \\ E = (\{E_i\}, \{\varphi_{ij}\}) &\mapsto \Phi_\eta(E) := (\{E_i\}, \{\varphi_{ij}\eta_{ij}\}), \end{aligned} \tag{2.13}$$

where $\widetilde{\mathrm{VB}}_\zeta(X)$ denotes the set of ζ-twisted Hilbert bundles on X (not quotiented out up to isomorphism). Since $\mathbb{P}(E) = \mathbb{P}(\Phi_\eta(E))$, the isomorphism class of

$\mathbb{P}(E)$ only depends on $[\zeta] \in \check{H}^2(\mathfrak{U}; \underline{\mathrm{U}}(1)) \cong H^3(X;\mathbb{Z})$ (see [4]). It follows that $H^3(X;\mathbb{Z})$ classifies projective Hilbert bundles on X.

If $\check{\delta}^1\eta = 1$, then, since any two ζ-twisted bundles are isomorphic, there exists an isomorphism $f = \{f_i\}: E \to \Phi_\eta(E)$. This means that $f_i: E_i \to E_i$ and $\varphi_{ij}\eta_{ij}f_i = f_i\varphi_{ij}$, hence f induces an automorphism $\bar{f}: \mathbb{P}(E) \to \mathbb{P}(E)$. Let us see that any automorphism $\bar{f}$ can be realized in this way from suitable η and f. In fact, by local triviality, we can lift $\bar{f}$ to $f_i: E_i \to E_i$ for each i. Since the family $\{f_i\}$ glues to $\bar{f}$, there exists η_{ij} such $f_j\varphi_{ij} = \varphi_{ij}f_i\eta_{ij}$. The latter condition necessarily implies $\check{\delta}^1\eta = 1$. Moreover, the only freedom we had in constructing the cocycle η was the choice of the lifts f_i. Any other choice is of the form $f_i\xi_i$, that replaces η by $\eta \cdot \check{\delta}^0\xi$. Therefore, the following map is well-defined:

$$\begin{aligned} \Phi\colon \mathrm{Aut}(\mathbb{P}(E)) &\to H^2(X;\mathbb{Z}) \\ \bar{f} &\mapsto [\{\eta_{ij}\}]. \end{aligned} \tag{2.14}$$

It is easy to prove that it is a group homomorphism. Moreover, it follows from the previous construction that $\bar{f} \in \mathrm{Aut}(\mathbb{P}(E))$ lifts to an automorphism of E if and only if $\Phi(\bar{f}) = 0$, therefore $\Phi(\bar{f})$ can be thought of as the obstruction to the existence of such a lift. This remark leads quite easily to the following lemma, that also follows from the fact that $\mathrm{PU}(\mathcal{H})$ is an Eilenberg-MacLane space $K(\mathbb{Z}, 2)$ (see [4]).

Definition of twisted K-theory. We fix a cocycle $\zeta \in \check{Z}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$ and a ζ-twisted Hilbert bundle $E = (\{E_i\}, \{\varphi_{ij}\})$, inducing the corresponding projective bundle $\mathbb{P}(E)$. We denote by $P_{\mathbb{P}(E)}$ the bundle of projective reference frames of $\mathbb{P}(E)$ and by $\mathrm{Fred}(\mathcal{H})$ the space of Fredholm operators acting on $\mathcal{H}$. We have a natural adjoint action of $\mathrm{PU}(\mathcal{H})$ on $\mathrm{Fred}(\mathcal{H})$ by conjugation, that we denote by $\rho\colon \mathrm{PU}(\mathcal{H}) \to C^0(\mathrm{Fred}(\mathcal{H}))$, hence we construct the associated $\mathrm{Fred}(\mathcal{H})$–bundle $F_{\mathbb{P}(E)} := P_{\mathbb{P}(E)} \times_\rho \mathrm{Fred}(\mathcal{H})$. We denote by $\Gamma(F_{\mathbb{P}(E)})$ its set of global sections and by $\bar{\Gamma}(F_{\mathbb{P}(E)})$ the corresponding quotient with respect to homotopy of sections. The latter carries a natural abelian group structure, induced by composition of Fredholm operators.

Definition 2.3.14. The *twisted K-theory group $K_\zeta(X)$* is defined as the abelian group $\bar{\Gamma}(F_{\mathbb{P}(E)})$ for any ζ-twisted Hilbert bundle E.

Since the space of bounded invertible operators in $\mathcal{H}$ is contractible (like $\mathrm{U}(\mathcal{H})$), a section of $F_{\mathbb{P}(E)}$ which is point-wise invertible, is always homotopic to the identity. Therefore, if a section is point-wise invertible in a subset of X, we consider it trivial on such a subset. This fact justifies the following definition.

Definition 2.3.15. A section of $F_{\mathbb{P}(E)}$ is called *compactly supported* if it is point-wise invertible in the complement of a compact subset of X. We denote by $\Gamma_{\mathrm{cpt}}(F_{\mathbb{P}(E)})$ and $\bar{\Gamma}_{\mathrm{cpt}}(F_{\mathbb{P}(E)})$, respectively, the space of compactly-supported sections of $F_{\mathbb{P}(E)}$ and its quotient up to compactly-supported homotopy. We define the *compactly supported twisted K-theory group* $K_{\zeta,\mathrm{cpt}}(X)$ as the abelian group $\bar{\Gamma}_{\mathrm{cpt}}(F_{\mathbb{P}(E)})$ for any ζ-twisted Hilbert bundle E.[1]

2.4 Differential extensions

Definition 2.4.1. (smooth refinement) Given a cohomology theory E^*, a smooth refinement $\hat{E}^*$ is a functor $\hat{E}\colon \mathrm{Diff} \to \mathrm{Grps}$ with transformations I, R such that

$$\begin{tikzcd} \hat{E}(M) \arrow[r, "I"] \arrow[d, "R"'] & E^*(M) \arrow[d] \\ \Omega^*_{d=0}(M,V) \arrow[r] & E^*_{dR}(M) \end{tikzcd}$$

where $V = E^*(pt) \otimes \mathbb{R}$ is the graded non-torsion cohomology of E on the one-point space and such that there is a transformation

$$a\colon \Omega^{*-1}(M)/\mathrm{Im}(d) \to \hat{E}^*(M)$$

that gives an exact sequence

$$\begin{tikzcd} E^{*-1}(M) \arrow[r, "ch"'] & \Omega^{*-1}(M)/\mathrm{Im}(d) \arrow[r, "a"'] \arrow[dr, "d"'] & \hat{E}(M) \arrow[r, "I"'] \arrow[d, "R"'] & H^*(M) \\ & & \Omega^*_{d=0}(M) & \end{tikzcd}$$

For the case where the cohomology theory is K-theory, an extension model is given as follows. A differential K-cocycle of X is a quadruple

$$\check{E} = (E, h^E, \nabla^E, \omega)$$

where E is a complex vector bundle over X with a Hermitian metric h^E and a Hermitian connection ∇^E and $\omega \in \Omega^{\mathrm{odd}}(X)/\mathrm{Im}(d)$. A K-relation among three differential K-cocycles

$$\check{E}_1 = (E_1, h^{E_1}, \nabla^{E_1}, \omega_1) \quad \check{E}_2 = (E_2, h^{E_2},\ \nabla^{E_2}, \omega_2) \quad \check{E}_3 = (E_3, h^{E_3}, \nabla^{E_3}, \omega_3)$$

is given by a short exact sequence of Hermitian vector bundles

$$0 \to E_1 \to E_2 \to E_3 \to 0,$$

and an element

$$CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \in \Omega^{\mathrm{odd}}(X)/\mathrm{Im}(d)$$

such that $w_2 = \omega_1 + \omega_3 + CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$.

[1] When X is compact, Definitions 2.3.14 and 2.3.15 are equivalent. Actually, we will never apply Definition 6.1.5 when X is not compact, hence it would be sufficient to state definition 2.3.15 for every (locally compact) space.

Definition 2.4.2. (differential K-theory) The differential K-theory of X, denoted by $\check{K}^0(X)$, is the quotient of the free abelian group generated by differential K-cocycles of X, by the relation

$$[\check{E}_2] = [\check{E}_1] + [\check{E}_3]$$

whenever there is a K-relation amongst $\check{E}_1$, $\check{E}_2$ and $\check{E}_3$.

This model in terms of the extension definition we gave earlier would look like this:

$$\begin{tikzcd} \hat{K}^0(X) \arrow[r, "I"] \arrow[d, "\check{ch}"'] & K^*(X) \arrow[d, "ch"'] \\ \Omega_0^{ev}(X) \arrow[r] & H^{ev}(X,\mathbb{R}) \end{tikzcd}$$

There exist two natural homomorphisms

$$\check{K}^0(X) \to K^0(X)$$

given by the forgetful map $[(E, h^E, \nabla^E, \omega)] \mapsto [E]$, and

$$\check{ch}\colon \check{K}^0(X) \to \Omega_0^{ev}(X)$$

given by $[(E, h^E, \nabla^E, \omega)] \mapsto ch(E, \nabla^E) - d\omega$.

2.5 Spectral sequences

Definition 2.5.1. (Differential bigraded module) A differential bigraded module over a ring R, is a collection of R–modules, $\{E^{p,q}\}$, where p and q are integers, together with an R–linear mapping, $d\colon E^{*,*} \to E^{*,*}$, the differential, of bidegree $(s, 1-s)$ or $(s, s-1)$, for some integer s, and satisfying $d \circ d = 0$.

Definition 2.5.2. (Spectral sequence) A spectral sequence is a collection of differential bigraded R–modules $\{E_r^{*,*}, d_r\}$, where $r = 1, 2, \ldots$; the differentials are all of bidegree $(r, 1-r)$ and for all p, q, r, $E_{r+1}^{p,q}$ is isomorphic to $H(E_r^{p,q}, d_r)$, the homology with respect to the maps d_r.

- Notice how $E_r^{*,*}$ together with d_r determines E_{r+1} but they do not determine d_{r+1}.
- When the differentials d_r are zero, the E_{r+1} page is exactly E_r.
- Most spectral sequences encountered in practice have the property that $d_r = 0$ for all r bigger than a certain value, say N. This means that $E_\infty = E_N$ and we say that the sequence collapses at N.

Definition 2.5.3. (convergence) A spectral sequence $\{E_r^{*,*}, d_r\}$, is said to converge to H^*, a graded R–module, if there is a filtration F of H^*,

$$\{0\} \subset F^p H^p \subset \cdots \subset F^0 H^p = H^p$$

such that

$$E_\infty^{p,q} = F^p H^{p+q}/F^{p+1}H^{p+q}$$

where $E_\infty^{*,*}$ is the limit term of the spectral sequence.

Theorem 2.5.4. *(The Cohomological Leray-Serre Spectral Sequence) Let R be a commutative ring with unit. Suppose $F \to E \to B$ is a fibration, where B is path-connected and F is connected. Then there is a first quadrant spectral sequence of algebras, $\{E^{*,*}, d_r\}$, converging to $H^*(E; R)$ as an algebra, with*

$$E_2^{p,q} = H^p(B; H^q(F; R)) \tag{2.15}$$

denoting the cohomology of the space B with local coefficients in the cohomology of the fibre F.

Proposition 2.5.5. *If on top of the requirements in the previous theorem $\pi_1(B)$ acts trivially on $H^*(F; R)$, then there is a first quadrant spectral sequence $\{E_r^{p,q}, d_r\}$, with*

$$E_2^{p,q} = H^p(B; H^q(F; R))$$

converging to $H^*(E)$. Here the cohomology of B is untwisted.

If the base space B of a fibration $F \to E \to B$ is a finite-dimensional CW-complex then we can extend the Serre spectral sequence to any generalized cohomology theory h^*.

Proposition 2.5.6. *Let $F \to E \to B$ be a fibration where B is a finite-dimensional CW-complex. Then we have a spectral sequence $\{E_r^{*,*}, d_r\}$ with*

$$E_2^{p,q} = H^p(B; h^q(F))$$

converging to $h^*(E)$. Again, here the E_2-page is given by cohomology with local coefficients.

The Atiyah-Hirzebruch spectral sequence for K-theory, is the spectral sequence from the previous proposition for the fibration $pt \to X \to X$ in the case $h^* = K^*$. A more general spectral sequence (Theorem 20.4.1 in [34]) that we already mentioned in the introduction is:

Theorem 2.5.7. *(Serre spectral sequence) Let B be a CW complex with p-skeleton B^p and let X be an excellent spectrum over B. Let $j_p\colon B^p \to B$ be the inclusion, let $X^p = j_{p!}j_p^*X$, and let $i_p\colon X^p \to X^{p+1}$ be the induced inclusion of spectra over B. Let J^* be a parametrized cohomology theory over B. Then there is a conditionally convergent spectral sequence*

$$E_1^{p,q} = \prod_{p-cells\ e} J^{p+q}(e_!e^*X^p, \partial e_! \partial e^* X^{p-1}) \Longrightarrow J^{p+q}(X)$$

The sequence converges strongly if the derived E_∞ terms RE_∞ vanish. If the theory is represented by an excellent spectrum J over B, then

$$E_2^{p,q} = H^p(B; \mathcal{L}^q(X, J))$$

We only give the statement of this theorem without going into detail of the terms used since we will not be using it in what follows. However, it is worth mentioning that in the case of twisted K-theory and a fibration $F \to E \to B$, if we have a twist for the base B, this spectral sequence converges to the twisted K-theory of E with respect to the pullback twist.

Chapter 3

Twistings for fiber bundles over the circle

In this section we construct the class of a twisting for the total space of a fiber bundle $M \to X \to S^1$ from classes in $H^2(M)$ and $H^1(S^1)$. To achieve this, first we give a description of $H^3(X)$ through the Serre spectral sequence, then we build a projective bundle over X that represents the sought class and eventually we give a cohomological description of it. Finally, as an addition that will not be transcendental for our objective in the last three sections we explore $H^1(S^1; H^2(M))$ through the construction of the spectral sequence, prove an explicit description of $H^3(X)$ in a particular case and alternatives to the previous constructions using Čech cohomology.

3.1 The Serre spectral sequence

Let $M \hookrightarrow X \xrightarrow{\pi} S^1$ be a fiber bundle, where M is a manifold. Using the Serre spectral sequence, we determine $H^i(X)$ for $1 \leq i \leq 3$ up to extensions. Using Theorem 1.14 in [25], we have $E_2^{p,q} \cong H^p(S^1; H^q(M))$, where this is cohomology with local coefficients, so the second page is

$$\begin{array}{c|ccc}
3 & H^0(S^1; H^3(M)) & H^1(S^1; H^3(M)) & 0 \\
2 & H^0(S^1; H^2(M)) & H^1(S^1; H^2(M)) & 0 \\
1 & H^0(S^1; H^1(M)) & H^1(S^1; H^1(M)) & 0 \\
0 & H^0(S^1; H^0(M)) & H^1(S^1; H^0(M)) & 0 \\
\hline
 & 0 & 1 & 2
\end{array}$$

Since $d_k \colon E_k^{p,q} \to E_k^{p+k,q-k+1}$, we obtain $E_2^{p,q} = E_\infty^{p,q}$. We first determine $H^1(X)$ up to extensions. We have $E_\infty^{0,1} = F^0H^1(X)/F^1H^1(X)$ and $E_\infty^{1,0} = F^1H^1(X)/F^2H^1(X)$. Because $F^2H^1(X) = 0$, we obtain $E_\infty^{1,0} = F^1H^1(X) =$

$H^1(S^1; H^0(M))$ and $E_\infty^{0,1} = H^0(S^1; H^1(M)) = H^1(X)/H^1(S^1; H^0(M))$, thus

$$0 \to H^1(S^1; H^0(M)) \to H^1(X) \to H^0(S^1; H^1(M)) \to 0 \tag{3.1}$$

To calculate $H^2(X)$ up to extensions, we now consider $E_\infty^{0,2} = F^0H^2(X)/F^1H^2(X)$ and $E_\infty^{1,2} = F^1H^2(X)/F^2H^2(X)$. Because $F^2H^2(X) = 0$, we obtain $E_\infty^{1,1} = F^1H^2(X) = H^1(S^1; H^1(M))$ and $E_\infty^{0,2} = H^0(S^1; H^2(M)) = H^2(X)/H^1(S^1; H^1(M))$, thus

$$0 \to H^1(S^1; H^1(M)) \to H^2(X) \to H^0(S^1; H^2(M)) \to 0 \tag{3.2}$$

To compute $H^3(X)$ up to extensions, we use $E_\infty^{0,3} = F^0H^3(X)/F^1H^3(X)$ and $E_\infty^{1,2} = F^1H^3(X)/F^2H^3(X)$. Because $F^2H^3(X) = 0$, we obtain $E_\infty^{1,2} = F^1H^3(X) = H^1(S^1; H^2(M))$ and $E_\infty^{0,3} = H^0(S^1; H^3(M)) = H^3(X)/H^1(S^1; H^3(M))$, thus

$$0 \to H^1(S^1; H^2(M)) \to H^3(X) \to H^0(S^1; H^3(M)) \to 0. \tag{3.3}$$

In this chapter we will be particularly interested in twistings for X which are classified by elements of the subgroup $H^1(S^1; H^2(M))$. Note that

$$H^0(S^1; H^3(M)) \cong H^3(M)^{\mathbb{Z}}$$

for the usual action of the fundamental group of the base on the cohomology of the fiber. Moreover, the map $H^3(X) \to H^3(M)^{\mathbb{Z}}$ is the restriction of codomain of the map $H^3(X) \to H^3(M)$ induced by the inclusion of the fiber. Hence we can think of them as twistings on X which are trivial when restricted to the fiber.

3.2 Twistings via a mapping torus construction

In this section we construct a twisting over the total space of the fiber bundle mentioned in the previous section.

Let M be a manifold and $f\colon M \to M$ a homeomorphism. Recall that the mapping torus of f is the quotient $X = (I \times M)/\sim$, where $(0, x) \sim (1, f(x))$. This is the total space of a fiber bundle over S^1 with fiber M. In this section we construct a projective bundle over X from a map $M \to PU(\mathcal{H})$, whose total space will also be a mapping torus.

Lemma 3.2.1. *Let $p\colon E \to B$ be a fiber bundle with fiber F and $q\colon X \to B$ a quotient map. Then the map $q'\colon q^*E \to E$ in the corresponding pullback diagram is a quotient map.*

$$\begin{tikzcd} q^*E \arrow[r, "q'"] \arrow[d, "p'"'] \arrow[dr, phantom, "\ulcorner", very near start] & E \arrow[d, "p"] \\ X \arrow[r, "q"] & B \end{tikzcd}$$

Proof. Given $A \subseteq E$ such that $q'^{-1}(A)$ is open, we will show that A is open.

Let us assume first that the result holds for subsets of E where the bundle trivializes. Now let $A \subseteq E$ be such that $q'^{-1}(A)$ is open in E and let $\{U_i\}_{i\in J}$ be an open cover of B where the bundle trivializes. If $B_i = A \cap p^{-1}(U_i)$, we have that

$$q'^{-1}(B_i) = q'^{-1}(A) \cap q'^{-1}(p^{-1}(U_i))$$

is open in $q'^{-1}(p^{-1}(U_i))$, hence in q^*E. Therefore B_i is open in E by our assumption, so $A = \bigcup_{i\in J} B_i$ is open in E as well.

On the other hand, let $A \subseteq p^{-1}(U_i)$, with $\{U_i\}_{i\in J}$ as before, and let $h\colon p^{-1}(U_i) \xrightarrow{\cong} U_i \times F$ be a trivialization. Then A is open if and only if $h(A)$ is open. The trivialization h induces a trivialization of q^*E given by

$$\begin{gathered} h'\colon p'^{-1}(q^{-1}(U_i)) \to q^{-1}(U_i) \times F \\ (x, e) \mapsto (x, pr_2 h(e)) \end{gathered}$$

where pr_2 is the projection to the second coordinate, $x \in X$ and $e \in p^{-1}(U_i)$ satisfies $p(e) = q(x)$. This trivialization fits into a commutative diagram

$$\begin{tikzcd} q^{-1}(U_i) \times F \arrow[r, "q\times id"] \arrow[dr, phantom, "\ulcorner", very near start] & U_i \times F \\ p'^{-1}(q^{-1}(U_i)) \arrow[u, "h'"] \arrow[r, "q'"] & p^{-1}(U_i) \arrow[u, "h"'] \end{tikzcd}$$

This diagram is commutative because $pr_1 h(e) = p(e) = q(x)$. Finally, we prove that if $q'^{-1}(A)$, then A is open. Recall that A is open if and only if $h(A)$ is open, so it is enough to check that $h(A)$ is open. Due to the fact that $h(A) \subseteq U_i \times F$ and $q \times id$ is a quotient map, then $h(A)$ is open if only if $(q \times id)^{-1}(h(A))$ is open in $q^{-1}(U_i) \times F$. Now by commutativity we have

$$(q \times id)^{-1}(h(A)) = h'(q'^{-1}(A)).$$

The left side is open because h' is a homeomorphism and we suppose $q'^{-1}(A)$ open. $\square$

Given an element g in a topological group G, let us denote by $L_g\colon G \to G$ the map given by left multiplication with g.

Lemma 3.2.2. *Let $\pi\colon Y \to S^1$ be a principal G–bundle. Then Y is homeomorphic to the mapping torus of a map $L_g\colon G \to G$ for some $g \in G$ and π corresponds to the natural projection associated to the mapping torus.*

Proof. Let $\Phi\colon I \to S^1$ be the quotient map induced by the exponential map and $\partial I = \{0, 1\}$. We have a pullback diagram

$$\begin{tikzcd} Y' \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\hat{\pi}"'] \arrow[dr, phantom, "\ulcorner", very near start] & Y \arrow[d, "\pi"] \\ I \arrow[r, "\Phi"] & S^1 \end{tikzcd}$$

Let $x, y \in \pi^{-1}(\Phi(0))$, then there is a tuple $(0, x)$ in Y' if and only if there is a tuple $(1, x)$. On the other hand $\widetilde{\Phi}(0, x) = \widetilde{\Phi}(1, y)$ if and only if $x = y$. This is due to the fact that $\pi(x) = \pi(y) = \Phi(0) = \Phi(1)$. Moreover $\widetilde{\Phi}^{-1}(x) = \{(0, x), (1, x)\}$.

By Lemma 3.2.1, we know that $Y \cong Y'/\sim$, where $x \sim y$ if and only if $\widetilde{\Phi}(x) = \widetilde{\Phi}(y)$. Due to the fact that I is contractible, there is a G–principal bundle trivialization $\rho \colon Y' \to I \times G$ hence $Y \cong (I \times G)/\approx$, where $x \approx y$ if and only if $\widetilde{\Phi}\rho^{-1}(x) = \widetilde{\Phi}\rho^{-1}(y)$.

Recall that the restriction of $\widetilde{\Phi}$ to each fiber is a homeomorphism. In particular, we have homeomorphisms $\widetilde{\Phi}_j \colon \hat{\pi}^{-1}(j) \to \pi^{-1}(1)$ for $j = 0, 1$. Now we define the homeomorphism $h = \rho \restriction_{\hat{\pi}^{-1}(1)} \circ f \circ \rho^{-1} \restriction_{0 \times G} \colon 0 \times G \to 1 \times G$, where $f \colon \hat{\pi}^{-1}(0) \to \hat{\pi}^{-1}(1)$ is defined by $f(x) = \widetilde{\Phi}_1^{-1}(\widetilde{\Phi}_0(x))$. The map h must be of the form L_g for some $g \in G$ because ρ and $\widetilde{\Phi}$ are G–equivariant maps, being a trivialization of a principal G–bundle and a map of principal G–bundles, respectively.

Let (x, g) and (y, z) be elements of $I \times G$ such that $(x, g) \approx (y, z)$. Then $\widetilde{\Phi}\rho^{-1}(x, g) = \widetilde{\Phi}\rho^{-1}(y, z)$ and therefore

$$\pi\widetilde{\Phi}\rho^{-1}(x, g) = \pi\widetilde{\Phi}\rho^{-1}(y, z)$$

which equals

$$\Phi\hat{\pi}\rho^{-1}(x, g) = \Phi\hat{\pi}\rho^{-1}(y, z)$$

We can simplify further

$$\Phi \mathrm{pr}_1(x, g) = \Phi \mathrm{pr}_1(y, z)$$

that is, $\Phi(x) = \Phi(y)$. If $x = y$, then $\widetilde{\Phi}\rho^{-1}(x, g) = \widetilde{\Phi}\rho^{-1}(x, z)$ implies $g = z$ because the restriction of $\widetilde{\Phi}\rho^{-1}$ to $\{x\} \times G$ is a homeomorphism. On the other hand, $(0, g) \approx (1, z)$ if and only if $\widetilde{\Phi}\rho^{-1}(0, g) = \widetilde{\Phi}\rho^{-1}(1, z)$. In the notation we introduced in the previous paragraph, this is

$$\widetilde{\Phi}_0\rho^{-1}|_{0 \times G}(0, g) = \widetilde{\Phi}_1\rho^{-1}|_{1 \times G}(1, z)$$

which holds if and only if $(1, z) = h(0, g)$ □

Lemma 3.2.3. *Let $\pi \colon Y \to S^1$ be a principal G–bundle and let $X \to S^1$ be the associated bundle with fiber F. Then X is homeomorphic to the mapping torus of an action map $L_g \colon F \to F$ for some $g \in G$ and π corresponds to the natural projection associated to the mapping torus.*

Proof. By Lemma 3.2.2, there exists $g \in G$ such that Y is homeomorphic to the mapping torus of $L_g \colon G \to G$, that we denote by $Y \cong (I \times G)/\sim_g$. Then

$$X \cong [(I \times G)/\sim_g] \times_G F$$

On the other hand, let $(I \times F)/\sim_g$ represent the mapping torus of $L_g \colon F \to F$. Now we define

$$\begin{aligned} \Psi \colon\ [(I \times G)/\sim_g] \times_G F &\to (I \times F)/\sim_g \\ [[x, h], f] &\mapsto [x, hf] \end{aligned}$$

The map Ψ is well-defined because if $((0,h),f)$ and $((1,ghh'),h'^{-1}f)$ represent the same element in the domain, then

$$\Psi([[0,h],f]) = [0,hf] = [1,g(hf)] = [1,(gh)f] = [1,(gh)h'h'^{-1}f] = \Psi([[1,ghh'],h'^{-1}f])$$

Now we define

$$\begin{aligned} \Psi^{-1}\colon (I\times F)/{\sim_g} &\to [(I\times G)/{\sim_g}]\times_G F \\ [x,f] &\mapsto [[x,g],g^{-1}f] \end{aligned}$$

The map Ψ^{-1} is well-defined because if $(0,f)\sim_g(1,g\cdot f)$ then

$$\Psi^{-1}([0,f]) = [[0,g],g^{-1}f] = [[1,gg],g^{-1}f) = [[1,g],f] = \Psi([1,g\cdot f]$$

It is straightforward to check that $\Psi^{-1}\Psi = id$ and $\Psi\Psi^{-1} = id$. □

Let $\pi\colon X\to S^1$ be a fiber bundle with fiber a compact manifold M and let $\Phi\colon I\to S^1$ be the quotient map induced by the exponential map. The corresponding pullback is represented by the following diagram.

$$\begin{tikzcd} X' \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\hat{\pi}"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d, "\pi"] \\ I \arrow[r, "\Phi"] & S^1 \end{tikzcd}$$

By Lemma 3.2.3, there is a homeomorphism $\varsigma\colon 0\times M\to 1\times M$, such that

$$X\cong (I\times M)/{\sim}$$

where $(0,x)\sim(1,y)$ if and only if $\varsigma(0,x)=(1,y)$. We will use the notation ς' for the second component of ς, that is $\varsigma(0,x)=(1,\varsigma'(x))=(1,y)$.

To motivate the following construction, let $\sigma\colon X\to BPU(\mathcal{H})$ be a representative of a class in $H^3(X;\mathbb{Z})$ and let $\mathcal{P}_\sigma$ a principal $PU(\mathcal{H})$–bundle classified by σ. We denote by $\mathbb{P}_\sigma(X)=\mathcal{P}_\sigma\times_{PU(\mathcal{H})}\mathrm{Fred}(\mathcal{H})$ the associated Fredholm bundle. As before, there is a pullback diagram

$$\begin{tikzcd} \mathbb{P}'_\sigma \arrow[r, "\widetilde{\widetilde{\Phi}}"] \arrow[d, "\hat{p}"] \arrow[dr, phantom, "\ulcorner", very near start] & \mathbb{P}_\sigma \arrow[d, "p"] \\ X' \arrow[r, "\widetilde{\Phi}"] & X \end{tikzcd}$$

Thus $\pi p\colon \mathbb{P}_\sigma\to S^1$ is a fiber bundle with fiber $M\times\mathrm{Fred}(\mathcal{H})$ and $\hat{\pi}\hat{p}$ his corresponding pullback. Using Lemma 3.2.3, $\mathbb{P}_\sigma$ can be obtained by a mapping torus from a homeomorphism $\epsilon\colon (0\times M)\times\mathrm{Fred}(\mathcal{H})\to(1\times M)\times\mathrm{Fred}(\mathcal{H})$.

Our strategy will be to reverse the logic here, first constructing a map

$$\begin{aligned} \epsilon\colon 0\times(M\times PU(\mathcal{H})) &\to 1\times(M\times PU(\mathcal{H})) \\ \epsilon(0,m,y) &= (\varsigma(0,m),\kappa_\lambda(m)y) \end{aligned}$$

where $\varsigma\colon 0\times M \to 1\times M$ is the homeomorphism considered previously and $\kappa_\lambda\colon M \to PU(\mathcal{H})$ is a representative of the class $-\lambda \in H^2(M;\mathbb{Z}) = [M, PU(\mathcal{H})]$. Next we will check that the mapping torus of ϵ is isomorphic to the composition of the fiber bundle $M \hookrightarrow X \xrightarrow{\pi} S^1$, where X is the mapping torus of ς, and a $PU(\mathcal{H})$–principal fiber bundle $\mathbb{P}(X) \xrightarrow{p} X$ (see Figure 3.1).

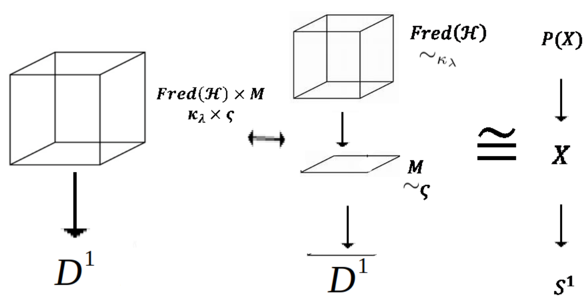


Figure 3.1: Composition bundles

We will do this by taking the quotient of $I\times M\times PU(\mathcal{H})$ by the relations $\sim_\varsigma$ and $\sim_{\kappa_\lambda}$. The first relation matches $(0,m,1_\mathcal{H})$ with $(1,\varsigma'(m),1_\mathcal{H})$ where $m\in M$ and $1_\mathcal{H}$ is the identity in PU($\mathcal{H}$). This leads us to define

$$\begin{aligned}\pi_1\colon (I\times M\times PU(\mathcal{H}))/\!\sim_\varsigma \to& S^1\\ [i,m,h]\mapsto& [i]\end{aligned}$$

It is straightforward to verify that π_1 restricted to $(I\times M\times 1_\mathcal{H})/\!\sim_\varsigma$ is isomorphic to $M\hookrightarrow X \xrightarrow{\pi} S^1$. In the same way we can define

$$\begin{aligned}p'\colon (I\times M\times PU(\mathcal{H}))/\sim_\varsigma \to& (I\times M\times 1_\mathcal{H})/\sim_\varsigma\\ [i,m,h]\mapsto& [i,m,1_\mathcal{H}]\end{aligned}$$

The map p' is a projection and the restriction to $((0,1)\times M\times PU(\mathcal{H}))/\!\sim_\varsigma$ is a fiber bundle with fiber $PU(\mathcal{H})$. However, it is not a fiber bundle since

$$p'^{-1}([]0,m,1_\mathcal{H}]) = \{[i,m,h]\in I\times M\times PU(\mathcal{H})/\!\sim_\varsigma \mid i=0,1\}$$

that is, two copies of $M\times PU(\mathcal{H})$ pairing to $M\times 1_\mathcal{H}$. To make p' into a fiber bundle, we use the relation $\sim_{\kappa_\lambda}$. We match $[0,m,h]$ with $[1,\varsigma'(m),\kappa_\lambda(m)(h)]$. This is well-defined because

$$\begin{aligned}[[0,m,1_\mathcal{H}]] &= [[1,\varsigma'(m),\kappa_\lambda(m)(1_\mathcal{H})]]\\ &= [[1,\varsigma'(m),\kappa_\lambda(m)1_\mathcal{H}{\kappa_\lambda}^{-1}(m)]]\\ &= [[1,\varsigma'(m),1_\mathcal{H}]]\end{aligned}$$

In other words, the equivalence relation $\sim_{\kappa_\lambda}$ coincides with $\sim_\varsigma$ in $I\times M\times 1_\mathcal{H}$. With this in mind we define

$$\begin{aligned}p_1\colon\ [(I\times M\times PU(\mathcal{H}))/\sim_\varsigma]/\sim_{\kappa_\lambda} \to& (I\times M\times 1_\mathcal{H})/\sim_\varsigma\\ [[i,m,h]]\mapsto& [i,m,1_\mathcal{H}]\end{aligned}$$

It remains to prove that there is a local trivialization around $[[i, m, 1_{\mathcal{H}}]]$ with $i = 0, 1$. Without loss of generality we will use $[[0, m, 1_{\mathcal{H}}]]$. Let $U \subset S^1$ be an open neighbourhood of $[0]$. Since $\Phi\colon I \to S^1$ is a quotient map, we have $\Phi^{-1}(U) = U_0 \cup U_1$, where U_0 and U_1 are disjoint semi-open intervals with $0 \in U_0$ and $1 \in U_1$.

Next take the open subset $V = \pi_1^{-1}(U) \subseteq (I \times M \times 1_{\mathcal{H}})/\sim_\varsigma$. It is a neighborhood of $[[0, m, 1_{\mathcal{H}}]]$. Now we define

$$\begin{aligned} \xi\colon p_1^{-1}(V) \to & V \times PU(\mathcal{H}) \\ [[i, m, h]] \mapsto & \begin{cases} ([i, m, 1_{\mathcal{H}}], h) & \text{if } [i, m, 1_{\mathcal{H}}] \in \pi_1^{-1}(U_0) \\ ([i, m, 1_{\mathcal{H}}], \kappa_\lambda^{-1}(\varsigma'^{-1}(m))(h)) & \text{if } [i, m, 1_{\mathcal{H}}] \in \pi_1^{-1}(U_1) \end{cases} \end{aligned}$$

To see that it is well defined, note that the different choice of representative $(1, \varsigma'(m), \kappa_\lambda(m)h)$ for the element $[[0, m, h]]$ would lead us to

$$\begin{aligned} ([1, \varsigma'(m), 1_{\mathcal{H}}], \kappa_\lambda^{-1}(\varsigma'^{-1}(\varsigma'(m)))(\kappa_\lambda(m)h) &= ([0, m, 1_{\mathcal{H}}], \kappa_\lambda^{-1}(m)(\kappa_\lambda(m)h) \\ &= ([0, m, 1_{\mathcal{H}}], h) \end{aligned}$$

Having this $PU(\mathcal{H})$–principal bundle in mind we can define

$$p_a\colon (I \times M \times PU(\mathcal{H}))/\sim_\epsilon \to X \tag{3.4}$$

$$[t, m, h] \mapsto a[t, m] \tag{3.5}$$

where a is the homeomorphism between $(I \times M)/\sim_\varsigma$ and X. Now we construct the associated bundle

$$p\colon\ [(I \times M \times PU(\mathcal{H}))/\sim_\epsilon] \times_{PU(\mathcal{H})} \mathrm{Fred}(\mathcal{H}) \to X$$

and it is straightforward to check that πp is isomorphic to $(I \times M \times \mathrm{Fred}(\mathcal{H}))/\sim$ where $(0, m, y) \sim \epsilon(0, m, y)$, that is, the mapping torus of ϵ over S^1. Additionally, the $PU(\mathcal{H})$–principal fiber bundle $p_a\colon \mathcal{P}(X) \to X$ is classified by a class $\sigma \in H^3(X; \mathbb{Z})$.

Remark 3.2.4. (Independence of the representative) In the previous construction, we chose a representative $\kappa_\lambda\colon M \to PU(\mathcal{H})$ of an element of $H^2(M) \cong [M, PU(\mathcal{H})]$ to construct a $PU(\mathcal{H})$–principal bundle that we will call P_{κ_λ} in this remark.

If we chose another representative $\kappa_{\lambda'}\colon M \to PU(\mathcal{H})$ we would have a homotopy $H\colon M \times I \to PU(\mathcal{H})$ with $H_0 = \kappa_\lambda$ and $H_1 = \kappa_{\lambda'}$ and we can apply construction 3.4 again to obtain

$$P_H = I \times I \times M \times PU(\mathcal{H})/\sim$$

with the relation $(t, 0, m, y) \sim (t, 1, \varsigma(m), H(m, t)y)$ and

$$\begin{aligned} P_H \to & X \times I \\ [t, s, m, y] \mapsto & ([s, m], t) \end{aligned} \tag{3.6}$$

which by construction turns out to be a $PU(\mathcal{H})$–principal bundle over $X \times I$. A property that is useful to us is that $P_{\mathcal{H}} \restriction_{X\times 0} \cong P_{\kappa_\lambda}$ and $P_{\mathcal{H}} \restriction_{X\times 1} \cong P_{\kappa_{\lambda'}}$, finally applying Theorem 4.9.8 in [29] we get an isomorphism of $PU(\mathcal{H})$–principal bundles

$$P_{\kappa_\lambda} \cong P_{\kappa_{\lambda'}}$$

Hence this construction is independent of the choice of κ_λ up to isomorphism.

Remark 3.2.5. We will use the notation $D_+ = \Phi([0, \frac{1}{2}])$ and $D_- = \Phi([\frac{1}{2}, 1])$ but to avoid excessive notation we will also use the notation D_+ and D_- for $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, respectively. With this in mind we define trivializations of the fiber bundles p_1 and of the restriction π of π_1 to X. The names of these trivializations and other relevant maps are displayed in the following diagrams for convenience.

$$\begin{tikzcd}
(D_+ \times M) \times \mathrm{Fred}(\mathcal{H}) \arrow[dr, "q_+"'] & (\rho_+^{-1})^*\mathbb{P}(X)\restriction \arrow[l, "\alpha", "\cong"'] \arrow[d, "pr_1"] & \mathbb{P}(X)\restriction \arrow[l, "\widetilde{\rho_+}"'] \arrow[r, hook] \arrow[d, "p\restriction"] & \mathbb{P}(X) \arrow[d, "p"] \\
& D_+ \times M \times 1_{\mathcal{H}} & \pi^{-1}(D_+) \arrow[l, "\rho_+"'] \arrow[r, hook] & \pi^{-1}(S^1)
\end{tikzcd}$$

$$\begin{tikzcd}
(D_- \times M) \times \mathrm{Fred}(\mathcal{H}) \arrow[dr, "q_-"'] & (\rho_-^{-1})^*\mathbb{P}(X)\restriction \arrow[l, "\beta", "\cong"'] \arrow[d, "pr_1"] & \mathbb{P}(X)\restriction \arrow[l, "\widetilde{\rho_-}"'] \arrow[r, hook] \arrow[d, "p\restriction"] & \mathbb{P}(X) \arrow[d, "p"] \\
& D_- \times M \times 1_{\mathcal{H}} & \pi^{-1}(D_-) \arrow[l, "\rho_-"'] \arrow[r, hook] & \pi^{-1}(S^1)
\end{tikzcd}$$

Now we describe these maps

$$\begin{aligned} q_\pm \colon D_\pm \times M \times \mathrm{Fred}(\mathcal{H}) &\to D_\pm \times M \times 1_{\mathcal{H}} \\ (i, m, h) &\mapsto (i, m, 1_{\mathcal{H}}) \end{aligned}$$

$$\begin{aligned} \rho_+ \colon \pi^{-1}(D_+) &\to \left[0, \frac{1}{2}\right] \times M \times 1_{\mathcal{H}} \\ [[i, m, 1_{\mathcal{H}}]] &\mapsto \begin{cases} (i, m, 1_{\mathcal{H}}) & \text{if } i \neq 1 \\ (0, \varsigma'^{-1}(m), 1_{\mathcal{H}}) & \text{if } i = 1 \end{cases} \end{aligned}$$

$$\begin{aligned} \rho_- \colon \pi^{-1}(D_-) &\to \left[\frac{1}{2}, 1\right] \times M \times 1_{\mathcal{H}} \\ [[i, m, 1_{\mathcal{H}}]] &\mapsto \begin{cases} (i, m, 1_{\mathcal{H}}) & \text{if } i \neq 0 \\ (1, \varsigma'(m), 1_{\mathcal{H}}) & \text{if } i = 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \widetilde{\rho_+} \colon p^{-1}(\pi^{-1}(D_+)) &\to (\rho_+^{-1})^*(p \restriction_{\pi^{-1}(D_+)}) \\ [[i, m, h]] &\mapsto (\rho_+[[i, m, h]], [[i, m, h]]) \end{aligned}$$

$$\begin{aligned}\widetilde{\rho_-}\colon p^{-1}(\pi^{-1}(D_-)) &\to (\rho_-^{-1})^*(p\restriction_{\pi^{-1}(D_-)})\\ [[i,m,h]] &\mapsto (\rho_-[[i,m,h]],[[i,m,h]])\end{aligned}$$

To trivialize $(\rho_+^{-1})^*\mathbb{P}(X)$, we first point out that there is a homeomorphism

$$\begin{aligned}p_1^{-1}(\pi^{-1}(D_+)) &\to \pi^{-1}(D_+)\times \mathrm{Fred}(\mathcal{H})\\ [i,m,h] &\mapsto \begin{cases}([i,m],h) & \text{if } i\neq 1\\ ([i,m],\kappa_\lambda^{-1}(\varsigma'^{-1}(m))\cdot h) & \text{if } i=1\end{cases}\end{aligned}$$

We obtain then a trivialization for the pullback:

$$\begin{aligned}(\rho_+^{-1})^*[p_1^{-1}(\pi^{-1}(D_+))] &\to D_+\times M\times \mathrm{Fred}(\mathcal{H})\\ ((j,n,1_\mathcal{H}),[i,m,h]) &\mapsto \begin{cases}(j,n,h) & \text{if } i\neq 1\\ (j,n,\kappa_\lambda^{-1}(\varsigma'^{-1}(m))\cdot h) & \text{if } i=1\end{cases}\end{aligned}$$

Now we use $\rho_+([i,m,1_\mathcal{H}])=(j,n,1_\mathcal{H})$. If $i\neq 1$, then $(i,m)=(j,n)$. If $i=1$, then $(j,n)=(0,\varsigma'^{-1}(m))$, thus

$$(j,n,\kappa_\lambda^{-1}(\varsigma'^{-1}(m))\cdot h)=(0,\varsigma'^{-1}(m),\kappa_\lambda^{-1}(\varsigma'^{-1}(m))\cdot h)$$

Hence the trivialization α is described as

$$\begin{aligned}\alpha\colon (\rho_+^{-1})^*(p_1\restriction_{\pi^{-1}(D_+)}) &\to \left[0,\frac{1}{2}\right]\times M\times \mathrm{Fred}(\mathcal{H})\\ ((j,n,1_\mathcal{H}),[[i,m,h]]) &\mapsto \begin{cases}(i,m,h) & \text{if } i=j\\ (0,\varsigma'^{-1}(m),\kappa_\lambda^{-1}(\varsigma'^{-1}(m))h) & \text{if } i\neq j\end{cases}\end{aligned}$$

$$\begin{aligned}\beta\colon (\rho_-^{-1})^*(p_1\restriction_{\pi^{-1}(D_-)}) &\to \left[\frac{1}{2},1\right]\times M\times \mathrm{Fred}(\mathcal{H})\\ ((j,n,1_\mathcal{H}),[[i,m,h]]) &\mapsto \begin{cases}(i,m,h) & \text{if } i=j\\ (1,\varsigma'(m),\kappa_\lambda(\varsigma'(m))h) & \text{if } i\neq j\end{cases}\end{aligned}$$

Additionally, let t_0 and t_1 be local sections over $D_+\times M\times 1_\mathcal{H}$ and $D_-\times M\times 1_\mathcal{H}$ respectively, obtained by pullback from $\alpha\widetilde{\rho_+}$ and $\beta\widetilde{\rho_-}$ of a global section $t\colon \pi^{-1}(S^1)\to\mathbb{P}(X)$. They must have the form:

$$\begin{aligned}t_1\colon \left[\frac{1}{2},1\right]\times M\times 1_\mathcal{H} &\to \left[\frac{1}{2},1\right]\times M\times \mathrm{Fred}(\mathcal{H})\\ (i,m,1_\mathcal{H}) &\mapsto (i,m,s_1(i,m,1_\mathcal{H}))\end{aligned}$$

$$\begin{aligned}t_0\colon \left[0,\frac{1}{2}\right]\times M\times 1_\mathcal{H} &\to \left[0,\frac{1}{2}\right]\times M\times \mathrm{Fred}(\mathcal{H})\\ (i,m,1_\mathcal{H}) &\mapsto (i,m,s_0(i,m,1_\mathcal{H}))\end{aligned}$$

Then we have

$$\begin{aligned}
\alpha\widetilde{\rho}_+\widetilde{\rho}_-^{-1}\beta^{-1}t_1(1,m,1_{\mathcal{H}}) =&\alpha\widetilde{\rho}_+\widetilde{\rho}_-^{-1}\beta^{-1}(1,m,s_1(1,m,1_{\mathcal{H}}))\\
=&\alpha\widetilde{\rho}_+\widetilde{\rho}_-^{-1}((1,m,1_{\mathcal{H}}),[[1,m,s_1(1,m,1_{\mathcal{H}})]])\\
=&\alpha\widetilde{\rho}_+([[1,m,s_1\rho_-([[1,m,1_{\mathcal{H}}]])]])\\
=&\alpha\widetilde{\rho}_+([[1,m,s_1\rho_-([[0,\varsigma'^{-1}(m),1_{\mathcal{H}}]])]])\\
=&\alpha((0,\varsigma'^{-1}(m),1_{\mathcal{H}}),[[1,m,s_1\rho_-\rho_+^{-1}(0,\varsigma'^{-1}(m),1_{\mathcal{H}})]])\\
=&(0,\varsigma'^{-1}(m),\kappa_\lambda^{-1}(\varsigma'^{-1}(m))s_1\rho_-\rho_+^{-1}(0,\varsigma'^{-1}(m),1_{\mathcal{H}})).
\end{aligned}$$

Since $s_0(0,n,1_{\mathcal{H}}) = \kappa_\lambda^{-1}(n)s_1\rho_-\rho_+^{-1}(0,n,1_{\mathcal{H}})$, the relation between t_0 and t_1 is

$$t_0 = \epsilon^{-1}t_1\rho_-\rho_+^{-1} = \epsilon^{-1}t_1\varsigma \tag{3.7}$$

We check that this is indeed the case:

$$\begin{aligned}
\epsilon^{-1}t_1\rho_-\rho_+^{-1}(0,n,1_{\mathcal{H}}) &= \epsilon^{-1}t_1(1,\varsigma'(n),1_{\mathcal{H}})\\
&= \epsilon^{-1}(1,\varsigma'(n),s_1(1,\varsigma'(n),1_{\mathcal{H}}))\\
&= (0,\varsigma'^{-1}(\varsigma'(n)),\kappa_\lambda^{-1}(\varsigma'^{-1}(\varsigma'(n)))s_1(1,\varsigma'(n),1_{\mathcal{H}}))\\
&= (0,n,\kappa_\lambda^{-1}(n)s_1(1,\varsigma'(n),1_{\mathcal{H}}))\\
&= (0,n,s_0(0,n,1_{\mathcal{H}}))\\
&= t_0(0,n,1_{\mathcal{H}})
\end{aligned}$$

On the other hand

$$\begin{aligned}
\alpha\widetilde{\rho}_+\widetilde{\rho}_-^{-1}\beta^{-1}t_1\left(\frac{1}{2},m,1_{\mathcal{H}}\right) =&\alpha\widetilde{\rho}_+\widetilde{\rho}_-^{-1}\beta^{-1}\left(\frac{1}{2},m,s_1\left(\frac{1}{2},m,1_{\mathcal{H}}\right)\right)\\
=&\alpha\widetilde{\rho}_+\widetilde{\rho}_-^{-1}\left(\left(\frac{1}{2},m,1_{\mathcal{H}}\right),\left[\left[\frac{1}{2},m,s_1\left(\frac{1}{2},m,1_{\mathcal{H}}\right)\right]\right]\right)\\
=&\alpha\widetilde{\rho}_+\left(\left[\left[\frac{1}{2},m,s_1\rho_-\left(\left[\left[\frac{1}{2},m,1_{\mathcal{H}}\right]\right]\right)\right]\right]\right)\\
=&\alpha\left(\left(\frac{1}{2},m,1_{\mathcal{H}}\right),\left[\left[\frac{1}{2},m,s_1\rho_-\rho_+^{-1}\left(\frac{1}{2},m,1_{\mathcal{H}}\right)\right]\right]\right)\\
=&\alpha\left(\left(\frac{1}{2},m,1_{\mathcal{H}}\right),\left[\left[\frac{1}{2},m,s_1\left(\frac{1}{2},m,1_{\mathcal{H}}\right)\right]\right]\right)\\
=&\left(\frac{1}{2},m,s_1\left(\frac{1}{2},m,1_{\mathcal{H}}\right)\right).
\end{aligned}$$

3.3 Cohomological interpretation

In this section we want to describe the inclusion map $H^1(S^1;H^2(M)) \hookrightarrow H^3(X)$ and relate it to the construction from the previous section. Let us start by remembering a classic result about fiber bundles over homology spheres.

Proposition 3.3.1 (Wang sequence)**.** *For a fibration $F \hookrightarrow X \xrightarrow{\pi} B$, where B is a simply-connected homology n-sphere and F is path-connected, we have a long exact sequence*

$$\cdots \to H^k(X) \to H^k(F) \to H^{k-n+1}(F) \to H^{k+1}(X) \to \cdots \tag{3.8}$$

This long exact sequence follows from the Serre spectral sequence, but for our purposes we show a different way to achieve this result when $B = S^1$ and π is a fiber bundle, by using the Mayer-Vietoris exact sequence.

Let $e = (1,0) \in S^1$ and let w be the antipodal point to e. Given the open cover $\{U, V\}$ of S^1, where $U = S^1 - \{e\}$ and $V = S^1 - \{w\}$, we can define homotopy equivalences ϕ_i and homeomorphisms ξ_i

$$\phi_1 \colon \pi^{-1}(U) \to \pi^{-1}(n), \qquad \phi_2 \colon \pi^{-1}(V) \to \pi^{-1}(s)$$
$$\xi_1 \colon \pi^{-1}(n) \to F \times \{n\}, \qquad \xi_2 \colon \pi^{-1}(s) \to F \times \{s\}$$

such that $\phi_1 \restriction_{\pi^{-1}(n)} = id$, $\phi_2 \restriction_{\pi^{-1}(s)} = id$ and $\phi_2\phi_1 \restriction_{\pi^{-1}(s)}$ coincides with the action of a generator g of $\pi_1(S^1)$ over the fiber. Here n denotes the point $(0,1)$ and $s = -n$. We denote $u = \xi_1\phi_1\xi_2^{-1}$ and $v = \xi_2\phi_2\xi_1^{-1}$.

Now we write the Mayer-Vietoris exact sequence associated to the covering of the total space X by $\pi^{-1}(U)$ and $\pi^{-1}(V)$. This is:

$$\cdots \to H^k(X) \to H^k(\pi^{-1}(U)) \oplus H^k(\pi^{-1}(V)) \xrightarrow{i_U - i_V} H^k(\pi^{-1}(U) \cap \pi^{-1}(V)) \to H^{k+1}(X) \to \cdots \tag{3.9}$$

Using the above homotopy equivalences and homeomorphisms, this sequence becomes:

$$\cdots \to H^k(X) \to H^k(F \times \{n\}) \oplus H^k(F \times \{s\}) \to H^k(F \times \{n\}) \oplus H^k(F \times \{s\}) \to H^{k+1}(X) \to \cdots \tag{3.10}$$

To describe the map that corresponds to $i_U - i_V$ under these homotopy equivalences and homeomorphisms, we use the following two diagrams:

$$\begin{tikzcd} H^k(\pi^{-1}(U)) \arrow[r, "i_U^*"] & H^k(\pi^{-1}(U) \cap \pi^{-1}(V)) \arrow[d, "(\xi_1^{-1} \oplus \xi_2^{-1})^*"] \\ H^k(F \times \{n\}) \arrow[u, "(\xi_1\phi_1)^*"] \arrow[r, dashed, "(id, u^*)"] & H^k(F \times \{n\}) \oplus H^k(F \times \{s\}) \end{tikzcd}$$

$$\begin{tikzcd} H^k(\pi^{-1}(V)) \arrow[r, "i_V^*"] & H^k(\pi^{-1}(U) \cap \pi^{-1}(V)) \arrow[d, "(\xi_1^{-1} \oplus \xi_2^{-1})^*"] \\ H^k(F \times \{s\}) \arrow[u, "(\xi_2\phi_2)^*"] \arrow[r, dashed, "(v^*, id)"] & H^k(F \times \{n\}) \oplus H^k(F \times \{s\}) \end{tikzcd}$$

With this in mind we get the map

$$H^k(F \times \{n\}) \oplus H^k(F \times \{s\}) \xrightarrow{\begin{pmatrix} 1 & v^* \\ u^* & 1 \end{pmatrix}} H^k(F \times \{n\}) \oplus H^k(F \times \{s\}) \tag{3.11}$$

Intuitively we identify that there is an extra copy of $H^k(F)$ on each side here and we would like to remove it. Let us denote $f = v^*u^*$ and have a look at its kernel and its cokernel.

The kernel is the set of tuples (x, y) such that $x+v^*(y) = 0$ and $u^*(x)+y = 0$. So these tuples must have the form $(x, -u^*(x))$, where $x = v^*u^*(x)$. But v^*u^* is exactly the induced action of g conjugated by ξ_2. We take a class and move it along a path going from the point n to the point s avoiding e, and then from the point s to the point n avoiding w, so we are moving across the whole loop. Hence the kernel is $\mathrm{Ker}(f - I)$.

Similarly, the cokernel is the set of tuples (x, y) modulo the subgroup of tuples of the form $(x, y) = (a, u^*(a))+(v^*(b), b)$. But the maps $(x, y) \mapsto x-v^*(y)$ and $z \mapsto (z, 0)$ induce an isomorphism

$$\mathrm{Coker}\begin{pmatrix} 1 & v^* \\ u^* & 1 \end{pmatrix} \cong \mathrm{Coker}(f - I).$$

To see this, note that for elements of the form $(a, u^*(a))$ and $(v^*(b), b)$, we have

$$a - v^*u^*(a) = a - f(a)$$

$$v^*(b) - v^*(b) = 0$$

And for an element of the form $a - f(a)$, we have

$$(a - f(a), 0) = (a - v^*u^*(a), 0) = (a, u^*(a)) - (v^*u^*(a), u^*(a))$$

Thus, we can replace the map by $f - I\colon H^k(F \times \{n\}) \to H^k(F \times \{n\})$. Of course $H^k(F \times \{n\})$ is just $H^k(F)$ and we get the Wang sequence.

An important observation in the case $k = 2$ is that

$$\mathrm{Coker}(f - I) \cong H^2(F)_{\mathbb{Z}} \cong H^1(S^1; H^2(M))$$

where $H^2(F)_{\mathbb{Z}}$ denotes the coinvariants. Now we want to describe the connecting morphism $\overline{\gamma}\colon \mathrm{Coker}(f - I) \hookrightarrow H^3(X)$ in the long exact sequence. To achieve this, we use the following diagram.

$$\begin{tikzcd}
H^2(\pi^{-1}(U)) \oplus H^2(\pi^{-1}(V)) \arrow[rr] \arrow[d, "\phi^*=(\phi_1^{-1}\xi_1^{-1},\phi_2^{-1}\xi_2^{-1})^*"', "\cong"] & & H^2(\pi^{-1}(U) \cap \pi^{-1}(V)) \arrow[d, "(\xi_1^{-1},\xi_2^{-1})^*"', "\cong"] \arrow[dr, "\delta"] & \\
H^2(F \times \{n\}) \oplus H^2(F \times \{s\}) \arrow[rr, "\psi"] & & H^2(F \times \{n\}) \oplus H^2(F \times \{s\}) \arrow[r, "\delta'"] \arrow[d, "q"] & H^3(X) \\
& 0 \arrow[r] & \mathrm{Coker}(\psi) \arrow[ur, "\gamma'"] & \\
& H^2(F) \arrow[r, "f-I"] & H^2(F) \arrow[u, "\omega"] \arrow[uur, "\gamma"'] &
\end{tikzcd} \tag{3.12}$$

Here $q(z,w) = [z,w]$, $\gamma'[z,w] = \delta'(z,w)$ and $\omega(a) = [a,0]$. Note that ω is the appropriate map by the previous considerations that identify the cokernels of ψ and $f - I$. With this we have a better expression for $\overline{\gamma}$, namely

$$\overline{\gamma}(a) = \delta'\omega(a) = \delta(\xi_1^*(a), 0).$$

Now we describe δ to improve this formula, our idea will be use the description in the following diagram:

$$\begin{tikzcd}
H^k(C_g) \arrow[r, "q^*"] & H^k(X) \\
H^k(\Sigma(\pi^{-1}(U \cap V))) \arrow[u, "\cong"] & \\
H^{k-1}(\pi^{-1}(U \cap V)) \arrow[u, "\cong"] \arrow[uur, "\delta"'] &
\end{tikzcd} \tag{3.13}$$

First we regard the Mayer-Vietoris sequence for $k \neq 1$ in another way:

$$\begin{tikzcd}
H^{k-1}(\pi^{-1}(U \cap V)) \arrow[r] \arrow[d, "\cong"] & H^k(X) \arrow[r] \arrow[dr, "g^*"] & H^k(\pi^{-1}(U)) \oplus H^k(\pi^{-1}(V)) \arrow[d, "\cong"] \\
H^k(\Sigma(\pi^{-1}(U \cap V))) \arrow[ur] & & H^k(\pi^{-1}(U) \vee \pi^{-1}(V))
\end{tikzcd} \tag{3.14}$$

where $g\colon \pi^{-1}(U) \vee \pi^{-1}(V) \to X$ is the inclusion in each summand. Secondly, we take its homotopy cofiber C_g and we point out that we have a homeomorphism:

$$e\colon \Sigma(\pi^{-1}(U \cap V)) \xrightarrow{\cong} C_g$$
$$[x,t] \mapsto \begin{cases} * & \text{if } x = * \\ [x, 1-2t] & \text{if } 0 \leq t \leq \frac{1}{2} \\ [x, 2t-1] & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

There are two different ways of identifying C_g that we describe now. The first description is:

$$C_g = X \vee [(\pi^{-1}(U) \vee \pi^{-1}(V)) \times I]/\sim$$

where $(z,1) \sim *$ and $(z,0) \sim g(z)$. The second description is:

$$C_g = C\pi^{-1}(U) \vee C\pi^{-1}(V)/\sim$$

where $[a,0] \sim [a,0]$ for $a \in \pi^{-1}(U \cap V)$. With either description, we define the map

$$\begin{aligned} q\colon X &\to C_g \\ x &\mapsto [x] \end{aligned} \tag{3.15}$$

and consider the bijections

$$\begin{aligned} H^3(C_g) &\cong [C_g, BPU(\mathcal{H})]_* \cong [C_g, BPU(\mathcal{H})] \\ H^2(\pi^{-1}(U \cap V)) &\cong [\pi^{-1}(U \cap V), PU(\mathcal{H})] \end{aligned}$$

Given the class of $h\colon \pi^{-1}(U \cap V) \to PU(\mathcal{H})$, we consider $\Sigma h\colon \Sigma(\pi^{-1}(U \cap V)) \to \Sigma PU(\mathcal{H})$. Now we use the fact that there is a homotopy equivalence $PU(\mathcal{H}) \xrightarrow{\simeq} \Omega BPU(\mathcal{H})$ and taking the adjoint map we obtain:

$$\Sigma(\pi^{-1}(U \cap V)) \to \Sigma PU(\mathcal{H}) \to BPU(\mathcal{H})$$

To obtain a bundle interpretation of an element in $H^3(\Sigma(\pi^{-1}(U \cap V)))$, we need two observations:

- The $PU(\mathcal{H})$–principal bundle P over $\Sigma PU(\mathcal{H}) = C_+PU(\mathcal{H}) \cup_{PU(\mathcal{H})} C_-PU(\mathcal{H})$ obtained through the above adjoint map is

$$P = \frac{C_+PU(\mathcal{H}) \times PU(\mathcal{H}) \amalg C_-PU(\mathcal{H}) \times PU(\mathcal{H})}{\sim},$$

  where $([x, 0], z) \sim ([x, 0], xz)$, hence the clutching map is the identity map.

- The restriction of Σh to $\pi^{-1}(U \cap V)$ is h, thus

$$\Sigma h^* P = \frac{C\pi^{-1}(U \cap V) \times PU(\mathcal{H}) \amalg C\pi^{-1}(U \cap V) \times PU(\mathcal{H})}{([a, 0], z) \sim ([a, 0], h(a)z)},$$

The corresponding principal $PU(\mathcal{H})$–bundle Q over C_g induced by $\Sigma h^* P$ through the homeomorphism e is given by:

$$Q = \frac{C\pi^{-1}(U) \times PU(\mathcal{H}) \amalg C\pi^{-1}(V) \times PU(\mathcal{H})}{([a, 0], z) \sim ([a, 0], h(a)z)},$$

since $e \restriction_{\pi^{-1}(U \cap V)} = id$. Finally we pull Q back along q.

$$\begin{array}{ccc} q^*Q & \longrightarrow & Q \\ \downarrow & & \downarrow \\ X & \xrightarrow{q} & C_g \end{array} \tag{3.16}$$

Here $q(\pi^{-1}(U)) \subset C\pi^{-1}(U)$ and $q(\pi^{-1}(V)) \subset C\pi^{-1}(V)$, hence $q^*Q \restriction_{\pi^{-1}(U)}$ and $q^*Q \restriction_{\pi^{-1}(V)}$ are trivial. The bundle q^*Q is the result of gluing two trivial bundles on $\pi^{-1}(U \cap V)$ using the map

$$\begin{tikzcd} \pi^{-1}(U \cap V) \arrow[r, "q=id"] \arrow[rr, bend right, "h"'] & \pi^{-1}(U \cap V) \arrow[r, "h"] & PU(\mathcal{H}) \end{tikzcd} \tag{3.17}$$

$$q^*Q = \frac{\pi^{-1}(U) \times PU(\mathcal{H}) \amalg \pi^{-1}(V) \times PU(\mathcal{H})}{(a,z) \sim (a, h(a)z)}, \tag{3.18}$$

Finally we match the interpretation from Equation 3.18 with the bundle described in Equation 3.4. For this, we will again represent X as a torus mapping of ς and choose a map $h\colon \pi^{-1}(U \cap V) \to PU(\mathcal{H})$ which on a connected component of $\pi^{-1}(U \cap V)$ represents a map homotopically equivalent to λ and on the other component to the constant map.

$$H\colon (I \times M \times PU(\mathcal{H})/\simeq_\epsilon \to q^*Q$$
$$[t,m,z] \mapsto \begin{cases} [t,m,z]_U & \text{if } [t,m,z] \in p_a^{-1}(\pi^{-1}(D_+)) \\ [t,m,z]_V & \text{if } [t,m,z] \in p_a^{-1}(\pi^{-1}(D_-)) \end{cases}$$

By the way they were constructed, it is easy to verify that this map is a morphism of $PU(\mathcal{H})$–principal bundles. Therefore they are isomorphic. It only rests to check that it is well-defined, that is:

$$\begin{aligned} H(0,m,y) &= [0,m,y]_U \\ &= [0,m,h(0,m)y]_V \\ &= [\varsigma(0,m), \kappa_\lambda(m)y]_V \\ &= [1, \varsigma'(m), \kappa_\lambda(m)y]_V \\ &= H(1, \varsigma'(m), \kappa_\lambda(m)y) \end{aligned}$$

.

3.4 Another description of $E_\infty^{1,2}$

In this section we get a description of $E_\infty^{1,2} = H^1(S^1; H^2(M))$ in terms of relative cohomology groups. To achieve this, we will construct $E_2^{1,2}$ from the E_1-page of the Serre spectral sequence.

First of all we give a filtration for $B = S^1$ where B^0 is the one-point space and $B^1 = S^1$. This induces a filtration in X given by $X_i = \pi^{-1}(B^i)$. So $X_0 = M$ and $X_i = \emptyset$ for $i < 0$. To construct the first page we consider the staircase diagram

$$\begin{array}{ccccccccccc}
\to H^2(X_1,X_0) & \longrightarrow & H^2(X_1) & \longrightarrow & H^3(X_2,X_1) & \longrightarrow & H^3(X_2) & \longrightarrow & H^4(X_3,X_2) \to \\
 & & \downarrow & & & & \downarrow & & \\
\to H^2(X_0,X_{-1}) & \longrightarrow & H^2(X_0) & \longrightarrow & H^3(X_1,X_0) & \longrightarrow & H^3(X_1) & \longrightarrow & H^4(X_2,X_1) \to \\
 & & \downarrow & & & & \downarrow & & \\
\to H^2(X_0,X_{-1}) & \longrightarrow & H^2(X_0) & \longrightarrow & H^3(X_1,X_0) & \longrightarrow & H^3(X_1) & \longrightarrow & H^4(X_2,X_1) \to
\end{array}$$

In our case, it has the form

$$
\begin{tikzcd}
\arrow[r] & H^2(X,M) \arrow[r] & H^2(X) \arrow[r] \arrow[d, "i^*"] & H^3(X,X) \arrow[r] & H^3(X) \arrow[r] \arrow[d] & H^4(X,X) \arrow[r] & {} \\
\arrow[r] & H^2(M,\emptyset) \arrow[r, "j^*"] & H^2(M) \arrow[r, "\delta"] \arrow[d] & H^3(X,M) \arrow[r, "j^*"] & H^3(X) \arrow[r, "\delta"] \arrow[d, "i^*"] & H^4(X,X) \arrow[r] & {} \\
\arrow[r] & H^2(\emptyset,\emptyset) \arrow[r] & H^2(\emptyset) \arrow[r] & H^3(M,\emptyset) \arrow[r] & H^3(M) \arrow[r] & H^4(X,M) \arrow[r] & {}
\end{tikzcd}
$$

We now define $E_1^{n,p} = H^n(X_p, X_{p-1})$ and $d_1 = j^* \circ \delta$. With this notation $H^n(X)$ is filtered by the subgroups $F_p^n = \operatorname{Ker}(H^n(X) \to H^n(X_{p-1}))$ and $E_\infty^{n,p} \cong F_p^n / F_{p+1}^n$. In Section 3.1 we concluded that $E_\infty^{n,p} = E_2^{n,p}$ and $E_\infty^{1,2} = H^1(S^1; H^2(M))$, so we will calculate $E_2^{1,2}$ from $E_1^{1,2}$. We have

$$E_1^{0,2} \xrightarrow{d_1} E_1^{1,2} \xrightarrow{d_1} E_1^{2,2}$$

that is

$$H^2(M,\emptyset) \xrightarrow{j^* \circ \delta} H^3(X,M) \xrightarrow{j^* \circ \delta} H^4(X,X) = 0$$

Due to the fact that $j^* = id$, we have $E_2^{1,2} = \operatorname{Coker}(\delta)$. From the long exact sequence of the pair

$$\to H^2(M) \xrightarrow{\delta} H^3(X,M) \xrightarrow{\hat{j}^*} H^3(X) \to$$

we obtain

$$
\begin{tikzcd}
 & \dfrac{H^3(X,M)}{\operatorname{Im}(\delta)} = \dfrac{H^3(X,M)}{\operatorname{Ker}(\hat{j}^*)} \arrow[d, hook] = E_2^{1,2} \\
H^3(X,M) \arrow[ur] \arrow[r, "\hat{j}^*"'] & H^3(X)
\end{tikzcd}
$$

3.5 The case of a trivial action

Here we describe the isomorphism $\Psi\colon \operatorname{Hom}(H_1(S^1, B^0), H^2(M)) \to H^3(X,M)$ in the case when $\pi_1(X)$ acts trivially over $H^2(M)$. Let $\pi\colon X \to B$ be a fiber bundle with fiber M, let B_p be the p-skeleton of B and let $X_p = \pi^{-1}(B_p)$. Let $\Phi\colon D^p \to B_p$ be the characteristic map for the p-cell e_α^p of B. Let $\widetilde{D}_\alpha^p = \Phi_\alpha^*(X_p)$ and let $\widetilde{S}_\alpha^{p-1}$ be the part of $\widetilde{D}_\alpha^p$ over ∂D_α^p.

$$
\begin{tikzcd}
\widetilde{D}_\alpha^p \arrow[r, "\widetilde{\Phi}_\alpha"] \arrow[d] & X^p \arrow[d, "\pi"] \\
D_\alpha^p \arrow[r, "\Phi_\alpha"] & B_p
\end{tikzcd}
$$

The isomorphism Ψ is constructed via the following conmutative diagram (see Proposition 1.14 in [25]).

$$\begin{array}{ccc}\prod_\alpha H^{p+q}\left(\widetilde{D}^p_\alpha, \widetilde{S}^{p-1}_\alpha\right) & \xleftarrow[\widetilde{\Phi}_*]{\cong} & H^{p+q}\left(X_p, X_{p-1}\right) \\ {\scriptstyle \prod_\alpha \epsilon^p_\alpha}\uparrow\cong & & {\scriptstyle \Psi}\uparrow\cong \\ \prod_\alpha H^q(M) & \xleftarrow[\cong]{} & \mathrm{Hom}\left(H_p\left(B_p, B_{p-1}\right), H^q(M)\right)\end{array}$$

We are interested in the case where M is a manifold and $B = S^1$. We consider the cellular decomposition for S^1 with one 0-cell D^0 and one 1-cell D^1 which is attached by the constant map $\partial D^1 = \{D^0_1, D^0_{-1}\} \to D^0$. In particular there is a homeomorphism $\rho\colon \widetilde{D}^1 \to D^1 \times M$. We also add the hypothesis of trivial action of $\pi_1(X)$ in $H^2(X)$.

The leftmost vertical map turns up out of

$$\begin{aligned} H^2\left(\widetilde{D}^0_1\right) &\cong H^2\left(\widetilde{S}^0, \widetilde{D}^0_{-1}\right) \\ &\cong H^3\left(\widetilde{D}^1, \widetilde{S}^0\right) \end{aligned} \tag{3.19}$$

The isomorphism ϵ^1_i in the diagram is completed with

$$H^2(\widetilde{D}^0_1) \cong H^2(F_1) \cong H^2(M)$$

where $F_1 = \widetilde{\Phi}(\widetilde{D}^0_1)$, the first isomorphism is induced by $\widetilde{\Phi}$ and the second one is given by the hypothesis of trivial action.

Consider the fibration $\widetilde{D}^1 \to D^1$, now we determine Ψ explicitly in terms of Čech cohomology. By the previous isomorphisms we have

$$\begin{aligned} \check{H}^1(M; \underline{U(1)}) &\cong \check{H}^1(F_1; \underline{U(1)}) \\ &\cong \check{H}^1\left(\widetilde{D}^0_1; \underline{U(1)}\right) \\ &\cong \check{H}^1\left(\widetilde{S}^0_, \widetilde{D}^0_{-1}; \underline{U(1)}\right) \\ &\cong \check{H}^2\left(\widetilde{D}^1, \widetilde{S}^0; \underline{U(1)}\right) \end{aligned} \tag{3.20}$$

Let $\{U_i\}$ be an open cover of M and $\overline{\phi} \in \check{H}^1\left(M; \underline{U(1)}\right)$, where ϕ is a class representative with $\{\phi_{ij}\colon U_{ij} \to U(1)\}$. The first isomorphism follows from the hypothesis of trivial action, so we will use the same notation ϕ but the image has the form $\phi = \{\phi_{ij}\colon D^0 \times U_{ij} \to U(1)\}$. The second isomorphism is induced by $\widetilde{\Phi}$, hence we still use the same notation $\phi = \{\phi_{ij}\colon D^0_1 \times U_{ij} \to U(1)\}$. The third isomorphism comes from excision, so its image in $\check{H}^1\left(S^0 \times M, D^0_{-1} \times M; \underline{U(1)}\right)$ is the class of

$$\phi' = \left\{\phi'_{ij}\colon S^0 \times U_{ij} \to U(1)\right\},$$

where $\phi'(y) = \phi(y)$ if $y \in D^0_1 \times M$ and $\phi'(y) = 1$ if $y \in D^0_{-1} \times M$. The fourth isomorphism is induced by a connecting morphism, so we use the following diagram to describe it:

$$
\begin{tikzcd}
 & \check{C}^2\left(D^1\times M, D^0_{-1}\times M; \underline{U(1)}\right) & \check{C}^2\left(D^1\times M, S^0\times M; \underline{U(1)}\right) \arrow[l] \\
\check{C}^1\left(S^0\times M, D^0_{-1}\times M; \underline{U(1)}\right) \arrow[urr, dashed] & \check{C}^1\left(D^1\times M, D^0_{-1}\times M; \underline{U(1)}\right) \arrow[l] \arrow[u]
\end{tikzcd}
$$

Since $\phi' \in \check{C}^1\left(S^0\times M, D^0_{-1}\times M; \underline{U(1)}\right)$, we get a lifting $\phi'' = \left\{\phi''_{ij} \colon D^1\times U_{ij} \to U(1)\right\}$ in $\check{C}^1\left(D^1\times M, D^0_{-1}\times M; \underline{U(1)}\right)$ defined as a family of homotopies between $\phi'_{ij} \restriction (D^0_1\times M)$ and $\phi'_{ij} \restriction (D^0_{-1}\times M)$. Now we have $\partial\phi'' \in \check{C}^2\left(D^1\times M, D^0_{-1}\times M; \underline{U(1)}\right)$ defined by

$$\partial\phi'' = \gamma = \left\{\gamma_{ijk} \colon D^1\times U_{ijk} \to U(1)\right\}$$

where

$$\gamma_{ijk} = \phi''_{ij} \restriction (D^1\times U_{ijk}) - \phi''_{ik} \restriction (D^1\times U_{ijk}) + \phi''_{jk} \restriction (D^1\times U_{ijk}),$$

Since $\phi'' \restriction (S^0\times U_{ij}) = \phi'$ and $\partial\phi' = 0$, we have that $\gamma \in \check{C}^2\left(D^1\times M, S^0\times M; \underline{U(1)}\right)$ and it is a cocycle.

Finally the isomorphism $\widetilde{\Phi}_*$ turns up out of the excision property. Since B_{p-1} is a deformation retract of a neighborhood N in B_p, we have that X_{p-1} is a deformation retract of $\pi^{-1}(N)$ and so:

$$
\begin{aligned}
\check{H}^2\left(X_p, X_{p-1}; \underline{U(1)}\right) &\simeq \check{H}^2\left(X_p, \pi^{-1}(N); \underline{U(1)}\right) \\
&\simeq \check{H}^2\left(D^1\times M, \widetilde{\Phi}^{-1}(\pi^{-1}(N)); \underline{U(1)}\right) \qquad (3.21)\\
&\simeq \check{H}^2\left(D^1\times M, S^0\times M; \underline{U(1)}\right).
\end{aligned}
$$

3.6 An interpretation with Čech cohomology

Let $M \hookrightarrow X \to S^1$ be a fiber bundle where M is manifold and let $\{B_1, B_0\}$ be a cellular descomposition of S^1 as in the previous section. Let $\partial I = \{0,1\}$ and let N be an open neighborhood of B_0 in S^1. Consider the following diagram of pullbacks.

$$
\begin{tikzcd}
 & X' \arrow[rr, "\widetilde{\Phi}"] \arrow[dd, "\hat{\pi}"] \arrow[ddrr, phantom, "\ulcorner"] & & X \arrow[dd, "\pi"] \\
Z \arrow[ur] \arrow[dd] \arrow[dr, phantom, "\ulcorner"] & & & \\
 & I \arrow[rr, "\Phi"] & & S^1 \\
V \arrow[ur] \arrow[rr] & & N \arrow[ur, "i"] &
\end{tikzcd}
$$

Here we take $\rho\colon X' \to I \times M$, $\varsigma\colon 0 \times M \to 1 \times M$ and $f\colon \hat{\pi}^{-1}(0) \to \hat{\pi}^{-1}(1)$ as in Lemma 3.2.2.

$$\begin{aligned}
\check{H}^1\left(S^1;\underline{\mathbb{Z}}\right) \otimes \check{H}^1\left(M;\underline{U(1)}\right) &\cong \check{H}^1\left(I, S^0;\underline{\mathbb{Z}}\right) \otimes \check{H}^1\left(M;\underline{U(1)}\right) \\
&\cong \check{H}^2\left(I \times M, S^0 \times M;\underline{U(1)}\right) \\
&\cong \check{H}^2\left(I \times M, V \times M;\underline{U(1)}\right) \\
&\cong \check{H}^2\left(X', Z;\underline{U(1)}\right) \\
&\cong \check{H}^2\left(X, M;\underline{U(1)}\right).
\end{aligned} \tag{3.22}$$

Let $\{T_+, T_-\}$ be the open cover of S^1 (see Figure 3.2) corresponding to hemispheres with non-empty intersection, composed of two disjoint components that we denote by $(T_+ \cap T_-)_{-1}$ and $(T_+ \cap T_-)_1$. Let h be the Čech 1-cocycle whose generator class is defined by

$$\begin{aligned}
h\colon T_+ \cap T_- &\to \mathbb{Z} \\
(T_+ \cap T_-)_{-1} &\mapsto 0 \\
(T_+ \cap T_-)_1 &\mapsto 1
\end{aligned} \tag{3.23}$$

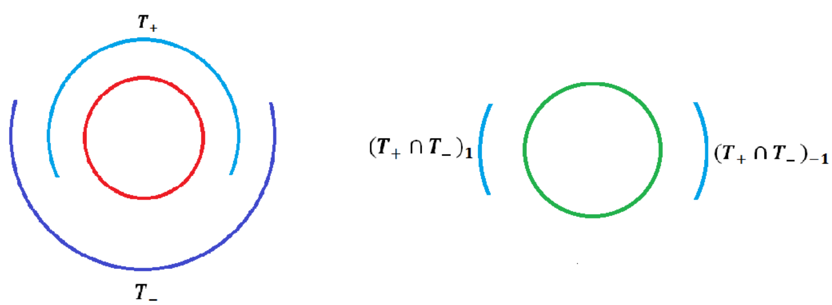


Figure 3.2: Cover of S^1

In the same way we take an open cover $\{U_i\}$ of M with $i \in I$ and a Čech 1-cocycle $g = \{g_{ij}\colon U_{ij} \to U(1)\}$. Let $\{D^1, D^0\}$ be a cellular decomposition for S^1 where D^0 is given by a point in $(T_+ \cap T_-)_{-1}$. The induced cover for D^1 (see Figure 3.3) has two non-connected open sets as well, namely $\{T'_+ = T^a_+ \cup T^b_+, T'_- = T^a_- \cup T^b_-\}$. Here $0 \in T^a_+, T^a_-$ and $1 \in T^b_+, T^b_-$. And their intersection is

$$T'_+ \cap T'_- = (T_+ \cap T_-)_1 \amalg (T_+ \cap T_-)_{-1_a} \amalg (T_+ \cap T_-)_{-1_b}$$

The first isomorphism in Equation (3.22) is induced by

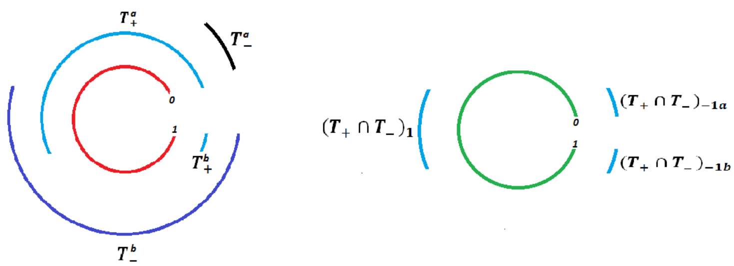


Figure 3.3: Induced cover of D^1

$$q \otimes id\colon (D^1, S^0) \times M \to (D^1/S^0, S^0/S^0) \times M$$

Thus the induced image of $h \otimes g$ through $(q \otimes id)_*$ is the map $h' \otimes g$ defined by

$$\begin{aligned} h' \otimes g\colon (T'_+ \cap T'_-) \otimes \{U_{ij}\} &\to \underline{\mathbb{Z}} \otimes \underline{U(1)} \\ (T_+ \cap T_-)_1 \otimes U_{ij} &\mapsto 1 \otimes g_{ij}(U_{ij}) \\ (T_+ \cap T_-)_{-1_a} \otimes U_{ij} &\mapsto 0 \otimes g_{ij}(U_{ij}) \\ (T_+ \cap T_-)_{-1_b} \otimes U_{ij} &\mapsto 0 \otimes g_{ij}(U_{ij}) \end{aligned} \tag{3.24}$$

For the second isomorphism, let $\pi_1\colon D^1 \times M \to D^1$ and $\pi_2\colon D^1 \times M \to M$ be the projection maps. We have a cup product

$$H^1(D^1 \times M, S^0 \times M; \underline{\mathbb{Z}}) \otimes H^1(D^1 \times M; \underline{U(1)}) \to H^2(D^1 \times M, S^0 \times M; \underline{\mathbb{Z}} \otimes \underline{U(1)})$$

and the last term is isomorphic to $H^2(D^1 \times M, S^0 \times M; \underline{U(1)})$. The induced open cover for $D^1 \times M$ is $\{T'_k \times U_i\}$ hence the image of $h' \otimes g$ is a cup product between $h' \circ \pi_1$ and $g \circ \pi_2$. Nevertheless we will use the open cover

$$\{U_i^k = T_k^a \times \kappa^{-s(k)}(U_i) \cup T_k^b \times \kappa^{-s(k)+1}(U_i)\}$$

with $i \in I$, $k \in \{-, +\}$, $s(+) = 0$ and $s(-) = 1$. The cocycle considered previously in this cover is a cup product $\tilde{h}' \smile \tilde{g}$, where $\tilde{h}'$ and $\tilde{g}$ are the cocycles $h' \circ \pi_1$ and $g \circ \pi_2$ on the new open cover. We point out that by construction $\tilde{g} \restriction_{\pi^{-1}(b)} = g$ for all $b \in (T_+ \cap T_-)_1$ due to the fact that both open covers are the same there.

$$\begin{aligned} \tilde{h}' \smile \tilde{g}\colon \{F_{t_1} \cap F_{t_2} \cap F_{t_3}\} &\to \underline{\mathbb{Z}} \otimes \underline{U(1)} \\ U_i^{k_1} \cap U_j^{k_2} \cap U_k^{k_3} &\mapsto \tilde{h}'(U_i^{k_1} \cap U_j^{k_2}) \otimes \tilde{g}(U_j^{k_2} \cap U_k^{k_3}) \\ U_i^{k_1} \cap U_j^{k_2} \cap \pi_1^{-1}(V) &\mapsto \tilde{h}'(U_i^{k_1} \cap U_j^{k_2}) \otimes \tilde{g}(U_j^{k_2} \cap \pi_1^{-1}(V)) \\ \pi_1^{-1}(V) \cap U_j^{k_2} \cap U_k^{k_3} &\mapsto \tilde{h}'(\pi_1^{-1}(V) \cap U_j^{k_2}) \otimes \tilde{g}(U_j^{k_2} \cap U_k^{k_3}) \end{aligned} \tag{3.25}$$

In the third isomorphism V represents an open neighborhood of S^0 in D^1 which deformation retracts onto S^0. This isomorphism is obtained using the long exact sequences of the triple since $H^2(V \times M, S^0 \times M; \underline{U(1)}) = 0$. In our example $\tilde{h}' \smile \tilde{g}$ is still a cocycle for

$$\check{H}^2\left(D^1 \times M, V \times M; \underline{U(1)}\right) \cong \check{H}^2\left(X', Z; \underline{U(1)}\right).$$

In relation to the fourth isomorphism, there is a homeomorphism $\phi\colon X' \to D^1 \times M$ which induces

$$\check{H}^2\left(D^1 \times M, V \times M; \underline{U(1)}\right) \cong \check{H}^2\left(X', Z; \underline{U(1)}\right)$$

An open cover for X' is $\{V_i^k = \phi^{-1}(F_t)\}$ and the cocycle asociated to $\tilde{h}' \smile \tilde{g}$ is:

$$\begin{aligned} \rho\colon \{V_i^{k_1} \cap V_j^{k_2} \cap V_k^{k_3}\} &\to \underline{\mathbb{Z}} \otimes \underline{U(1)} \\ V_i^{k_1} \cap V_j^{k_2} \cap V_k^{k_3} &\mapsto \tilde{h}'(\phi(V_i^{k_1} \cap V_j^{k_2})) \otimes \tilde{g}(\phi(V_j^{k_2} \cap V_k^{k_3})) \end{aligned} \tag{3.26}$$

Finally, in the last isomorphism, we verify that $F\restriction\colon X' - Z \to X - p^{-1}(N)$ is a homeomorphism. Here N deformation retracts onto D_0, hence so does $p^{-1}(N)$ over $p^{-1}(D^0)$. By excision

$$\check{H}^2\left(X', Z; \underline{U(1)}\right) \cong \check{H}^2\left(X, p^{-1}(N); \underline{U(1)}\right)$$

Next using the long exact sequence of the triple we have

$$\check{H}^2\left(X, p^{-1}(N); \underline{U(1)}\right) \cong \check{H}^2\left(X, M; \underline{U(1)}\right)$$

The cocycle σ asociated to ρ is defined similarly.

Chapter 4

Some computations of twisted K-theory

In this chapter we perform some calculations of twisted K-theory. For this purpose we first obtain a formula up to extensions that generalizes some results of Harju and Mickelsson (see Theorem 4.2 in [24]), then through the Atiyah-Hirzebruch spectral sequence we obtain formulas to calculate the twisted K-theory of closed connected 3-manifolds and we finally apply these two results in some examples.

4.1 The twisted K-theory of fiber bundles over the circle

Let $\pi\colon X \to S^1$ be a fiber bundle whose fiber is a compact manifold M and let $\sigma \to X$ be the principal $PU(\mathcal{H})$–bundle constructed in Section 3.2. If we know the K-theory of the fiber M, we can use a Mayer-Vietoris sequence for closed subspaces to obtain a description of a K-theory of X up to extensions. To calculate ${}^{\sigma}K(X)$ we use the closed hemisphere partition $\{D_+, D_-\}$ of the base, where $D_+ \cap D_- = \{-1, 1\}$. To achieve this, we need to trivialize the fiber bundle over each hemisphere and the twisting at each $(\pi^{-1}(D_+), \pi^{-1}(D_-))$. Let $\rho_i\colon \pi^{-1}(D_i) \to D_i \times M$ be the trivializations corresponding to regarding X as a mapping torus. With this in mind, we have

$$
\begin{tikzcd}
 & & & \sigma \arrow[ddd] & & & \\
(\rho_+^{-1})^*\sigma_+ \arrow[r, "\widetilde{\rho_+^{-1}}", "\cong"'] \arrow[dd] & \sigma_+ \arrow[urr, hook, bend left, "\widetilde{i_1'}"] \arrow[dd] & & & & \sigma_- \arrow[ull, hook', bend right, "\widetilde{i_2'}"'] \arrow[dd] & (\rho_-^{-1})^*\sigma_- \arrow[l, "\widetilde{\rho_-^{-1}}"', "\cong"] \arrow[dd] \\
 & & & \pi^{-1}(S^1) & & & \\
D_+ \times M \arrow[r, "\rho_+^{-1}", "\cong"'] & \pi^{-1}(D_+) \arrow[urr, hook, bend left, "i_1'"] & & \pi^{-1}(D_{+-}) \arrow[ll, hook', "i_1"] \arrow[rr, hook, "i_2"'] \arrow[d, shift right, "\rho_+"'] \arrow[d, shift left, "\rho_-"] & & \pi^{-1}(D_-) \arrow[ull, hook', bend right, "i_2'"'] & D_- \times M \arrow[l, "\rho_-^{-1}"', "\cong"] \\
 & & & D_{+-} \times M \arrow[lllu, hook', bend left, "j_1"] \arrow[rrru, hook, bend right, "j_2"'] & & &
\end{tikzcd}
$$

We focus on the intersection:

$$
\begin{tikzcd}
 & (\rho_+^{-1})^*\sigma_{+-} \arrow[dl] \arrow[dd] & & (\rho_-^{-1})^*\sigma_{+-} \arrow[ll, "\widetilde{\rho_+\rho_-^{-1}}"'] \arrow[dl] \arrow[dd] \\
(\rho_+^{-1})^*\sigma_+ \arrow[dd] & & (\rho_-^{-1})^*\sigma_- \arrow[dd] & \\
 & D_{+-} \times M \arrow[dl, hook', "j_1"'] \arrow[dr, hook, "j_2"] & & D_{+-} \times M \arrow[ll, "\rho_+\rho_-^{-1}=(id,\kappa^{-1})"'] \arrow[dl, hook', "j_2"] \\
D_+ \times M & & D_- \times M \arrow[ll, "(i_1'\rho_+^{-1})^{-1}i_2'\rho_-^{-1}"] &
\end{tikzcd}
$$

Then, the first part of the Mayer-Vietoris sequence is:

$$
\begin{tikzcd}
{}^{\sigma_+}K(\pi^{-1}(D_+)) \arrow[d, "(\rho_+^{-1})^*"] & \oplus & {}^{\sigma_-}K(\pi^{-1}(D_-)) \arrow[r, "i_1^*-i_2^*"] \arrow[d, "(\rho_-^{-1})^*"] & {}^{\sigma_{+-}}K(\pi^{-1}(D_{+-})) \arrow[d, "(\rho_+^{-1}\restriction)^*"] \\
{}^{(\rho_+^{-1})^*\sigma_+}K(D_+ \times M) & \oplus & {}^{(\rho_-^{-1})^*\sigma_-}K(D_- \times M) \arrow[r, "j_1^*-(\widetilde{\rho_+\rho_-^{-1}})_*j_2^*"'] & {}^{(\rho_+^{-1})^*\sigma_{+-}}K(D_{+-} \times M)
\end{tikzcd}
$$

Thus $(\rho_+\rho_-^{-1}) \restriction_{-1\times M}= id$ and $(\rho_+\rho_-^{-1}) \restriction_{1\times M}= \varsigma^{-1}$, where ς is the homeomorphism from Lemma 3.2.3. Therefore from the description of twisted K-theory in terms of sections:

$$(j_1^* - (\widetilde{\rho_+\rho_-^{-1}})_* j_2^*)(x,y) = (x \restriction - id_*(y \restriction), x \restriction - \varsigma_*^{-1}(y \restriction)),$$

Here $x \restriction - id_*(y \restriction)$ lies over $-1 \times M$ and $x \restriction - \varsigma_*^{-1}(y \restriction)$ lies over $1 \times M$.

To trivialize $(\rho_+^{-1})^*\sigma_+$ and $(\rho_-^{-1})^*\sigma_-$ we will use the description of σ as a mapping torus from Section 3.2. This description of σ is useful because we have a description over $D_+ \cap D_-$ and we see that its restriction to any fiber $\pi^{-1}(y)$ with $y \in S^1$ is trivial. Additionally each D_i are contractible so for $x \in D_{+-}$

$$\begin{tikzcd}
\pi^{-1}(x) \arrow[r, hook, "j'"] \arrow[d] & \pi^{-1}(D_i) \arrow[d] \\
x \arrow[r, hook, "j"] & D_i
\end{tikzcd}
\qquad
\begin{tikzcd}
D_i \times M & \pi^{-1}(D_i) \arrow[l, "\rho_i"', "\cong"] \\
x \times M \arrow[u, hook] & \pi^{-1}(x) \arrow[l, "\rho_i\restriction"', "\cong"] \arrow[u, hook, "j'"]
\end{tikzcd}$$

The map j' is a homotopy equivalence, thus so is $x \times M \xrightarrow{\rho_i j'(\rho_i\restriction)^{-1}} D_i \times M$. Hence we can define bundle trivializations α, β (as in Remark 3.2.5) shown in the following diagram

$$\begin{tikzcd}
(D_+ \times M) \times \mathrm{Fred}(\mathcal{H}) \arrow[d] & (\rho_+^{-1})^*\sigma_+ \arrow[l, "\alpha"', "\cong"] \arrow[r, "\widetilde{\rho_+^{-1}}"] \arrow[d] & \sigma_+ \arrow[r, hook, "\widetilde{i_1'}"] \arrow[d] & \sigma \arrow[d] \\
D_+ \times M & D_+ \times M \arrow[l, "="'] \arrow[r, "\rho_+^{-1}"] & \pi^{-1}(D_+) \arrow[r, hook, "i_1'"] & \pi^{-1}(S^1) \\
(D_- \times M) \times \mathrm{Fred}(\mathcal{H}) \arrow[d] & (\rho_-^{-1})^*\sigma_- \arrow[l, "\beta"', "\cong"] \arrow[r, "\widetilde{\rho_-^{-1}}"] \arrow[d] & \sigma_- \arrow[r, hook, "\widetilde{i_2'}"] \arrow[d] & \sigma \arrow[d] \\
D_- \times M & D_- \times M \arrow[l, "="'] \arrow[r, "\rho_-^{-1}"] & \pi^{-1}(D_-) \arrow[r, hook, "i_1'"] & \pi^{-1}(S^1)
\end{tikzcd}$$

We focus on the intersection and using for notation $\mathcal{F} = \mathrm{Fred}(\mathcal{H})$, we have a diagram

$$\begin{tikzcd}
& (D_{+-} \times M) \times \mathcal{F} \arrow[rr, "\alpha^{-1}"] \arrow[dl] \arrow[dd] & & (\rho_+^{-1})^*\sigma_{+-} \arrow[dl] \arrow[dd] \\
(D_+ \times M) \times \mathcal{F} \arrow[dd] & & (D_- \times M) \times \mathcal{F} \arrow[dd] & \\
& D_{+-} \times M \arrow[rr, "="] \arrow[dl, hook, "j_1"'] \arrow[dr, hook, "j_2"] & & D_{+-} \times M \arrow[dl] \\
D_+ \times M & & D_- \times M \arrow[ll, "(i_1'\rho_+^{-1})^{-1} i_2' \rho_-^{-1}"] &
\end{tikzcd}$$

The map $(\rho_+^{-1})^*\sigma_{+-} \to (D_- \times M) \times \mathcal{F}$ on the top right is represented in the following diagram:

$$\begin{tikzcd}
(D_- \times M) \times \mathcal{F} \arrow[d] & (\rho_-^{-1})^*\sigma_{+-} \arrow[l, hook', "\widetilde{j_2\beta}"'] \arrow[d] & \sigma_{+-} \arrow[l, "\widetilde{\rho_-}"'] \arrow[d] & (\rho_+^{-1})^*\sigma_{+-} \arrow[l, hook', "\widetilde{\rho_+^{-1}}"'] \arrow[d] \\
D_- \times M & D_{+-} \times M \arrow[l, hook', "j_2 id"'] & \pi^{-1}(D_{+-}) \arrow[l, "\rho_-"'] & D_{+-} \times M \arrow[l, hook', "\rho_+^{-1}"']
\end{tikzcd}$$

Then the second part of the Mayer-Vietoris sequence has the form:

$$\begin{tikzcd}
{}^{(\rho_+^{-1})^*\sigma_+}K(D_+ \times M) \oplus {}^{(\rho_-^{-1})^*\sigma_-} K(D_- \times M) \arrow[r, "j_1^* - (\widetilde{\rho_+\rho_-^{-1}})_* j_2^*"'] \arrow[d, "\alpha_* \quad \beta_*"] & {}^{(\rho_+^{-1})^*\sigma_{+-}}K(D_{+-} \times M) \arrow[d, "\alpha\restriction_*"] \\
K(D_+ \times M) \oplus K(D_- \times M) \arrow[r, "j_{1*} - \alpha_*(\widetilde{\rho_+\rho_-^{-1}})_* \beta_*^{-1} j_{2*}"'] & K(D_{+-} \times M)
\end{tikzcd}$$

If we identify $[0,1/2]$ and $[1/2,1]$ with D_+ and D_- respectively through the exponential map, using the description of twisted K-theory in terms of sections and the mapping torus construction (see Remark 3.2.5), we have:

$$\alpha(\widetilde{\rho_+\rho_-^{-1}})\beta^{-1}\left(\frac{1}{2}, m, s\left(\frac{1}{2}, m\right)\right) = \left(\frac{1}{2}, m, s\left(\frac{1}{2}, m\right)\right)$$

$$\alpha(\widetilde{\rho_+\rho_-^{-1}})\beta^{-1}(1, m, s(1,m)) = (0, \varsigma'^{-1}(m), \kappa_\lambda^{-1}(\varsigma'^{-1}(m)) s \rho_- \rho_+^{-1}(0, \varsigma'^{-1}(m)))$$

Finally, using the class $\lambda \in H^2(M;\mathbb{Z})$ induced by κ_λ^{-1} and defining $a\colon = (j_{1*} - \alpha_*(\widetilde{\rho_+\rho_-^{-1}})_* \beta_*^{-1} j_{2*})$, we can rewrite the map between the untwisted K-theory groups as follows:

$$a(x,y) = (x - y, x - \lambda \otimes \varsigma^* y). \tag{4.1}$$

The tensor product corresponds to the product in the usual ring structure in ordinary K-theory, hence we will use the notation $\cdot$ in the following arguments instead of $\otimes$. This will also be convenient for the analogous map in K^{-1}. We also use the notation ς instead of ς' for simplicity in what follows.

Theorem 4.1.1. *Let $M \hookrightarrow X \xrightarrow{\pi} S^1$ be a fiber bundle where M is a compact manifold and X obtained as the mapping torus of a homeomorphism $\varsigma\colon M \to M$. Given a class $[\lambda] \in H^2(M)$ represented by a complex line bundle λ, let $\sigma \in H^3(X)$ be the image of the class of $[\lambda]$ under the inclusion $H^2(M)_\mathbb{Z} \cong H^1(S^1; H^2(M)) \to H^3(X)$. Then the twisted K-theory group ${}^\sigma K^*(X)$, for $* = 0, 1$, is isomorphic to an extension of*

$$\{x \in K^*(M) \mid x = \lambda \cdot \varsigma^* x\} \text{ by } \frac{K^{*+1}(M)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^{*+1}(M)\}}. \tag{4.2}$$

Proof. Using the decomposition $\{D_+, D_-\}$ in terms of closed subspaces of S^1, we obtain the Mayer-Vietoris sequence

$$\begin{tikzcd}
{}^\sigma K^0(X) \arrow[r, "c_0"] & K^0(D_+ \times M) \oplus K^0(D_- \times M) \arrow[r, "a_0"] & K^0(D_{+-} \times M) \arrow[d, "b_0"] \\
K^1(D_{+-} \times M) \arrow[u, "b_1"] & K^1(D_+ \times M) \oplus K^1(D_- \times M) \arrow[l, "a_1"'] & {}^\sigma K^1(X) \arrow[l, "c_1"']
\end{tikzcd}$$

Thus there are group isomorphisms

$$\begin{aligned} {}^\sigma K^*(X) &\cong \frac{K^{*+1}(D_{+-} \times M)}{\mathrm{Im}(a_{*+1})} \oplus_\rho \mathrm{Im}(c_*) \\ &\cong \frac{K^{*+1}(M) \oplus K^{*+1}(M)}{\mathrm{Im}(a_{*+1})} \oplus_\rho \mathrm{Ker}(a_*) \end{aligned} \tag{4.3}$$

where the right side denotes group extensions associated to some cocycle ρ in group cohomology. The map a_* is given by Equation (4.1), that is, if we consider a class $(x, y) \in K^*(D_0 \times M) \oplus K^*(D_1 \times M)$ for $* \in \{0, 1\}$, the gluing maps a_* are given by

$$a_*(x, y) = (x - y, x - \lambda \cdot \varsigma^* y).$$

With this in mind we can rewrite (4.3) as

$$\mathrm{Ker}(a_*) = \{x \in K^*(M) \mid x = \lambda \cdot \varsigma^* x\}$$

Now consider the morphism

$$\frac{K^{*+1}(M) \oplus K^{*+1}(M)}{\mathrm{Im}(a_{*+1})} \longrightarrow \frac{K^{*+1}(M)}{\{z - \lambda \cdot \varsigma^*(z) \mid z \in K^{*+1}(M)\}}$$

that sends the class of (x, y) to the class of $y - x$. This is well-defined since the class of the element (x, x) is sent to the class of 0 and the class of the element $(-y, -\lambda \cdot \varsigma^*(y))$ to the class of the element $y - \lambda \cdot \varsigma^*(y)$, which is the trivial class. It is easy to check that it is bijective and this proves the theorem. □

Note that this result could be described alternatively stating that ${}^\sigma K^*(X)$ is an extension of $\mathrm{Ker}(1 - \lambda \cdot \varsigma^*)$ by $\mathrm{Coker}(1 - \lambda \cdot \varsigma^*)$, where the first $1 - \lambda \cdot \varsigma^*$ is the map defined on $K^*(M)$ and the second one on $K^{*+1}(M)$.

Remark 4.1.2. (Independence of the representative) In the previous theorem we chose a representative λ of an element in $H^2(M)$. If we chose another representative λ', this would be isomorphic to λ as line bundles, hence it would define the same element in $K(M)$. Therefore the computation would yield the same result. This is consistent with Remark 3.2.4, since performing the computation with λ' would amount to computing the twisted K-theory with respect to an isomorphic $PU(\mathcal{H})$–principal bundle.

4.2 Twisted K-theory for 3-manifolds

In this section we use the Atiyah-Hirzebruch spectral sequence to calculate the twisted K-theory of closed connected 3-manifolds X up to extensions. Indeed we will obtain it in terms of cohomology groups.

4.2.1 Orientable 3-manifolds

Proposition 4.2.1. *Let M be a closed, path connected and orientable 3-dimensional manifold and consider $n\omega \in H^3(M;\mathbb{Z})$ where ω is the Poincaré dual of the unit element in $H_0(M;\mathbb{Z})$ and $n \in \mathbb{Z}$. If $n \neq 0$, we have*

$$^{n\omega}K^1(X) \cong H^1(X;\mathbb{Z}) \oplus \mathbb{Z}/n \tag{4.4}$$

$$^{n\omega}K^0(X) \cong H^2(X;\mathbb{Z}) \tag{4.5}$$

Proof. We consider the second page of the Atiyah-Hirzebruch spectral sequence which is given by the formula

$$E_2^{pq} = H^p(X; K^q(\text{point}))$$

See Proposition 4.1 in [4]. Because $K^q(\text{point}) = 0$ for q odd, we have that the even differentials $d_2, d_4, \ldots$ are zero, in particular $E_2^{p,q} = E_3^{p,q}$.

$$\begin{array}{r|ccccc}
3 & 0 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z} & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z} & 0 \\
-3 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Now we need to apply the differential $d_3\colon H^p(X) \to H^{p+3}(X)$, which is given by Proposition 4.6 in [5], namely

$$d_3(x) = Sq^3_{\mathbb{Z}}(x) - n\omega x$$

where $Sq^3_{\mathbb{Z}}\colon H^p(X) \to H^{p+3}(X)$ is the integral Steenrod square. Recall that $Sq^3_{\mathbb{Z}} = \beta Sq^2$, where β is the Bockstein homomorphism associated to the short exact sequence $0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$. In this case, the only differential that could possibly be zero is $d_3\colon H^0(X) \to H^3(X)$. Since Sq^2 is trivial on $H^0(X)$, we have $d_3(1) = n$. Assume $n \neq 0$, then the fourth page has the form:

$$\begin{array}{r|ccccc}
3 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z}/n & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z}/n & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 0 & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z}/n & 0 \\
-3 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Due to the fact that $d_n\colon E_n^{p,q} \to E_n^{p+n,q-n+1}$ advances at least four terms to the right if $n \geq 4$, we obtain $E_4^{p,q} = E_\infty^{p,q}$. To calculate ${}^{n\omega}K^1(X)$, we use

$$\begin{aligned} 0 = E_\infty^{0,1} &= F^0K^1(X)/F^1K^1(X) \\ H^1(X;\mathbb{Z}) = E_\infty^{1,0} &= F^1K^1(X)/F^2K^1(X) \\ 0 = E_\infty^{2,-1} &= F^2K^1(X)/F^3K^1(X) \\ \mathbb{Z}/n = E_\infty^{3,-2} &= F^3K^1(X)/F^4K^1(X) \end{aligned}$$

Because $F^4K^1(X) = 0$, we obtain $F^2K^1 = \mathbb{Z}/n$ and $F^1K^1 = K^1$, thus we have an exact sequence

$$0 \to \mathbb{Z}/n \to {}^{n\omega}K^1(X) \to H^1(X;\mathbb{Z}) \to 0 \tag{4.6}$$

Since $H^1(X;\mathbb{Z})$ is finitely generated and torsion-free, it is free abelian, hence the sequence splits. In the same way

$${}^{n\omega}K^0(X) \cong H^2(X;\mathbb{Z}). \tag{4.7}$$

And this finishes the proof. □

4.2.2 Non-orientable 3-manifolds

On the other hand, for the non-orientable case, first we need to describe the group $H^3(X;\mathbb{Z})$. Using Poincare Duality for non-orientable manifolds (see for instance Theorem 3H.6 in [27]), $H^3(X,\mathbb{Z}) \cong H_0(X;\underline{\mathbb{Z}})$, where $\underline{\mathbb{Z}}$ indicates that this is homology with local coefficients, more precisely, it is the coefficient system where the class of a loop in $\pi_1(X)$ acts trivially if it preserves orientation and by multiplication by -1 otherwise. It is well-known that $H_0(X;\underline{\mathbb{Z}})$ are the coinvariants of $\pi_1(X)$ acting over $\mathbb{Z}$. Since we are assuming that X is non-orientable, there is at least an element $g \in \pi(X)$ which does not preserve orientation and so we have:

$$H_0(X;\underline{\mathbb{Z}}) \cong \frac{\mathbb{Z}}{\langle n - g(n)\rangle} \cong \frac{\mathbb{Z}}{\langle n - (-n)\rangle} \cong \mathbb{Z}/2$$

Proposition 4.2.2. *Let M be a closed, path-connected and non-orientable 3-dimensional manifold and let $\omega \in H^3(M;\mathbb{Z})$ be the non-trivial class. Then*

$${}^{\omega}K^0(X) \cong \mathbb{Z} \oplus H^2(X;\mathbb{Z}) \tag{4.8}$$

$${}^{\omega}K^1(X) \cong H^1(X;\mathbb{Z}) \tag{4.9}$$

Proof. Similarly to the case of orientable manifolds, we will use the Atiyah-Hirzebruch spectral sequence, only in this case the second page has the form

$$\begin{array}{r|ccccc} 3 & 0 & 0 & 0 & 0 & 0 \\ 2 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z}/2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z}/2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ -2 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \mathbb{Z}/2 & 0 \\ -3 & 0 & 0 & 0 & 0 & 0 \end{array}$$

We again have $E_2 = E_3$ and we need to compute the differential $d_3\colon H^0(X) \to H_3(X)$. Again we have $d_3(x) = Sq^3_{\mathbb{Z}}(x) - \omega x$, but $Sq^3_{\mathbb{Z}}$ vanishes on $H^0(X)$, hence $d_3(1) = \omega$ and ω represents the nontrivial element of $\mathbb{Z}/2$.

$$\begin{array}{r|ccccc}
3 & 0 & 0 & 0 & 0 & 0 \\
2 & 2\mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \frac{\mathbb{Z}/2}{\mathbb{Z}/2} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \frac{\mathbb{Z}/2}{\mathbb{Z}/2} & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-2 & 2\mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & \frac{\mathbb{Z}/2}{\mathbb{Z}/2} & 0 \\
-3 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Due to the fact that $d_n\colon E_n^{p,q} \to E_n^{p+n,q-n+1}$ advances at least four terms to the right if $n \geq 4$, we obtain $E_4^{p,q} = E_\infty^{p,q}$, hence $E_\infty^{p,q}$ has the form

$$\begin{array}{r|ccccc}
3 & 0 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & H^1(X;\mathbb{Z}) & H^2(X;\mathbb{Z}) & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 0
\end{array}$$

To calculate ${}^{\omega}K^0(X)$, we use

$$\begin{aligned}
\mathbb{Z} = E_\infty^{0,0} &= F^0K^0(X)/F^1K^0(X) \\
0 = E_\infty^{1,-1} &= F^1K^0(X)/F^2K^0(X) \\
H^2(X;\mathbb{Z}) = E_\infty^{2,-2} &= F^2K^0(X)/F^3K^0(X) \\
0 = E_\infty^{3,-3} &= F^3K^1(X)/F^4K^0(X)
\end{aligned}$$

Because $F^4K^0(X) = 0$, we obtain $F^1K^0 = H^2(X;\mathbb{Z})$, thus we have an exact sequence

$$0 \to H^2(X;\mathbb{Z}) \to {}^{\omega}K^0(X) \to \mathbb{Z} \to 0 \tag{4.10}$$

which splits. In the same way

$${}^{\omega}K^1(X) \cong H^1(X;\mathbb{Z}). \tag{4.11}$$

And this concludes the proof. □

4.3 Computations and comparisons

In this section we compute the twisted K-theory of some 3-manifolds using the two methods shown in the previous sections for proper contrast.

4.3.1 The case of $S^1 \times S^2$

Using Proposition 4.2.1, if $n \neq 0$ we have

$$^{n\omega}K^0(S^1 \times S^2) \cong H^2(S^1 \times S^2; \mathbb{Z})$$

$$^{n\omega}K^1(S^1 \times S^2) \cong H^1(S^1 \times S^2; \mathbb{Z}) \oplus \mathbb{Z}/n$$

To put it explicitly, we will apply the Künneth formula to obtain $H^1(S^1 \times S^2; \mathbb{Z})$ and $H^2(S^1 \times S^2; \mathbb{Z})$

$$H^1(S^1 \times S^2; \mathbb{Z}) \cong H^0(S^2; \mathbb{Z}) \otimes H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}$$

$$H^2(S^1 \times S^2; \mathbb{Z}) \cong H^0(S^1; \mathbb{Z}) \otimes H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}.$$

Then

$$\begin{aligned} ^{n\omega}K^0(S^1 \times S^2) &\cong \mathbb{Z}, \\ ^{n\omega}K^1(S^1 \times S^2) &\cong \mathbb{Z} \oplus \mathbb{Z}/n. \end{aligned} \tag{4.12}$$

Now we will calculate twisted K-theory using Theorem 4.1.1. First we need to understand

$$\{x \in K^*(S^2) \mid x = \lambda \cdot \varsigma^* x\} \text{ and } \frac{K^{*+1}(S^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^{*+1}(S^2)\}}.$$

Initially we will consider it for K^0. The K-theory ring for S^2 is $\frac{\mathbb{Z}[t]}{(t-1)^2}$ (see Corollary 2.3 in [26]), so any element has the form $x_1 + x_2 t$ or equivalently (x_1, x_2), where 1 represents the rank 1 trivial bundle and t the tautological lineal bundle L. Then

$$\{(x_1, x_2) \in K^0(S^2) \mid (x_1, x_2) = \lambda \cdot \varsigma^*(x_1, x_2)\}$$

We choose $\lambda = L^n \in H^2(S^2; \mathbb{Z})$, then $L = (0, 1)$ and $L^n = (1-n, n)$. In this case ς is the identity map, hence

$$\lambda(x_1, x_2) = ((1-n)x_1 - nx_2, (1-n)x_2 + nx_1 + 2nx_2), \tag{4.13}$$

thus we need to solve

$$\begin{aligned} x_1 &= (1-n)x_1 - nx_2 \\ x_2 &= (1-n)x_2 + nx_1 + 2nx_2 \end{aligned}$$

and we obtain $x_1 = -x_2$, hence

$$\{x \in K^0(S^2) \mid x = \lambda \cdot \varsigma^* x\} \cong \mathbb{Z}.$$

On the other hand, for the second term in (1.1) we have $(x_1, x_2) - \lambda\varsigma^*(x_1, x_2) = (n(x_1 + x_2), -n(x_1 + x_2))$. Now

$$\frac{\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(n,-n)} = \frac{\mathbb{Z}(1,-1) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(n,-n)}$$

Therefore

$$\frac{K^0(S^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^0(S^2)\}} \cong = \mathbb{Z} \oplus \mathbb{Z}/n$$

Due to the fact that $K^1(S^2) = 0$ the terms in (1.1) related to it are trivial. Then by Theorem 4.1.1

$$\begin{aligned} 0 \to \mathbb{Z}/n \oplus \mathbb{Z} \to {}^{\sigma_n}K^1(X) \to 0 \to 0 \\ 0 \to 0 \to {}^{\sigma_n}K^0(X) \to \mathbb{Z} \to 0 \end{aligned} \tag{4.14}$$

Finally we obtain

$$\begin{aligned} {}^{\sigma_n}K^0(S^1 \times S^2) \cong \mathbb{Z}, \\ {}^{\sigma_n}K^1(S^1 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}/n. \end{aligned} \tag{4.15}$$

where $\sigma_n \in H^3(S^1 \times S^2; \mathbb{Z})$ is the class that corresponds to L_n, which in turn corresponds to $n \in H^2(S^2; \mathbb{Z})$.

4.3.2 The case of $\mathbb{R} \times_\rho S^2$

Let $\rho\colon \pi_1(S^1) \times S^2 \to S^2$ be the action given by $\rho(z, x) = x$ if z is even and $\rho(z, x) = -x$ if z is odd. We consider the space $X = \mathbb{R} \times_\rho S^2$, which is a nonorientable 3-manifold.

First of all we will calculate the cohomology groups of X. Using the Serre spectral sequence, we obtained in a previous section the Equations 3.1, 3.2, 3.3, which we repeat here for convenience. Note that they are cohomology groups with local coefficients.

$$0 \to H^1(S^1; H^0(S^2)) \to H^1(X) \to H^0(S^1; H^1(S^2)) \to 0$$

$$0 \to H^1(S^1; H^1(S^2)) \to H^2(X) \to H^0(S^1; H^2(S^2)) \to 0$$

$$0 \to H^1(S^1; H^2(S^2)) \to H^3(X) \to H^0(S^1; H^3(S^2)) \to 0.$$

The groups $H^0(S^1; H^1(S^2))$, $H^1(S^1; H^1(S^2))$, $H^0(S^1; H^3(S^2))$ vanish. On the other hand $H^0(S^1; H^2(S^2))$ are the invariants of $H^2(S^2)$ under the action of $\pi_1(S^1)$. Nevertheless, this group vanishes as well. $H^1(S^1; H^0(S^2))$ and $H^1(S^1, H^2(S^2))$ are the coinvariants of $\pi_1(S^1)$ acting over $H^0(S^2)$ and $H^2(S^2)$ respectively, so we have

$$H^0(S^2)_{\pi_1(S^1)} \cong \frac{\mathbb{Z}}{\langle n - g(n) \rangle} \cong \frac{\mathbb{Z}}{\langle n - (n) \rangle} \cong \mathbb{Z}$$

$$H^2(S^2)_{\pi_1(S^1)} \cong \frac{\mathbb{Z}}{\langle n - g(n) \rangle} \cong \frac{\mathbb{Z}}{\langle n - (-n) \rangle} \cong \frac{\mathbb{Z}}{2\mathbb{Z}} \cong \mathbb{Z}/2$$

where $g \in \pi_1(S^1)$. Therefore, $H^1(X; \mathbb{Z}) \cong \mathbb{Z}$, $H^2(X; \mathbb{Z}) = 0$, $H^3(X; \mathbb{Z}) = \mathbb{Z}/2$. To twist K-theory we will use the generator $\sigma \in H^3(X; \mathbb{Z})$, hence using

Proposition 4.2.2 we obtain ${}^{\sigma}K^1(X) \cong H^1(X;\mathbb{Z})$ and ${}^{\sigma}K^0(X) \cong H^2(X;\mathbb{Z}) \oplus \mathbb{Z}$. So finally we have

$$\begin{aligned} {}^{\sigma}K^1(X) &\cong \mathbb{Z} \\ {}^{\sigma}K^0(X) &\cong \mathbb{Z} \end{aligned}$$

Now we will calculate twisted K-theory using Theorem 4.1.1. The fiber of $X \to S^1$ is S^2. First we need to understand

$$\{x \in K^*(S^2) \mid x = \lambda \cdot \varsigma^* x\} \text{ and } \frac{K^{*+1}(S^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^{*+1}(S^2)\}}.$$

In the same way as in the previous example, the fiber S^2 has K-theory ring $\dfrac{\mathbb{Z}[t]}{(t-1)^2}$, hence any element has a form $x_1 + x_2 t$ or equivalently (x_1, x_2), where 1 represent the rank 1 trivial bundle and t the tautological lineal bundle L, then we will determine

$$\{(x_1, x_2) \in K^0(S^2) \mid (x_1, x_2) = \lambda \cdot \varsigma^*(x_1, x_2)\}$$

We choose a generator $\lambda \in H^2(S^2;\mathbb{Z})$, then λ represents $(0,1)$ in K-theory. Next ς acts via the antipodal map on S^2 so it takes the class of the tautological vector bundle t to $2 - t$ in K-theory. Hence $\varsigma^*(x_1 + x_2 t) = x_1 + x_2(2-t) = (x_1 + 2x_2, -x_2)$. Thus

$$\lambda\varsigma^*(x_1, x_2) = \lambda(x_1 + 2x_2, -x_2) = (0,1)(x_1 + 2x_2, -x_2) = (x_2, x_1)$$

. Therefore

$$\{(x_1, x_2) \in K^0(M) \mid (x_1, x_2) = \lambda \cdot \varsigma^*(x_1, x_2)\} = \{(x_1, x_2) \in K^0(M) \mid x_1 = x_2\} \cong \mathbb{Z}.$$

On the other hand, for the second term in (1.1) we have

$$(x_1, x_2) - \lambda\varsigma^*(x_1, x_2) = (x_1 - x_2, x_2 - x_1),$$

Therefore

$$\frac{K^0(S^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^0(S^2)\}} \cong \frac{\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(1,-1)} = \frac{\mathbb{Z}(1,-1) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(1,-1)}$$

Thus this other terms is isomorphic to $= \mathbb{Z}$. Due to the fact that $K^1(S^2) = 0$ the terms in (1.1) related to it are null. Then by Theorem 4.1.1 we have:

$$\begin{aligned} 0 \to \mathbb{Z} \to {}^{\sigma}K^1(X) \to 0 \to 0 \\ 0 \to 0 \to {}^{\sigma}K^0(X) \to \mathbb{Z} \to 0 \end{aligned} \tag{4.16}$$

And so finally

$$\begin{aligned} {}^{\sigma}K^0(\mathbb{R} \times_\rho S^2) &\cong \mathbb{Z}, \\ {}^{\sigma}K^1(\mathbb{R} \times_\rho S^2) &\cong \mathbb{Z}. \end{aligned} \tag{4.17}$$

where $\sigma \in H^3(\mathbb{R} \times_\rho S^2;\mathbb{Z})$ is the non-trivial class, which comes from the generator of $H^2(S^2;\mathbb{Z})$ corresponding to $\lambda = L$.

4.3.3 The case of $S^1 \times \mathbb{T}^2$

We use here the notation $\mathbb{T} = S^1$ to distinguish the other two factors in $X = S^1 \times \mathbb{T}^2$. Using Proposition 4.2.1, we have

$$^{n\omega}K^0(S^1 \times \mathbb{T}^2) \cong H^2(S^1 \times \mathbb{T}^2; \mathbb{Z})$$

$$^{n\omega}K^1(S^1 \times \mathbb{T}^2) \cong H^1(S^1 \times \mathbb{T}^2; \mathbb{Z}) \oplus \mathbb{Z}/n$$

To put it explicitly we will apply the Künneth formula to obtain $H^1(S^1 \times \mathbb{T}^2; \mathbb{Z})$ and $H^2(S^1 \times \mathbb{T}^2; \mathbb{Z})$.

$$H^1(S^1 \times \mathbb{T}^2; \mathbb{Z}) \cong H^0(S^1; \mathbb{Z}) \otimes H^1(\mathbb{T}^2; \mathbb{Z}) \oplus H^1(S^1; \mathbb{Z}) \otimes H^0(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}^{\oplus 3}$$

$$H^2(S^1 \times \mathbb{T}^2; \mathbb{Z}) \cong H^0(S^1; \mathbb{Z}) \otimes H^2(\mathbb{T}^2; \mathbb{Z}) \oplus H^1(S^1; \mathbb{Z}) \otimes H^1(\mathbb{T}^2; \mathbb{Z}) \cong \mathbb{Z}^{\oplus 3}$$

Then

$$\begin{aligned} ^{n\omega}K^0(S^1 \times \mathbb{T}^2) &\cong \mathbb{Z}^{\oplus 3}, \\ ^{n\omega}K^1(S^1 \times \mathbb{T}^2) &\cong \mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/n. \end{aligned} \tag{4.18}$$

Initially we will calculate $K^*(\mathbb{T}^2)$ using Künneth formula for K-theory (see Lemma 1 in [1]).

$$K^*(S^1) \otimes K^*(S^1) \cong K^*(S^1 \times S^1)$$

Here $\otimes$ indicates the $\mathbb{Z}/2$–graded tensor product, that is:

$$K^0(S^1) \otimes K^0(S^1) \oplus K^1(S^1) \otimes K^1(S^1) \cong K^0(S^1 \times S^1)$$

$$K^0(S^1) \otimes K^1(S^1) \oplus K^1(S^1) \otimes K^0(S^1) \cong K^1(S^1 \times S^1)$$

Analyzing each component, denote by $p_i \colon X_1 \times X_2 \to X_i$ the projection maps.

$$\begin{aligned} K^*(S^1) \otimes K^*(S^1) \to & K^*(S^1 \times S^1) \\ x \otimes y \mapsto & \, p_1^*(x) \otimes p_2^*(x) \end{aligned} \tag{4.19}$$

A more explicitly description follow from two facts, first the identity element e of $K^*(S^1 \times S^1)$ belongs $K^0(S^1) \otimes K^0(S^1)$. Second, for any two classes a, b in $K^1(S^1)$ we have $a \cdot b = 0$. This last fact is obtained from the isomorphism

$$K^1(S^1) \cong \widetilde{K}^0(S^2) \cong \widetilde{H}^2(S^2; \mathbb{Z})$$

which is multiplicative, since it comes from the Chern character (see Proposition 4.3 in [26]). If we regard $K^0(S^1 \times S^1)$ as the direct sum coming from the Künneth theorem above, given elements of the form $[(a_0 \otimes b_0), (a_1 \otimes b_1)]$ and $[(c_0 \otimes d_0), (c_1 \otimes d_1)]$ in $K^0(S^1 \times S^1)$ we have:

$$\begin{aligned} & [(a_0 \otimes b_0), (a_1 \otimes b_1)] \cdot [(c_0 \otimes d_0), (c_1 \otimes d_1)] \\ =& [(a_0c_0 \otimes b_0d_0), (a_0c_1 \otimes b_0d_1)] + [-(a_1c_1 \otimes b_1d_1), (a_1c_0 \otimes b_1d_0)] \\ =& [(a_0c_0 \otimes b_0d_0), (a_0c_1 \otimes b_0d_1)] + [(0, 0), (a_1c_0 \otimes b_1d_0)] \\ =& [(a_0c_0 \otimes b_0d_0), (a_0c_1 \otimes b_0d_1) + (a_1c_0 \otimes b_1d_0)] \end{aligned}$$

Then $K^0(S^1 \times S^1) \cong \frac{\mathbb{Z}[t]}{(t^2)}$, with $1 = [(1 \otimes 1), (0 \otimes 0)]$ and $t = [(0 \otimes 0), (1 \otimes 1)]$. Similarly, given$[(a_0 \otimes b_0), (a_1 \otimes b_1)] \in K^0(S^1 \times S^1)$ and $[(c_0 \otimes d_1), (c_1 \otimes d_0)] \in K^1(S^1 \times S^1)$ we have:

$$\begin{aligned} &[(a_0 \otimes b_0), (a_1 \otimes b_1)] \cdot [(c_0 \otimes d_1), (c_1 \otimes d_0)] \\ =&[(a_0c_0 \otimes b_0d_1), (a_0c_1 \otimes b_0d_0)] + [-(a_1c_1 \otimes b_1d_0), (a_1c_0 \otimes b_1d_1)] \\ =&[(a_0c_0 \otimes b_0d_1), (a_0c_1 \otimes b_0d_0)] + [-(0 \otimes b_1d_0), (a_1c_0 \otimes 0)] \\ =&[(a_0c_0 \otimes b_0d_1), (a_0c_1 \otimes b_0d_0)] \in K^1(S^1 \times S^1) \end{aligned} \tag{4.20}$$

The class of a vector bundle over $S^1 \times S^1$ has the shape $\alpha t + \beta \in K^0(S^1 \times S^1)$ with α, $\beta \in \mathbb{Z}$. Due to the fact that $H^2(S^1 \times S^1; \mathbb{Z}) \cong \mathbb{Z}$ there is a generator line bundle L, our next step is to find the polynomial that represents L. If we denote by 1_{S^1} the trivial line bundle over S^1, from Equation (4.19) we get that $1 = p_1^*(1_{S^1}) \otimes p_1^*(1_{S^1})$, so 1 represents the trivial line bundle over $S^1 \times S^1$.

From the cofibration sequence $S^1 \vee S^1 \xrightarrow{i} S^1 \times S^1 \xrightarrow{q} S^2$ we get a long exact sequence

$$\begin{tikzcd} \widetilde{K}^0(S^2) \arrow[r, "q^*"] & \widetilde{K}^0(S^1 \times S^1) \arrow[r, "i^*"] & \widetilde{K}^0(S^1 \vee S^1) \arrow[d] \\ \widetilde{K}^1(S^1 \vee S^1) \arrow[u] & \widetilde{K}^1(S^1 \times S^1) \arrow[l] & \widetilde{K}^1(S^2) \arrow[l] \end{tikzcd} \tag{4.21}$$

hence an exact sequence

$$\widetilde{K}^0(S^2) \cong \mathbb{Z} \xrightarrow[\cong]{q^*} \widetilde{K}^0(S^1 \times S^1) \cong \mathbb{Z} \xrightarrow{i^*} \widetilde{K}^0(S^1 \vee S^1) = 0 \tag{4.22}$$

It is known that $K^0(S^2) \cong \frac{\mathbb{Z}(x)}{(x-1)^2}$ and $\widetilde{K}^0(S^2)$ is the ideal generated by $x - 1$. Then $\widetilde{K}^0(S^1 \times S^1) = \langle t \rangle$, that means $q^*(x - 1) = t$, so the virtual dimension of t is zero. Additionally it induces an isomorphism $K^0(S^2) \simeq K^0(S^1 \times S^1)$ of groups. With this in mind our candidate to represent the class of line bundle generator is $t + 1$. Indeed the multiplicative subgroup $K^0(S^1 \times S^1)_{dim=1}$ of one-dimensional virtual bundles classes is generated by $t + 1$ and isomorphic to $\mathbb{Z}$, because $(t + 1)^n = nt + 1$ for $n \in \mathbb{Z}$. Now

$$\begin{aligned} \mathbb{Z} \cong \mathrm{Vect}_{\mathbb{C}}(S^2) \xrightarrow{\cong} & K^0(S^2)_{dim=1} \hookrightarrow K^0(S^2) \\ L \mapsto & x \mapsto x \end{aligned} \tag{4.23}$$

From this we get a ring isomorphism

$$\begin{aligned} K^0(S^2) \xrightarrow{\cong} & K^0(S^1 \times S^1) \\ x \mapsto & t + 1 \\ 1 \mapsto & 1. \end{aligned} \tag{4.24}$$

Hence $t+1$ represent the class of the line bundle generator L over $S^1 \times S^1$. In fact there is another isomorphism with $x \mapsto -t+1$, but a change of variable would lead to our choice.

Now to calculate ${}^{\sigma}K(S^1 \times \mathbb{T}^2)$ we use Theorem 4.1.1 with $\varsigma = id$. We need

$$\{x \in K^*(\mathbb{T}^2) \mid x = \lambda \cdot x\} \text{ and } \frac{K^{*+1}(\mathbb{T}^2)}{\{y - \lambda \cdot y \mid y \in K^{*+1}(\mathbb{T}^2)\}}.$$

Initially we will consider it for K^0. The K-theory ring of the fiber T^2 is $\dfrac{\mathbb{Z}[t]}{(t^2)}$, so any element has the form $x_1 + x_2 t$ or equivalently (x_1, x_2). Then we need to determine

$$\{(x_1, x_2) \in K^0(M) \mid (x_1, x_2) = \lambda \cdot (x_1, x_2)\}$$

We choose $\lambda = L^n \in H^2(\mathbb{T}^2)$, then $\lambda(x_1, x_2) = (1, n)(x_1, x_2) = (x_1, nx_1 + x_2)$. Thus we need to solve

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= nx_1 + x_2 \end{aligned}$$

from where $x_1 = 0$ and the desired group is

$$\{(x_1, x_2) \in K^0(\mathbb{T}^2) \mid x_1 = 0\} \cong \mathbb{Z}$$

For the other group, we compute

$$(x_1, x_2) - \lambda(x_1, x_2) = (0, -nx_1)$$

Therefore

$$\frac{K^0(\mathbb{T}^2)}{\{y - \lambda \cdot y \mid y \in K^0(\mathbb{T}^2)\}} \cong \frac{\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(0,-n)} \cong \mathbb{Z}/n \oplus \mathbb{Z}$$

Now we will describe it for K^1. For this purpose, it is advantageous to use the notation $\lambda = (1 \otimes 1, n \otimes 1)$ and $(x_1, x_2) = [(a_0 \otimes b_1), (a_1 \otimes b_0)]$ coming from Künneth theorem for K-theory.

$$\{(x_1, x_2) \in K^1(\mathbb{T}^2) \mid (x_1, x_2) = \lambda \cdot (x_1, x_2)\}$$

In Equation (4.20) we showed that

$$\lambda(x_1, x_2) = (1 \otimes 1, n \otimes 1)[(c_0 \otimes d_1), (c_1 \otimes d_0)] = [(c_0 \otimes d_1), (c_1 \otimes d_0)] = (x_1, x_2)$$

Thus $\{(x_1, x_2) \in K^1(\mathbb{T}^2) \mid (x_1, x_2) = \lambda(x_1, x_2)\} \cong \mathbb{Z} \oplus \mathbb{Z}$. On the other hand, for the second term, we have $(x_1, x_2) - \lambda(x_1, x_2) = (0, 0)$, thus

$$\frac{K^1(\mathbb{T}^2)}{\{y - \lambda \cdot y \mid y \in K^1(\mathbb{T}^2)\}} = K^1(\mathbb{T}^2) = 0$$

Then by Theorem 4.1.1, we have:

$$\begin{aligned} 0 \to \mathbb{Z}/n \oplus \mathbb{Z} \to {}^{\sigma_n}K^1(S^1 \times \mathbb{T}^2) \to \mathbb{Z} \oplus \mathbb{Z} \to 0 \\ 0 \to \mathbb{Z} \oplus \mathbb{Z} \to {}^{\sigma_n}K^0(S^1 \times \mathbb{T}^2) \to \mathbb{Z} \to 0 \end{aligned} \tag{4.25}$$

Finally

$$\begin{aligned} {}^{\sigma_n}K^0(S^1 \times \mathbb{T}^2) \cong \mathbb{Z}^{\oplus 3}, \\ {}^{\sigma_n}K^1(S^1 \times \mathbb{T}^2) \cong \mathbb{Z}^{\oplus 3} \oplus \mathbb{Z}/n. \end{aligned} \tag{4.26}$$

where $\sigma_n \in H^3(S^1 \times \mathbb{T}^2; \mathbb{Z})$ is the class that corresponds to $\lambda = L^n \in H^2(\mathbb{T}^2; \mathbb{Z})$.

4.3.4 The case of $\mathbb{R} \times_{\rho'} \mathbb{T}^2$

Let $\rho' \colon \pi_1(S^1) \times \mathbb{T}^2 \to \mathbb{T}^2$ be the action determined by

$$\rho'(1, (x, y)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} (x, y) = (x^a y^c, x^b y^d)$$

where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ belongs to $GL_2(\mathbb{Z})$. That is:

$$\rho'(z, (x, y)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^z (x, y) = \begin{bmatrix} a_z & b_z \\ c_z & d_z \end{bmatrix} (x, y) = (x^{a_z} y^{c_Z}, x^{b_z} y^{d_z})$$

Here $x = e^{i\theta_1}$ and $y = e^{i\theta_2}$. We will compute the twisted K-theory of $X = \mathbb{R} \times_{\rho'} \mathbb{T}^2$ with respect to certain twists.

First of all we will calculate the different cohomology groups of X. Using the short exact sequences obtained with the Serre spectral sequence in a previous section (Equations (3.1), (3.2), (3.3)), we have:

$$0 \to H^1(S^1; H^0(\mathbb{T}^2)) \to H^1(X) \to H^0(S^1; H^1(\mathbb{T}^2)) \to 0$$

$$0 \to H^1(S^1; H^1(\mathbb{T}^2)) \to H^2(X) \to H^0(S^1; H^2(\mathbb{T}^2)) \to 0$$

$$0 \to H^1(S^1; H^2(\mathbb{T}^2)) \to H^3(X) \to H^0(S^1; H^3(\mathbb{T}^2)) \to 0.$$

where we are using cohomology with local coefficients. We have $H^0(S^1; H^3(\mathbb{T}^2)) = 0$. On the other hand, $H^0(S^1; H^1(\mathbb{T}^2))$ and $H^0(S^1; H^2(\mathbb{T}^2))$ are the invariants of the action of $\pi_1(S^1)$ over $H^1(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$. To calculate them, let us first see what the action looks like. By identifying $H_1(\mathbb{T}^2) \cong \pi_1(\mathbb{T}^2$ and using the standard generators of this fundamental groups, we obtain the action of $\mathbb{Z}$ over $H_1(\mathbb{T}^2) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$

$$1 \cdot (r, s) = 1 \cdot (r\alpha + s\beta) = r(a\alpha + b\beta) + s(c\alpha + d\beta) = (ra + sc, rb + sd)$$

We have the dual action on $H^1(\mathbb{T}^2) = \mathbb{Z}\rho \oplus \mathbb{Z}\mu$, since $H^1(\mathbb{T}^2) \cong \mathrm{Hom}(H_1(\mathbb{T}^2), \mathbb{Z})$. Then

$$1 \cdot (r, s) = 1 \cdot (r\rho + s\mu) = r(a\rho + c\mu) + s(b\rho + d\mu) = (ra + sb, rc + sd)$$

The latter can be interpreted as:

$$1 \cdot (r, s) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = (ra + sb, rc + sd)$$

The action on $H^2(\mathbb{T}^2) = \mathbb{Z}\rho\mu$ is given by:

$$1 \cdot (\rho, \mu) = 1 \cdot (a\rho + c\mu)(b\rho + d\mu) = ad\rho\mu + cb\mu\rho = (ad - cb)\rho\mu = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \rho\mu$$

For $H^1(\mathbb{T}^2)$, the non-zero invariants are the eigenvectors of the eigenvalue 1 for M. We describe the different cases:

- If $\det(M) = 1$ and $a + d \neq 2$, then $H^1(\mathbb{T}^2)^{\mathbb{Z}} = 0$.
- If $\det(M) = 1$ and $a + d = 2$, then $H^1(\mathbb{T}^2)^{\mathbb{Z}} = \mathbb{Z}$.
- If $\det(M) = -1$ and $a \neq -d$, then $H^1(\mathbb{T}^2)^{\mathbb{Z}} = 0$.
- If $\det(M) = -1$ and $a = -d$, then $H^1(\mathbb{T}^2)^{\mathbb{Z}} = \mathbb{Z}$.

For $H^2(\mathbb{T}^2)$ the invariants are

- If $\det(M) = -1$, then $H^2(\mathbb{T}^2)^{\mathbb{Z}} = 0$.
- If $\det(M) = 1$, then $H^2(\mathbb{T}^2)^{\mathbb{Z}} = \mathbb{Z}$.

The groups $H^1(S^1; H^0(\mathbb{T}^2))$, $H^1(S^1; H^1(\mathbb{T}^2))$ and $H^1(S^1; H^2(\mathbb{T}^2))$ are the coinvariants of the action of $\pi_1(S^1)$ over $H^0(\mathbb{T}^2)$, $H^1(\mathbb{T}^2)$ and $H^2(\mathbb{T}^2)$, respectively, so we have:

$$H^1(S^1; H^0(\mathbb{T}^2)) \cong \frac{\mathbb{Z}}{\langle n - g(n) \rangle} \cong \frac{\mathbb{Z}}{\langle n - n \rangle} \cong \mathbb{Z}$$

$$\begin{aligned} H^1(S^1; H^1(\mathbb{T}^2)) &\cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (n, m) - g(n, m) \rangle} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (n, m) - (an + bm, cn + dm) \rangle} \\ &\cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (1 - a, c)n - (-b, 1 - d)m \rangle} \cong H^1(\mathbb{T}^2)_{\mathbb{Z}} \end{aligned}$$

$$H^1(S^1; H^2(\mathbb{T}^2)) \cong \begin{cases} \dfrac{\mathbb{Z}}{\langle n - g(n) \rangle} \cong \dfrac{\mathbb{Z}}{\langle n - n \rangle} \cong \mathbb{Z} & \text{if } \det(M) = 1 \\[2ex] \dfrac{\mathbb{Z}}{\langle n - g(n) \rangle} \cong \dfrac{\mathbb{Z}}{\langle n - (-n) \rangle} \cong \mathbb{Z}/2 & \text{if } \det(M) = -1 \end{cases}$$

where g belongs to $\pi(S^1)$. Therefore when $\det(M) = 1$, we have:

$$H^j(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } j = 1 \text{ and } a+d = 2 \\ \mathbb{Z} & \text{if } j = 1 \text{ and } a+d \neq 2 \\ \mathbb{Z}\oplus H^1(\mathbb{T}^2)_{\mathbb{Z}} & \text{if } j = 2 \\ \mathbb{Z} & \text{if } j = 3 \\ 0 & \text{if } j \geq 4 \end{cases}$$

In particular, X is an orientable manifold when $\det(M) = 1$. When $\det(M) = -1$, we have:

$$H^j(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } j = 0 \\ \mathbb{Z}\oplus\mathbb{Z} & \text{if } j = 1 \text{ and } a = -d \\ \mathbb{Z} & \text{if } j = 1 \text{ and } a \neq -d \\ H^1(\mathbb{T}^2)_{\mathbb{Z}} & \text{if } j = 2 \\ \mathbb{Z}/2 & \text{if } j = 3 \\ 0 & \text{if } j \geq 4 \end{cases}$$

and X is a nonorientable manifold.

When $\det(M) = -1$, we twist K-theory using the generator $\sigma \in H^3(X;\mathbb{Z})$. Since X is a closed nonorientable manifold, we will be using Proposition 4.2.2 first. We have ${}^{\sigma}K^1(X) \cong H^1(X;\mathbb{Z})$ and ${}^{\sigma}K^0(X) \cong H^2(X;\mathbb{Z}) \oplus \mathbb{Z}$, hence

$${}^{\sigma}K^1(\mathbb{R}\times_\rho \mathbb{T}^2) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & \text{if } a = -d \\ \mathbb{Z} & \text{if } a \neq -d \end{cases}$$

$${}^{\sigma}K^0(\mathbb{R}\times_\rho \mathbb{T}^2) \cong \mathbb{Z}\oplus H^1(\mathbb{T}^2)_{\mathbb{Z}}$$

When $\det(M) = 1$, we twist K-theory using $n\sigma \in H^3(X;\mathbb{Z})$ where σ is a generator. Since X is a closed orientable manifold, we will be using Proposition 4.2.1. We have ${}^{\sigma}K^1(X) \cong H^1(X;\mathbb{Z}) \oplus \mathbb{Z}/n$ and ${}^{\sigma}K^0(X) \cong H^2(X;\mathbb{Z})$, hence

$${}^{n\sigma}K^1(\mathbb{R}\times_\rho \mathbb{T}^2) \cong \begin{cases} \mathbb{Z}^{\oplus 2}\oplus\mathbb{Z}/n & \text{if } a+d = 2 \\ \mathbb{Z}\oplus\mathbb{Z}/n & \text{if } a+d \neq 2 \end{cases}$$

$${}^{n\sigma}K^0(\mathbb{R}\times_\rho \mathbb{T}^2) \cong \mathbb{Z}\oplus H^1(\mathbb{T}^2)_{\mathbb{Z}}$$

Now we will calculate ${}^{\sigma}K^*(\mathbb{R}\times_\rho\mathbb{T}^2)$ using Theorem 4.1.1 when $\det(M) = -1$. So initially we need

$$\{x \in K^*(\mathbb{T}^2) \mid x = \lambda\cdot\varsigma^* x\} \text{ and } \frac{K^{*+1}(\mathbb{T}^2)}{\{y - \lambda\cdot\varsigma^*(y) \mid y \in K^{*+1}(\mathbb{T}^2)\}}.$$

In the same way from previous example the fiber $\mathbb{T}^2$ has K-theory ring $\dfrac{\mathbb{Z}[t]}{(t^2)}$. We start determining

$$\{(x_1, x_2) \in K^0(\mathbb{T}^2) \mid (x_1, x_2) = \lambda\otimes\varsigma^*(x_1, x_2)\}$$

We choose the generator $\lambda = L \in H^2(\mathbb{T}^2;\mathbb{Z})$ which corresponds to $\lambda = (1,1)$ in $K^0(\mathbb{T}^2)$. Next, since ς acts reversing the orientation of $\mathbb{T}^2$, it takes the generator line bundle $t+1$ to $-t+1$. Then

$$\varsigma^*(x_1 + x_2 t) = \varsigma^*(x_1 - x_2 + x_2(t+1)) = x_1 - x_2 + x_2(-t+1) = (x_1, -x_2)$$

Hence

$$\lambda\varsigma^*(x_1,x_2) = \lambda(x_1,-x_2) = (1,1)(x_1,-x_2) = (x_1, x_1 - x_2)$$

and we need to solve

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= x_1 - x_2 \end{aligned}$$

from where $x_1 = 2x_2$. Therefore

$$\{(x_1,x_2) \in K^0(\mathbb{T}^2) \mid (x_1,x_2) = \lambda \otimes \varsigma^*(x_1,x_2)\} = \{(x_1,x_2) \in K^0(\mathbb{T}^2) \mid x_1 = 2x_2\} \cong \mathbb{Z}$$

On the other hand, for the second term we need to compute

$$(x_1,x_2) - \lambda\varsigma^*(x_1,x_2) = (0, 2x_2 - x_1)$$

Then

$$\frac{K^0(\mathbb{T}^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^0(\mathbb{T}^2)\}} \cong \frac{\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(0,1)} \cong \mathbb{Z}$$

Now we will describe it for K^1, for this purpose it is advantageous to use the notation:

$$\begin{aligned} \lambda &= (1 \otimes 1, 1 \otimes 1) \\ (x_1,x_2) &= [(a_0 \otimes b_1), (a_1 \otimes b_0)] \\ (y_1,y_2) &= [(c_0 \otimes d_1), (c_1 \otimes d_0)] \end{aligned}$$

coming from Künneth theorem for K-theory. To determine

$$\{(x_1,x_2) \in K^1(\mathbb{T}^2) \mid (x_1,x_2) = \lambda \cdot \varsigma^*(x_1,x_2)\}$$

we note that in Equation (4.20) we showed that

$$\lambda(y_1,y_2) = (1 \otimes 1, 1 \otimes 1)[(c_0 \otimes d_1),(c_1 \otimes d_0)] = [(c_0 \otimes d_1),(c_1 \otimes d_0)] = (y_1,y_2)$$

On the other hand, from the Chern character isomorphism we can deduce how ς acts on $K^1(\mathbb{T}^2)$ by analizing the action of ς on $H^1(\mathbb{T}^2)$. Hence

$$\varsigma^*(x_1,x_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (ax_1 + bx_2, cx_1 + dx_2)$$

and so

$$\lambda\varsigma^*(x_1,x_2) = \lambda(ax_1 + bx_2, cx_1 + dx_2) = (ax_1 + bx_2, cx_1 + dx_2)$$

Thus the desired set is

$$\{(x_1,x_2)\in K^1(\mathbb{T}^2)\mid (x_1,x_2)=(ax_1+bx_2,cx_1+dx_2)\}=\begin{cases}\mathbb{Z} & \text{if } a=-d\\ 0 & \text{if } a\neq -d\end{cases}$$

On the other hand, for the second term we compute

$$(x_1,x_2)-\lambda\varsigma^*(x_1,x_2)=(x_1-ax_1-bx_2,x_2-cx_1-dx_2)$$

Now

$$\frac{K^1(\mathbb{T}^2)}{\{y-\lambda\cdot\varsigma^*(y)\mid y\in K^1(\mathbb{T}^2)\}}\cong H^1(\mathbb{T}^2)_{\mathbb{Z}}$$

Then by Theorem 4.1.1, we have:

$$\begin{cases}0\to\mathbb{Z}\to{}^{\sigma}K^1(\mathbb{R}\times_\rho\mathbb{T}^2)\to\mathbb{Z}\to 0 & \text{if } a=-d\\[2ex] 0\to\mathbb{Z}\to{}^{\sigma}K^1(\mathbb{R}\times_\rho\mathbb{T}^2)\to 0\to 0 & \text{if } a\neq -d\end{cases}$$

$$0\to H^1(\mathbb{T}^2)_{\mathbb{Z}}\to{}^{\sigma}K^0(\mathbb{R}\times_\rho\mathbb{T}^2)\to\mathbb{Z}\to 0$$

Finally ${}^{\sigma}K^1(\mathbb{R}\times_\rho\mathbb{T}^2)\cong\mathbb{Z}^{\oplus 2}$ if $a=-d$ or ${}^{\sigma}K^1(\mathbb{R}\times_\rho\mathbb{T}^2)\cong\mathbb{Z}$ if $a\neq -d$ and ${}^{\sigma}K^0(\mathbb{R}\times_\rho\mathbb{T}^2)\cong\mathbb{Z}\oplus H^1(\mathbb{T}^2)_{\mathbb{Z}}$ where $\sigma\in H^3(\mathbb{R}\times\rho\mathbb{T}^2;\mathbb{Z})$ is the non-trivial class, which corresponds to the generator $\lambda\in H^2(\mathbb{T}^2;\mathbb{Z})$. We highlight that this coincides with the previous computation.

Now we will calculate ${}^{\sigma}K^*(\mathbb{R}\times_\rho\mathbb{T}^2)$ using Theorem 4.1.1 when $\det(M)=1$. So initially we need

$$\{x\in K^*(\mathbb{T}^2)\mid x=\lambda\cdot\varsigma^*x\}\text{ and }\frac{K^{*+1}(\mathbb{T}^2)}{\{y-\lambda\cdot\varsigma^*(y)\mid y\in K^{*+1}(\mathbb{T}^2)\}}.$$

In the same way from previous example the fiber $\mathbb{T}^2$ has K-theory ring $\dfrac{\mathbb{Z}[t]}{(t^2)}$. We start determining

$$\{(x_1,x_2)\in K^0(\mathbb{T}^2)\mid (x_1,x_2)=\lambda\otimes\varsigma^*(x_1,x_2)\}$$

We choose the element $\lambda=L^n\in H^2(\mathbb{T}^2;\mathbb{Z})$ which corresponds to $\lambda=(1,n)$ in $K^0(\mathbb{T}^2)$. Next, since ς acts maintaining the orientation of $\mathbb{T}^2$, it takes the element $t+1$ to $t+1$. Then

$$\varsigma^*(x_1+x_2t)=(x_1,x_2)$$

Hence

$$\lambda\varsigma^*(x_1,x_2)=\lambda(x_1,x_2)=(1,n)(x_1,x_2)=(x_1,nx_1+x_2)$$

and we need to solve

$$\begin{aligned}x_1&=x_1\\ x_2&=nx_1+x_2\end{aligned}$$

from where $x_1 = 0$. Therefore

$$\{(x_1, x_2) \in K^0(\mathbb{T}^2) \mid (x_1, x_2) = \lambda \otimes \varsigma^*(x_1, x_2)\} = \{(x_1, x_2) \in K^0(\mathbb{T}^2) \mid x_1 = 0\} \cong \mathbb{Z}$$

On the other hand, for the second term we need to compute

$$(x_1, x_2) - \lambda\varsigma^*(x_1, x_2) = (0, nx_1)$$

Then

$$\frac{K^0(\mathbb{T}^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^0(\mathbb{T}^2)\}} \cong \frac{\mathbb{Z}(1,0) \oplus \mathbb{Z}(0,1)}{\mathbb{Z}(0,-n)} \cong \mathbb{Z} \oplus \mathbb{Z}/n$$

Now we will describe it for K^1, for this purpose it is advantageous to use the notation:

$$\begin{gathered} \lambda = (1 \otimes 1, 1 \otimes 1) \\ (x_1, x_2) = [(a_0 \otimes b_1), (a_1 \otimes b_0)] \\ (y_1, y_2) = [(c_0 \otimes d_1), (c_1 \otimes d_0)] \end{gathered}$$

coming from Künneth theorem for K-theory. To determine

$$\{(x_1, x_2) \in K^1(\mathbb{T}^2) \mid (x_1, x_2) = \lambda \cdot \varsigma^*(x_1, x_2)\}$$

we note that in Equation (4.20) we showed that

$$\lambda(y_1, y_2) = (1 \otimes 1, 1 \otimes 1)[(c_0 \otimes d_1), (c_1 \otimes d_0)] = [(c_0 \otimes d_1), (c_1 \otimes d_0)] = (y_1, y_2)$$

On the other hand, from the Chern character isomorphism we can deduce how ς acts on $K^1(\mathbb{T}^2)$ by analizing the action of ς on $H^1(\mathbb{T}^2)$. Hence

$$\varsigma^*(x_1, x_2) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (ax_1 + bx_2, cx_1 + dx_2)$$

and so

$$\lambda\varsigma^*(x_1, x_2) = \lambda(ax_1 + bx_2, cx_1 + dx_2) = (ax_1 + bx_2, cx_1 + dx_2)$$

Thus the desired set is

$$\{(x_1, x_2) \in K^1(\mathbb{T}^2) \mid (x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2)\} = \left\{ \begin{array}{ll} \mathbb{Z} & \text{if } a = -d \\ 0 & \text{if } a \neq -d \end{array} \right.$$

On the other hand, for the second term we compute

$$(x_1, x_2) - \lambda\varsigma^*(x_1, x_2) = (x_1 - ax_1 - bx_2, x_2 - cx_1 - dx_2)$$

Now

$$\frac{K^1(\mathbb{T}^2)}{\{y - \lambda \cdot \varsigma^*(y) \mid y \in K^1(\mathbb{T}^2)\}} \cong H^1(\mathbb{T}^2)_{\mathbb{Z}}$$

Then by Theorem 4.1.1, we have:

$$\begin{cases} 0 \to \mathbb{Z} \oplus \mathbb{Z}/n \to {}^{n\sigma}K^1(\mathbb{R} \times_\rho \mathbb{T}^2) \to \mathbb{Z} \to 0 & \text{if } a = -d \\ 0 \to \mathbb{Z} \oplus \mathbb{Z}/n \to {}^{n\sigma}K^1(\mathbb{R} \times_\rho \mathbb{T}^2) \to 0 \to 0 & \text{if } a \neq -d \end{cases}$$

$$0 \to H^1(\mathbb{T}^2)_{\mathbb{Z}} \to {}^{n\sigma}K^0(\mathbb{R} \times_\rho \mathbb{T}^2) \to \mathbb{Z} \to 0$$

Finally ${}^{n\sigma}K^1(\mathbb{R} \times_\rho \mathbb{T}^2) \cong \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/n$ if $a = -d$ or ${}^{n\sigma}K^1(\mathbb{R} \times_\rho \mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}/n$ if $a = -d$ and ${}^{n\sigma}K^0(\mathbb{R} \times_\rho \mathbb{T}^2) \cong \mathbb{Z} \oplus H^1(\mathbb{T}^2)_{\mathbb{Z}}$ where $\sigma \in H^3(\mathbb{R} \times_\rho \mathbb{T}^2; \mathbb{Z})$ is the non-trivial class, which corresponds to the generator $\lambda \in H^2(\mathbb{T}^2; \mathbb{Z})$. We highlight that this coincides with the previous calculation.

Example 4.3.1. A particular example from the previous case is when

$$M = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the coinvariants $H^1(\mathbb{T}^2)_{\mathbb{Z}}$ are given by

$$H^1(\mathbb{T}^2)_{\mathbb{Z}} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (n,m) - g(n,m) \rangle} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{\langle (n,m) - (-n,m) \rangle} \cong \frac{\mathbb{Z} \oplus \mathbb{Z}}{2\mathbb{Z} \oplus 0} \cong \mathbb{Z} \oplus \mathbb{Z}/2$$

To twist K-theory we use the generator $\sigma \in H^3(X; \mathbb{Z}) \cong \mathbb{Z}/2$. Since X is a closed nonorientable manifold, we have

$$\begin{aligned} {}^{\sigma}K^1(\mathbb{R} \times_\rho \mathbb{T}^2) &\cong \mathbb{Z} \oplus \mathbb{Z} \\ {}^{\sigma}K^0(\mathbb{R} \times_\rho \mathbb{T}^2) &\cong \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/2 \end{aligned}$$

Chapter 5

The twisted K-theory of Borel constructions

In this chapter we will generalize Theorem 4.1.1 using the same construction for classifying spaces of free groups. Moreover, we construct a spectral sequence that converges to the twisted K-theory of the Borel construction of a G–space M for a certain twist. The E_2-page is given in terms of group cohomology of G with coefficients in the K-theory of M for a certain twisted action of G, not only the action induced by the action of G on M.

5.1 Constructing a $PU(\mathcal{H})$–principal bundle

Let G be a group and M a compact manifold with a G–action. Let $p\colon X = EG \times_G M \to BG$ be the fiber bundle obtained by the Borel construction. Our idea here is to construct a $PU(\mathcal{H})$–principal bundle over X given a map

$$\kappa\colon G \to \mathrm{MAP}(M, PU(\mathcal{H}))$$

with the properties:

1. $\kappa_{gh} = \kappa_h \circ L_{g^{-1}} \cdot \kappa_g$,
2. $\kappa_e = id$,

where e is the identity of G and id is the constant function to the identity of $PU(\mathcal{H})$. The notation $\cdot$ indicates pointwise multiplication using the product in $PU(\mathcal{H})$, namely given $m \in M$, the first property means

$$\kappa_{gh}(m) = \kappa_h(g^{-1}m)\kappa_g(m)$$

We will call such a map κ a **derivation of line bundles** over M. We define the $PU(\mathcal{H})$–principal bundle P over X by

$$P = EG \times_G M \times_\kappa PU(\mathcal{H})$$

Despite the notation, this does not refer to a Borel construction, although it can thought of as a twisted Borel construction. More explicitly, it is the quotient of $EG \times M \times PU(\mathcal{H})$ by the equivalence relation

$$(xg, m, u) \sim (x, gm, \kappa_{g^{-1}}(m)u)$$

and the bundle map $q\colon P \to X$ takes $[x, m, u]$ to $[x, m]$.

Let us check first that P is a fiber bundle. For $[x, n] \in X$ we define a trivialization ϕ over an advantageous open neighborhood $p^{-1}(U)$ of $[x, n]$. We take an open neighbourhood U of x in BG where the principal G–bundle $r\colon EG \to BG$ trivializes and so

$$r^{-1}(U) = \bigcup_{g\in G} V_g = \bigcup_{g\in G} V_e g$$

Then we have

$$p^{-1}(U) = \bigcup_{g\in G} V_g \times_G M$$

Now we will define the local trivialization. We denote by $p^{-1}(U) \times_\kappa PU((H))$ the subspace of $EG \times_G M \times_\kappa PU(\mathcal{H})$ of elements $[x, m, u]$ where $[x, m] \in p^{-1}(U)$. Assume that $(x, m) \in V_g \times F$

$$\begin{aligned} \phi\colon p^{-1}(U) \times_\kappa PU(\mathcal{H}) \to & p^{-1}(U) \times PU(\mathcal{H}) \\ [x, m, u] \mapsto & ([x, m], \kappa_{g^{-1}}(m)u). \end{aligned} \tag{5.1}$$

It is well-defined because if $(xh, m, u) \sim (x, hm, \kappa_{h^{-1}}(m)u)$ then $xh \in V_{gh}$ and so

$$\begin{aligned} \phi[xh, m, u] =& ([xh, m], \kappa_{h^{-1}g^{-1}}(m)u) \\ =& ([x, hm], \kappa_{h^{-1}g^{-1}}(m)u) \\ =& ([x, hm], \kappa_{g^{-1}} L_h(m) \kappa_{h^{-1}}(m)u) \\ =& ([x, hm], \kappa_{g^{-1}}(hm) \kappa_{h^{-1}}(m)u) \\ =& \phi[x, hm, \kappa_{h^{-1}}(m)u] \end{aligned}$$

Moreover ϕ is continuous due to the fact that the product and κ are continuous. The inverse for ϕ is defined by

$$\begin{aligned} p^{-1}(U) \times PU(\mathcal{H}) \to & p^{-1}(U) \times_\kappa PU(\mathcal{H}) \\ ([x, m], u) \mapsto & [x, m, \kappa_g L_g(m)u]. \end{aligned} \tag{5.2}$$

when $[x, m] \in V_g$. It is well-defined because if $(xh, m) \sim (x, hm)$ then

$$\begin{aligned}
\phi^{-1}([xh, m], u) =&([xh, m], \kappa_{gh} L_{gh}(m)u)\\
=&([x, hm], \kappa_{h^{-1}}(m)\kappa_{gh} L_{gh}(m)u)\\
=&([x, hm], \kappa_{h^{-1}}(m)\kappa_h L_{g^{-1}}(ghm)\kappa_g(ghm)u)\\
=&([x, hm], \kappa_{h^{-1}}(m)\kappa_h L_h(m)\kappa_g(ghm)u)\\
=&([x, hm], \kappa_{h^{-1}} L_{h^{-1}}(hm)\kappa_h(hm)\kappa_g(ghm)u)\\
=&([x, hm], \kappa_e(hm)\kappa_g L_g(hm)u)\\
=&([x, hm], \kappa_g L_g(hm)u)\\
=&\phi^{-1}([x, hm], u)
\end{aligned}$$

Finally, P is a $PU(\mathcal{H})$–principal bundle because if we choose two of these trivializations ϕ_1 and ϕ_2 over open neighborhoods $p^{-1}(U_1)$ and $p^{-1}(U_2)$ of $[x, n]$, we have

$$\begin{aligned}
\phi_2\phi_1^{-1}([m], u)) &= ([x, m], \kappa_{h^{-1}}(m)\kappa_g L_g(m)u)\\
&= ([x, m], \kappa_{h^{-1}} L_{g^{-1}}(gm)\kappa_g(gm)u)\\
&= ([x, m], \kappa_{gh^{-1}}(gm)u)
\end{aligned}$$

which means that the structural group for P is $PU(\mathcal{H})$.

Remark 5.1.1. The conditions for the derivation of line bundles κ may be too restrictive when G has torsion elements. If we denote by λ_f the line bundle over M associated to a map $M \to PU(\mathcal{H})$, and $g^n g^m = e$ we would have

$$\lambda_{\kappa_{g^m} L_{h^{-1}}} \otimes \lambda_{\kappa_{g^n}} = M \times \mathbb{C}$$

We will focus first in the case where G is free. In this case we can obtain the line bundle associated to $\kappa_{g_i^n}$ from the bundle associated to κ_{g_i}. This limitation stems from our approach to map G to bundles instead of isomorphism classes of bundles, but at this point we do not know how to construct a $PU(\mathcal{H})$–principal bundle from a similar map $G \to [M, PU(\mathcal{H})]$.

Remark 5.1.2. Note that the restriction of P to the fiber M is a trivial principal $PU(\mathcal{H})$–bundle.

5.2 The case of free groups

In this section we prove the following theorem.

Theorem 5.2.1. *Let G be a finitely generated free group with generators $\{g_i\}_{i\in J}$ which acts on a compact manifold M and let $M \hookrightarrow EG\times_G M \xrightarrow{\pi} BG$ be the fiber bundle associated to the Borel construction. Given a derivation of line bundles $\kappa\colon G \to MAP(M, PU(\mathcal{H}))$, if σ is the associated $PU(\mathcal{H})$–principal bundle, then* *the twisted K-theory group ${}^{\sigma}K^*(X)$, for $* = 0, 1$, is isomorphic to an extension group of*

$$\{x \in K^*(M) \mid x = \kappa(g_i) \cdot g_i^* x \text{ for all } i \in J\} \text{ by } \bigoplus_{i \in J} K_i^{*+1}(M)/N. \tag{5.3}$$

where N is the subgroup of tuples indexed by J with ith coordinate equal to $\kappa(g_i) \cdot g_i^* x - x \in K^{*+1}(M)$ for a certain $x \in K^{*+1}(M)$.

Proof. Due to the fact that G is a free group

$$BG \simeq \bigvee_{i \in J} S_i^1$$

Let y be the point where the circles are joined (see figure 5.1). Now we use the closed decomposition $\{D_+, D_-\}$ of $\bigvee_{i \in J} S_i^1$, where D_+ (blue part in Figure 5.1) is a contractible closed neighborhood of y and D_- is the closure of $\bigvee_{i \in J} S_i^1 - D_+$, which is homotopy equivalent to a disjoint union $\bigsqcup_{i \in J} t_i$ of points.

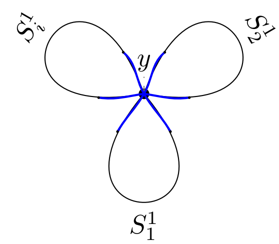


Figure 5.1: Guide to Theorem 5.2.1

We obtain a Mayer-Vietoris sequence

$$\begin{tikzcd}
{}^{\sigma}K^0(X) \arrow[r, "c_0"] & K^0(D_+ \times M) \oplus K^0(D_- \times M) \arrow[r, "a_0"] & K^0(D_{+-} \times M) \arrow[d, "b_0"] \\
K^1(D_{+-} \times M) \arrow[u, "b_1"] & K^1(D_+ \times M) \oplus K^1(D_- \times M) \arrow[l, "a_1"] & {}^{\sigma}K^1(X) \arrow[l, "c_1"]
\end{tikzcd}$$

Thus, there are the following group isomorphisms

$$\begin{aligned}
{}^{\sigma}K^*(X) &\cong \frac{K^{*+1}(D_{+-} \times M)}{\mathrm{Im}(a_{*+1})} \oplus_\rho \mathrm{Im}(c_*) \\
&\cong \frac{\bigoplus_{i \in J} [K^{*+1}(M) \oplus K^{*+1}(M)]}{\mathrm{Im}(a_{*+1})} \oplus_\rho \mathrm{Ker}(a_*)
\end{aligned} \tag{5.4}$$

where the terms on the right represent group extensions associated to some cocycle ρ in group cohomology.

The map a_* is computed in similar way to (4.1), that is, if we consider a class $(x, (y_i)_{i\in J}) \in K^*(D_+ \times M) \oplus K^*(D_- \times M)$ for $* \in \{0,1\}$ with $y_i \in K^*(t_i \times M)$, the gluing maps a_* are given by

$$a_*(x, (y_i)_i) = (x - y_i, x - \kappa(g_i) \cdot g_i^* y_i)_i.$$

With this in mind we can rewrite the kernel above:

$$\mathrm{Ker}(a_*) = \{x \in K^*(M) \mid x = \kappa(g_i) \cdot g_i^* x \text{ for all } i \in J\}$$

For $\mathrm{Coker}(a_*)$ we consider the following homomorphism:

$$\begin{aligned} \Psi\colon \frac{\bigoplus_{i\in J}[K^{*+1}(M) \oplus K^{*+1}(M)]}{\mathrm{Im}(a_{*+1})} &\longrightarrow \frac{\bigoplus_{i\in J} K_i^{*+1}(M)}{N} \\ [(x_i, y_i)_i] &\mapsto [(\kappa(g_i) \cdot g_i^* x_i - y_i)_i]. \end{aligned} \tag{5.5}$$

It is well-defined since the class of $(x - y_i, x - \kappa(g_i) \cdot g_i^* y_i)_i$ would be sent to the class of the element

$$(\kappa(g_i) \cdot g_i^* x - \kappa(g_i) \cdot g_i^* y_i - x + \kappa(g_i) \cdot g_i^* y_i)_i = (\kappa(g_i) \cdot g_i^* x - x)_i$$

which lies in N. It is clearly surjective. If an element $[(x_i, y_i)_i]$ is sent to the class of an element of N, then

$$\kappa(g_i) \cdot g_i^* x_i - y_i = \kappa(g_i) \cdot g_i^* x - x$$

and so

$$[(x_i, y_i)_i] = [(x - (x - x_i), x - \kappa(g_i) \cdot g_i^*(x - x_i))_i] = [a_{*+1}(x, (x - x_i)_i)]$$

This shows that Ψ is an isomorphism and proves the theorem. □

Remark 5.2.2. The terms in this theorem and in Theorem 4.1.1 are reminiscent of the cohomology of free groups. Namely, if G is a finitely generated free group with generators $\{g_i\}_{i\in J}$ and A is a $\mathbb{Z}G$–module, we have

$$H^0(G; A) \cong A^G = \{a \in A \mid g_i a = a \text{ for all } i \in J\}$$

$$H^1(G; A) \cong \left(\bigoplus_{i\in J} A\right) \Big/ N$$

where N is the subgroup of tuples indexed by J with ith coordinate equal to $g_i a - a$ for a certain $a \in A$. In the following section, we will see that it is possible to interpret these results in terms of a spectral sequence whose E_2-page is given in terms of group cohomology. These two theorems served as inspiration for the development of this spectral sequence.

Remark 5.2.3. It would be possible to prove a similar result for free groups which are not finitely generated using an open cover instead of a closed cover. However, we will not provide the details since such a result will follow from the spectral sequence in next section.

5.3 A spectral sequence for twisted K-theory

Let G be a discrete group and let us pick a model for BG which is a CW-complex. Let M be a compact manifold with a G–action and consider the fiber bundle $p\colon EG \times_G M \to BG$ obtained by the Borel construction. We have the following pushout diagram:

$$\begin{tikzcd}
\coprod_{\alpha\in J_k} S^{k-1}_\alpha \arrow[r, "\sqcup\varphi_\alpha"] \arrow[d] & BG^{(k-1)} \arrow[d] \\
\coprod_{\alpha\in J_k} D^k_\alpha \arrow[r, "\sqcup\phi_\alpha"] & BG^{(k)}
\end{tikzcd}$$

where $BG^{(k)}$ is the k-skeleton of BG, φ_α are the attaching maps of the k-cells and ϕ_α are their characteristic maps. We are denoting by J_k the set of k-cells of BG. The subspaces $H^{(k)} = p^{-1}(BG^{(k)})$ form a filtration of $H = EG \times_G M$ and we have

$$\begin{tikzcd}
\coprod_{\alpha\in J_k} S^{k-1}_\alpha \times p^{-1}(o_\alpha) \arrow[r, "\cong"] \arrow[d] & \coprod_{\alpha\in J} p^* S^{k-1}_\alpha \arrow[r] \arrow[d] & H^{(k-1)} \arrow[d] \\
\coprod_{\alpha\in J_k} D^k_\alpha \times p^{-1}(o_\alpha) \arrow[r, "\cong", "\sqcup\eta_\alpha"'] & \coprod_{\alpha\in J} p^* D^k_\alpha \arrow[r] & H^{(k)}
\end{tikzcd} \tag{5.6}$$

where o_α is the image of the center of D^k_α under the characteristic map ϕ_α in BG. We are going to consider the spectral sequence associated to the filtration of H by the subspaces $H^{(k)}$ in twisted K-theory. We consider a derivation of line bundles $\kappa\colon G \to \mathrm{MAP}(M, PU(\mathcal{H}))$ and the corresponding $PU(\mathcal{H})$–principal bundle P over H constructed in Section 5.1. Then the first page of the spectral sequence has the form

$$E_1^{k,m-k} = {}^PK^m\left(H^{(k)}, H^{(k-1)}\right) \cong \prod_{\alpha\in J_k} {}^{P'_\alpha}K^{m-k}(p^{-1}(o_\alpha))$$

for some twists P'_α which are restrictions of P in a certain sense. These twists are trivial, but not in a canonical way. We provide more details about this isomorphism. If we define F_α from the pullback

$$\begin{tikzcd}
p^* D^k_\alpha \arrow[r, "F_\alpha"] \arrow[d] & H \arrow[d, "p"] \\
D^k_\alpha \arrow[r, "\phi_\alpha"] & BG
\end{tikzcd}$$

by excision we have:

$${}^PK^m\left(H^{(k)}, H^{(k-1)}\right) \cong {}^{\sqcup F^*_\alpha P}K^m\left(\coprod p^* D^k_\alpha, \coprod p^* S^{k-1}_\alpha\right)$$

Using the homeomorphisms in Diagram (5.6), we get:

$$^{\sqcup F_\alpha^* P}K^m\left(\coprod p^* D_\alpha^k, \coprod p^* S_\alpha^{k-1}\right) \cong {}^{\sqcup \eta_\alpha^* F_\alpha^* P}K^m\left(\coprod D_\alpha^k \times p^{-1}(o_\alpha), \coprod S_\alpha^{k-1} \times p^{-1}(o_\alpha)\right)$$

and by the additivity property of twisted K-theory:

$$^{\sqcup \eta_\alpha^* F_\alpha^* P}K^m\left(\coprod D_\alpha^k \times p^{-1}(o_\alpha), \coprod S_\alpha^{k-1} \times p^{-1}(o_\alpha)\right) \cong \prod {}^{\eta_\alpha^* F_\alpha^* P}K^m\left(D_\alpha^k \times p^{-1}(o_\alpha), S_\alpha^{k-1} \times p^{-1}(o_\alpha)\right)$$

Now we use the isomorphism ${}^{Q}K^m(Z,B) \cong {}^{Q'}\widetilde{K}^m\left(\frac{Z}{B}\right)$ which holds when (Z,B) is a CW-pair (for a certain $PU(\mathcal{H})$–principal bundle $Q' \to Z/B$) and the homeomorphism

$$\frac{D_\alpha^k \times p^{-1}(o_\alpha)}{S_\alpha^{k-1} \times p^{-1}(o_\alpha)} \xrightarrow{\cong} S_\alpha^k \wedge p^{-1}(o_\alpha)_+$$

where W_+ denotes the disjoint union of W with the one-point space. Then we obtain

$$\prod {}^{\eta_\alpha^* F_\alpha^* P}K^m\left(D_\alpha^k \times p^{-1}(o_\alpha), S_\alpha^{k-1} \times p^{-1}(o_\alpha)\right) \cong \prod {}^{\eta_\alpha^* F_\alpha^* P}\widetilde{K}^m\left(S_\alpha^k \wedge p^{-1}(o_\alpha)_+\right)$$

On the right hand side the twist corresponds to that of the quotient, but we are abusing the notation for simplicity. The inclusion $r_\alpha\colon p^{-1}(o_\alpha)_+ \hookrightarrow S_\alpha^k \wedge p^{-1}(o_\alpha)_+$ induces an isomorphism:

$$\prod {}^{\eta_\alpha^* F_\alpha^* P}\widetilde{K}^m\left(S_\alpha^k \wedge p^{-1}(o_\alpha)_+\right) \cong \prod {}^{r_\alpha^* \eta_\alpha^* F_\alpha^* P}\widetilde{K}^{m-k}\left(p^{-1}(o_\alpha)_+\right)$$

We summarize some of the morphisms of the twist in the following diagram for convenience.

$$p^{-1}(o_\alpha) \hookrightarrow D_\alpha^k \times p^{-1}(o_\alpha) \xrightarrow[\eta_\alpha]{\cong} \phi_\alpha^* D_\alpha^k \xrightarrow{F_\alpha} H^{(k)}$$

and we denote this composition by $\eta_\alpha \restriction$. Using this notation we can write the E_1-page in a reduced way:

$$\prod {}^{r_\alpha^* \eta_\alpha^* F_\alpha^* P}K^{m-k}\left(p^{-1}(o_\alpha)\right) \cong \prod {}^{(\eta_\alpha \restriction)^* P}K^{m-k}\left(p^{-1}(o_\alpha)\right)$$

Rewriting the twist $(\eta_\alpha \restriction)^* P = P'_\alpha$, we get our first goal. The second step to simplify this spectral sequence is to find an explicit way to transform ${}^{(\eta_\alpha \restriction)^* P}K^{m-k}\left(p^{-1}(o_\alpha)\right)$ into $K^{m-k}(M)$. For this purpose, we are going to first fix a point $x_0 \in BG$ which corresponds to a 0-cell and a point $\tilde{x_0}$ on the fiber of x_0 in the universal principal G–bundle $q\colon EG \to BG$. An explicit description of this fiber gives rise to the following homeomorphism

$$\begin{aligned} p^{-1}(x_0) = \{[\widetilde{x}_0, z]/z \in M\} &\xrightarrow{\cong} M \\ [\widetilde{x}_0, z] &\mapsto z. \end{aligned} \tag{5.7}$$

Now for each cell D_α^k we choose a path β_α from x_0 to o_α and a lift $\widetilde{\beta_\alpha}$ as shown in the following diagram

$$\begin{tikzcd}
 & (EG, \widetilde{x}_0) \arrow[d] \\
(I,0) \arrow[ur, "\widetilde{\beta_\alpha}"] \arrow[r, "\beta_\alpha"] & (BG, x_0)
\end{tikzcd}$$

This lift induces a morphism

$$\begin{aligned} \widetilde{\widetilde{\beta_\alpha}} \colon I \times M &\to EG \times_G M \\ (t,z) &\mapsto [\widetilde{\beta_\alpha}(t), z]. \end{aligned} \tag{5.8}$$

Taking the end of the path through the fibers described by $\widetilde{\widetilde{\beta_\alpha}}$ we define h_α as the following composition

$$\begin{aligned} M \xrightarrow{\simeq} p^{-1}(x_0) &\xrightarrow{h_\alpha} p^{-1}(o_\alpha) \\ z_0 \mapsto [\widetilde{x}_0, z_0] &\mapsto [\widetilde{\widetilde{\beta_\alpha}}(1, z_0)]. \end{aligned} \tag{5.9}$$

Let α' be the 0-cell which corresponds to x_0. We indicate our route to follow next through the following maps:

$$\begin{tikzcd}
{}^{(\eta_\alpha \restriction)^* P} K^{m-k}\left(p^{-1}(o_\alpha)\right) \arrow[r, "h_\alpha^*", "\cong"'] & {}^{(h_\alpha^* \eta_\alpha \restriction)^* P} K^{m-k}\left(p^{-1}(x_0)\right) \arrow[r, "\cong"'] & {}^{(h_\alpha^* \eta_{\alpha'} \restriction)^* P} K^{m-k}\left(p^{-1}(x_0)\right) \arrow[d, "\cong"'] \\
 & & K^{m-k}(p^{-1}(x_0))
\end{tikzcd}$$

Now we consider the explicit description $P = EG \times_G M \times_\kappa PU(\mathcal{H})$ and define

$$\begin{aligned} \hat{\beta}_\alpha \colon I \times p^{-1}(x_0) \times PU(\mathcal{H}) &\to P \\ (t,z,p) &\mapsto [\widetilde{\beta_\alpha}(t), z, p]. \end{aligned} \tag{5.10}$$

We point out that $\hat{\beta}_\alpha(0,z,p) = [\widetilde{x}_0, z, p]$ lies over $[\widetilde{x}_0, z]$ and $\hat{\beta}_\alpha(1,z,p) = [\widetilde{\beta_\alpha}(1), z, p]$ lies over $[\widetilde{\beta_\alpha}(1), z] = \widetilde{\widetilde{\beta_\alpha}}(1,z) = h_\alpha(\widetilde{x}_0, z)$. Here it is important to highlight that we can construct this map $\hat{\beta}_\alpha$ thanks to the explicit construction of P. Using $\hat{\beta}_\alpha$ we define $\hat{h}$ as follows:

$$\begin{tikzcd}
\eta_{\alpha'} P = P \restriction_{p^{-1}(x_0)} \arrow[dr, dotted, "\hat{\hat{h}}"] \arrow[drr, bend left, "\hat{h}", "\cong"'] \arrow[ddr, bend right] & & \\
 & h_\alpha^* P \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & P \restriction_{p^{-1}(o_\alpha)} = \eta_\alpha^* P \arrow[d] \\
 & p^{-1}(x_0) \arrow[r, "h_\alpha"] & p^{-1}(o_\alpha)
\end{tikzcd}$$

Here we are using

$$
\begin{aligned}
\hat{h}_\alpha \colon P\restriction_{p^{-1}(x_0)} &\to P\restriction_{p^{-1}(o_\alpha)} \\
(\widetilde{x}_0, z, p) &\mapsto \hat{\beta}_\alpha(1)
\end{aligned}
\tag{5.11}
$$

From the pullback diagram we can extract a factorization of $\hat{h}$ through $\hat{\hat{h}}$ and this gives us the following composition

$$
\begin{tikzcd}
{}^{(\eta_\alpha\restriction)^* P}K^{m-k}\left(p^{-1}(o_\alpha)\right) \arrow[r, "h_\alpha^*", "\cong"'] & {}^{(h_\alpha^*\eta_\alpha\restriction)^* P}K^{m-k}\left(p^{-1}(x_0)\right) \arrow[r, "(\hat{\hat{h}}_\alpha^{-1})_*", "\cong"'] & {}^{(h_\alpha^*\eta_{\alpha'}\restriction)^* P}K^{m-k}\left(p^{-1}(x_0)\right) \arrow[d, "\cong"'] \\
& K^{m-k}(M) & K^{m-k}(p^{-1}(x_0)) \arrow[l, "\cong"']
\end{tikzcd}
$$

Adding the canonical maps to the diagram our second step is accomplished. The next step to achieve in our spectral sequence is to describe the differential map

$$d_1 \colon E_1^{k,m-k} \to E_1^{k+1,m-k}$$

in terms of Bredon cohomology of EG with coefficients in a certain system that we will describe later. This differential is defined as the composition of two maps from long exact sequences of the pair.

$$
\begin{tikzcd}
{}^P K^m(H^{(k)}, H^{(k-1)}) \arrow[rr, "d_1"] \arrow[dr, "j^*"'] & & {}^P K^{m+1}(H^{(k+1)}, H^{(k)}) \\
& {}^P K^m(H^{(k)}, w_0) \arrow[ur, "\delta"'] &
\end{tikzcd}
$$

Here $w_0 \in H^{(0)}$. We will expand this composition. First we use the following diagram

$$
\begin{tikzcd}
H^{(0)} \arrow[r] & \frac{H^{(k)}}{H^{(k-1)}} \arrow[r, two heads] \arrow[d, two heads] & \frac{H^{(k)}}{H^{(k-1)} \cup \left(\bigsqcup_{\beta \neq \alpha} \phi_\beta^* D_\beta^k \right)} \arrow[dl, "\cong"] \\
& \frac{D_\alpha^k \times p^{-1}(o_\alpha)}{S_\alpha^{k-1} \times p^{-1}(o_\alpha)} &
\end{tikzcd}
$$

Moving to twisted K-theory, we have:

$$
\begin{tikzcd}
{}^{P}K^m(H^{(k)}, H^{(k-1)}) \arrow[r, "j^*"] & {}^{P}K^m(H^{(k)}, w_0) \\
{}^{P}\widetilde{K}^m(H^{(k)}/H^{(k-1)}) \arrow[u, "\cong"] \arrow[ur, two heads] & \\
\prod_{\alpha} {}^{\eta_\alpha^* F_\alpha^* P}\widetilde{K}^m\left(\frac{D_\alpha^k \times p^{-1}(o_\alpha)}{S_\alpha^{k-1} \times p^{-1}(o_\alpha)}\right) \arrow[u, "\cong"] & \\
{}^{\vee_\alpha \eta_\alpha^* F_\alpha^* P}\widetilde{K}^m\left(\bigvee_\alpha S_\alpha^k \wedge p^{-1}(o_\alpha)_+\right) \arrow[u, "\cong"] & \\
\prod_{\alpha} {}^{\eta_\alpha^* F_\alpha^* P}\widetilde{K}^m(S_\alpha^k \wedge p^{-1}(o_\alpha)_+) \arrow[u, "\cong"] \arrow[uuuur, bend right, "q"'] &
\end{tikzcd}
$$

Now we will describe the connecting map δ. According to Theorem 8.2 in [42], we use the following diagram, where the middle horizontal line is a cofibration sequence.

$$
\begin{tikzcd}
 & & H^{(k+1)} \cup CH^{(k)} \arrow[d, two heads, "\text{by } CH^{(k)}"', "\simeq"] \arrow[dr, two heads, "\text{by } H^{(k+1)}"] & \\
H^{(k)} \arrow[r, hook] & H^{(k+1)} \arrow[r, two heads, "\text{by } H^{(k)}"'] & \frac{H^{(k+1)}}{H^{(k)}} \arrow[r, "\text{induces } \delta"'] & \Sigma H^{(k)} \\
 & & \frac{D_\beta^{k+1} \times p^{-1}(o_\beta)}{S_\beta^k \times p^{-1}(o_\beta)} \arrow[u, hook] \arrow[ur] &
\end{tikzcd}
$$

Now we feed it into twisted K-theory, again with similar abuses in the notation for twistings.

$$
\begin{tikzcd}
{}^P\widetilde{K}^{m+1}(\Sigma H^{(k)}) \arrow[d, "="] \arrow[dr, "\delta"] & \\
{}^PK^m(H^{(k)}, w_0) = {}^PK^{m+1}(\Sigma H^{(k)}, w_0) \arrow[r] \arrow[dd, "\cong"] \arrow[ddr, "((\Sigma\varphi_\beta)^*)_\beta"'] & {}^P\widetilde{K}^{m+1}\left(\frac{H^{(k+1)}}{H^{(k)}}\right) \arrow[d, "\cong"] \\
 & \prod_\beta {}^{\eta^*_\beta F^*_\beta P}\tilde{K}^{m+1}(S^{k+1}_\beta \wedge p^{-1}(o_\beta)_+) \arrow[d, "\cong"] \\
{}^P\widetilde{K}^m(H^{(k)}) \arrow[dr, "(\varphi^*_\beta)_\beta"'] & \prod_\beta {}^{\eta^*_\beta F^*_\beta P}\widetilde{K}^{m+1}(\Sigma(S^k_\beta \wedge p^{-1}(o_\beta)_+)) \arrow[d, "\cong"] \\
 & \prod_\beta {}^{\eta^*_\beta F^*_\beta P}\widetilde{K}^m(S^k_\beta \wedge p^{-1}(o_\beta)_+)
\end{tikzcd}
$$

Assembling these perspectives, we can summarize them in the following diagram.

$$
\begin{tikzcd}
D^{k+1}_\beta \times p^{-1}(o_\beta) \arrow[r] & H^{(k+1)} & & \\
S^k_\beta \times p^{-1}(o_\beta) \arrow[r] \arrow[u] \arrow[d] & H^{(k)} \arrow[u] \arrow[r, "q"] & H^{(k)}/H^{(k-1)} \arrow[r, phantom, "\cong"] & \bigvee_{\alpha \in J_k} S^k_\alpha \wedge p^{-1}(o_\alpha)_+ \arrow[d, "\text{quotient}"] \\
S^k_\beta \wedge p^{-1}(o_\beta)_+ \arrow[urr, "j^*"] \arrow[rrr, "r_{\beta,\alpha}"] & & & S^k_\alpha \wedge p^{-1}(o_\alpha)_+
\end{tikzcd}
$$

In particular, we will be using the maps

$$
\begin{aligned}
r^*_{\beta,\alpha} : {}^{\eta^*_\alpha F^*_\alpha P}\tilde{K}^m(S^k_\alpha \wedge p^{-1}(o_\alpha)_+) &\to {}^{\eta^*_\beta F^*_\beta P}\tilde{K}^m(S^k_\beta \wedge p^{-1}(o_\beta)_+) \\
x &\mapsto (q_\alpha \varphi_\beta)^*(x)
\end{aligned}
\tag{5.12}
$$

where q_α is the composition of q and the map labeled “quotient” in the diagram above. Now let us return to the objective of describing the differential d_1. With respect to the decomposition found previously, the differential d_1 corresponds to the lefthand vertical map in the following diagram:

$$\begin{tikzcd}
{}^{\coprod_\alpha \eta_\alpha^* F_\alpha^* P}K^m\left(\coprod_{\alpha\in J_k} D^k \times p^{-1}(o_\alpha), \coprod_{\alpha\in J_k} S^{k-1}\times p^{-1}(o_\alpha)\right) \arrow[r, "\cong"] \arrow[d, "r_{\beta,\alpha}"] & \prod_{\alpha\in J_k} K^m(M) \arrow[d, "\delta"] \\
{}^{\coprod_\beta \eta_\beta^* F_\beta^* P}K^m\left(\coprod_{\beta\in J_{k+1}} D^{k+1} \times p^{-1}(o_\beta), \coprod_{\beta\in J_{k+1}} S^{k}\times p^{-1}(o_\beta)\right) \arrow[r, "\cong"] & \prod_{\beta\in J_{k+1}} K^m(M)
\end{tikzcd}$$

where despite the abuse of notation, the vertical map on the left is given by:

$$[r_{\beta,\alpha}(x_\alpha)_\alpha]_\beta = \sum_{\partial\beta\cap\alpha\neq\emptyset} r_{\beta,\alpha}^*(x_\alpha) = \sum_{\partial\beta\cap\alpha\neq\emptyset} (q_\alpha\varphi_\beta)^*(x_\alpha) = \sum_{\partial\beta\cap\alpha\neq\emptyset} (q'_\alpha\varphi'_\beta)^*(x_\alpha)$$

Here $\partial\beta\cap\alpha\neq\emptyset$ means that the boundary of the $(k+1)$-cell labeled by β intersects the k-cell labeled by α in BG. The maps q'_α and φ'_β are given by:

$$\begin{tikzcd}
(S^k_\alpha)_+ \wedge M & H^{(k)} = EG^{(k)}\times_G M \arrow[l, "q_\alpha"'] & (S^k_\beta)_+\wedge M \arrow[l, "\varphi_\beta"'] \\
(S^k_\alpha\times G)_+ \wedge M \arrow[u] & EG^{(k)}\times M \arrow[u] \arrow[l, "q'_\alpha"] \arrow[dl, "q'_{g\alpha}"] & (S^k_\beta\times G)_+\wedge M \arrow[u] \arrow[l, "\varphi'_\beta"] \\
\coprod_{g\in G}(S^k_{g\alpha})_+\wedge M & & \coprod_{g\in G}(S^k_{g\beta})_+\wedge M \arrow[ul, "\varphi'_{g\beta}"']
\end{tikzcd}$$

We will reinterpret $\prod_{\alpha\in J_k} K^m(M)$ as the kth term in the cochain complex for Bredon Cohomology [10], with respect to certain a coefficient system, namely:

$$\begin{aligned}
\mathcal{M}^m\colon \mathcal{O}_G &\to \mathcal{A}b \\
G/H &\mapsto K^m(M)^H
\end{aligned} \tag{5.13}$$

where the action of G on $K^m(M)$ is given by

$$g\cdot z = \kappa_g \cdot L^*_{g^{-1}}(z).$$

Then we have a description

$$C^n_G(EG,\mathcal{M}^m) \cong \prod_{G/H\times D^n} \mathcal{M}^m(G/H)$$

where the product runs over the equivariant n-cells of EG. Since G acts freely on EG, this expression turns into it turn into

$$C^n_G(EG,\mathcal{M}^m) \cong \prod_{G/1\times D^n} \mathcal{M}^m(G/H) \cong \prod_{G/1\times D^n} K^m(M) = \prod_{\alpha\in J_n} K^m(M)$$

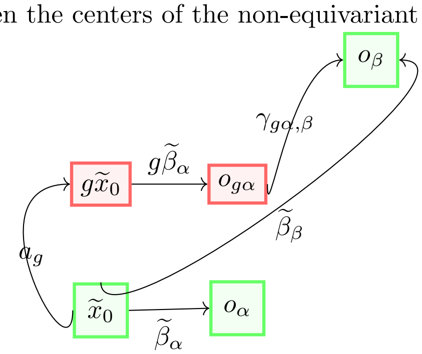
where the last equality holds because EG has an equivariant n-cell for each n-cell of BG. In what follows, let us denote $P_\alpha = \eta_\alpha^* F_\alpha^* P$. In order to identify the differential d_1 with the differential in the Bredon cohomology complex, we first perform the following steps.

$$\begin{tikzcd}
{}^{\bigsqcup_\alpha P_\alpha} K^m\left(\coprod_\alpha D^k \times p^{-1}(o_\alpha), \coprod_\alpha S^{k-1} \times p^{-1}(o_\alpha)\right) \arrow[r, "r^*_{\alpha,\beta}"] \arrow[d, "\cong"] & {}^{\bigsqcup_\beta P_\beta} K^m\left(\coprod_\beta D^{k+1} \times p^{-1}(o_\beta), \coprod_\beta S^k \times p^{-1}(o_\beta)\right) \arrow[d, "\cong"] \\
{}^{\bigvee_\alpha P_\alpha} K^m\left(\bigvee_\alpha S^k_\alpha \wedge p^{-1}(o_\alpha)_+\right) \arrow[r, "(q_\alpha\varphi_\beta)^*"] \arrow[d, "\cong"] & {}^{\bigvee_\beta P_\beta} K^m\left(\bigvee_\beta S^k_\beta \wedge p^{-1}(o_\beta)_+\right) \arrow[d, "\cong"] \\
\prod_\alpha {}^{P_\alpha} K^m(S^k_\alpha \wedge p^{-1}(o_\alpha)_+) \arrow[r, "(q_\alpha\varphi_\beta)^*"] \arrow[d, "\cong"', "(S^k\wedge(h_\alpha)_+)^* = hh^*_\alpha"] & \prod_\beta {}^{P_\beta} K^m(S^k_\beta \wedge p^{-1}(o_\beta)_+) \arrow[d, "\cong", "hh^*_\beta"'] \\
\prod_\alpha {}^{hh^*_\alpha P_\alpha} K^m(S^k_\alpha \wedge p^{-1}(x_0)_+) \arrow[r] & \prod_\beta {}^{hh^*_\beta P_\beta} K^m(S^k_\beta \wedge p^{-1}(x_0)_+)
\end{tikzcd}$$

Now we investigate the nature of the last horizontal arrow. To remove the dependence on the paths used to define h_α and h_β we intend to "lift" everything to EG in a certain sense. Note that we have a pullback diagram:

$$\begin{tikzcd}
EG \times M \arrow[r] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & EG \times_G M \arrow[d, "p"] \\
EG \arrow[r] & BG
\end{tikzcd}$$

and consider the following schematic diagram of paths in EG for a k-cell labeled by α and a $(k+1)$-cell labeled by β, both in BG, such that $g\alpha \cap \beta \neq \emptyset$ for some $g \in G$. We think of the equivariant cell in EG corresponding to α as a union of nonequivariant cells $g\alpha$. In the following we denote as $\gamma_{g\alpha,\beta}$ the radial path between the centers of the non-equivariant k-cell $g\alpha$ and the $(k+1)$-cell β.



Now we have "lifted" maps for each $g \in G$.

$$\begin{aligned} h^g_\alpha \colon p^{-1}(x_0) &\to q^{-1}(o_{g\alpha}) \\ [\widetilde{x}_0, z_0] &\mapsto (\widetilde{\beta}_\alpha(1)g, z_0), \end{aligned} \tag{5.14}$$

$$\begin{aligned} hh^g_\alpha \colon S^k \wedge p^{-1}(x_0) &\to S^k \wedge q^{-1}(o_{g\alpha}) \\ [y, [\widetilde{x}_0, z_0]] &\mapsto [y, (\widetilde{\beta}_\alpha(1)g, z_0)] \end{aligned} \tag{5.15}$$

and the maps induced by hh^g_α and $\gamma_{g\alpha,\beta}$ form a commutative diagram:

$$\begin{tikzcd}
{}^{P_\alpha}\widetilde{K}^m(S^k_\alpha \wedge q^{-1}(o_{1\alpha})_+) \arrow[d, "\cong"] \arrow[dr, "(hh^1_\alpha)^*"'] & {}^{P_{g\alpha}}\widetilde{K}^m(S^k_\alpha \wedge q^{-1}(o_{g\alpha})_+) \arrow[l, "L^*_g"'] \arrow[r, "(q_{g\alpha}\varphi_\beta)^*"] \arrow[d, "(hh^g_\alpha)^*"] & {}^{P_\beta}\widetilde{K}^m(S^k_\alpha \wedge q^{-1}(o_\beta)_+) \arrow[d, "(hh^1_\beta)^*"] \\
{}^{P_\alpha}\widetilde{K}^m(S^k_\alpha \wedge p^{-1}(o_\alpha)_+) \arrow[r, "hh^*_\alpha"'] & {}^{P_\alpha}\widetilde{K}^m(S^k_\alpha \wedge p^{-1}(x_0)_+) \arrow[r, dashed] \arrow[dr, "(a_g)_*"'] & {}^{P_\beta}\widetilde{K}^m(S^k_\beta \wedge p^{-1}(x_0)_+) \\
& & {}^{P_{g\alpha}}\tilde{K}^m(S^k_\alpha \wedge p^{-1}(x_0)_+) \arrow[u, "(g\tilde{\beta}_\alpha\cdot\gamma_{g\alpha,\beta})_*\cdot H_*"']
\end{tikzcd}$$

and we wish to determine the dashed line, hence we need to determine the composition of the inverse of $(hh^g_\alpha)^*$, the map $(q_{g\alpha}\varphi_\beta)^*$ and $(hh^1_\beta)^*$. Since EG is contractible, there is a canonical homotopy H rel ∂I from the path $a_g \cdot g\widetilde{\beta}_\alpha \cdot \gamma_{g\alpha,\beta}$ to $\widetilde{\beta}_\beta$. Here a_g is the unique lift of the loop represented by g in $\pi_1(BG, x_0)$ that starts at $\widetilde{x}_0$. This homotopy determines another homotopy from $L_{g^{-1}}hh_\alpha$ to hh^1_β which we will still denote by H. We assume from this point that the CW-structure on BG and the corresponding CW-structure on EG are regular in the sense of Section II.6 of [44]. Then following Theorem XII.6.12 in [44] in this twisted cohomological context (note that H_k in that theorem refers to a generalized homology theory, and see also the arguments in Sections XIII.4 and XIII.5 in [44]) we reduce the dashed line to $L^*_{g^{-1}}$ composed with the following maps:

$${}^{L^*_{g^{-1}}(hh^1_\alpha)^*P_\alpha}K^m(S^k_\alpha \wedge p^{-1}(x_0)_+) \xrightarrow{H_*} {}^{(hh^1_\beta)^*P_\beta}K^m(S^k_\beta \wedge p^{-1}(x_0)_+) \xrightarrow{[g\alpha\colon\beta]} {}^{(hh^1_\beta)^*P_\beta}K^m(S^k_\beta \wedge p^{-1}(x_0)_+)$$

When we trivialize the twists for the first two terms in this sequence to reduce to untwisted K-theory, by the way in which P was constructed, this composition will correspond to multiplication by κ_g. With this arguments we have shown that the isomorphism between $E_1^{k,m-k}$ and $C^k_G(EG;\mathcal{M}^{m-k})$ takes the differential of the spectral sequence to the differential of the Bredon cohomology complex, hence

$$E_2^{k,m-k} \cong H^k_G(EG;\mathcal{M}^{m-k})$$

Finally note that by the Remark in page I-22 of [10], we have an isomorphism of complexes

$$C^*_G(EG; \mathcal{M}^{m-k}) \cong C^*(BG; K^{m-k}(M))$$

where the right side denotes the complex that defines group cohomology with coefficients in the $\mathbb{Z}G$–module $K^{m-k}(M)$ for the action described earlier. Hence we have shown the following theorem.

Theorem 5.3.1. *Let G be a discrete group and let M be a compact manifold with a G–action. Given a derivation of line bundles $\kappa\colon G \to MAP(M, PU(\mathcal{H}))$, there is a spectral sequence*

$$E_2^{p,q} \cong H^p(BG; K^q(M)) \Longrightarrow {}^P K^{p+q}(EG \times_G M)$$

where P is the principal $PU(\mathcal{H})$–bundle associated to κ and the cohomology of BG has local coefficients for the action

$$g \cdot z = \kappa_g \cdot L^*_{g^{-1}}(z).$$

on $K^q(M)$.

Note that the action in our previous theorem when $G = \mathbb{Z}$ may seem different, but we chose in that case the class $\lambda \in H^2(M)$ corresponding to κ_λ^{-1}, so there was an inverse implicit in that statement. A similar comment applies for the case of free groups.

5.4 A computation using the spectral sequence

Let $\rho\colon \pi_1(B\mathbb{Z}^2) \times S^2 \to S^2$ be the action given by $\rho((x,y),z) = z$ if $x+y$ is even and $\rho((x,y),z) = -z$ if $x+y$ is odd. We consider the fiber bundle $S^2 \hookrightarrow E\mathbb{Z}^2 \times_\rho S^2 \to B\mathbb{Z}^2$, which is a Borel construction. And let $\kappa\colon \mathbb{Z}^2 \to \mathrm{MAP}(S^2, PU(\mathcal{H}))$ be the derivation of line bundles defined below in Equation (5.16). This derivation κ satisfies $\kappa_{(1,0)} = l$ and $\kappa_{(0,1)} = l$ where $l\colon S^2 \to PU(\mathcal{H})$ is the classifying map of the tautological complex line bundle. Since it may not be clear that defining κ in this way for $(1,0)$ and $(0,1)$ defines a unique derivation of line bundles, we will start by showing this. From $\kappa_{(1,0)}$ and $\kappa_{(0,1)}$ we get a general representation

$$\kappa_{(m,n)}(x) = \kappa_{(0,n)}((-1)^m x)\kappa_{(m,0)}(x) \tag{5.16}$$

And explicit descriptions of these factors are:

$$\kappa_{(m,0)}(x) = \prod_{i=1}^{m} l((-1)^{i+m}x) \quad \text{if } m > 0$$

$$\kappa_{(-m,0)}(x) = [\kappa_{(m,0)}((-1)^m x)]^{-1} \quad \text{if } m > 0$$

To check that this constitutes a derivation of line bundles we check

$$\begin{aligned}\kappa_{(0,0)}(x) &= \kappa_{(0,0)}((-1)^0 x)\kappa_{(0,0)}(x)\\ &= id(x)\end{aligned}$$

and

$$\begin{aligned}\kappa_{(a+m,b+n)}(x) &= \kappa_{(0,b+n)}((-1)^{a+m}x)\kappa_{(a+m,0)}(x)\\ &= \prod_{i=1}^{b+n} l((-1)^{i+m+n+a+b}x)\prod_{j=1}^{a+m} l((-1)^{j+n+a}x)\end{aligned}$$

$$\begin{aligned}\kappa_{(a,b)+(m,n)}(x) &= \kappa_{(m,n)}((-1)^{a+b}x)\kappa_{(a,b)}(x)\\ &= \kappa_{(0,n)}((-1)^{m+a+b}x)\kappa_{(m,0)}((-1)^{a+b}x)\kappa_{(0,b)}((-1)^{a}x)\kappa_{(a,0)}(x)\\ &= \prod_{i=1}^{n} l((-1)^{i+m+n+a+b}x)\prod_{j=1}^{m} l((-1)^{j+m+a+b}x)\prod_{t=1}^{b} l((-1)^{t+a+b}x)\prod_{s=1}^{a} l((-1)^{s+a}x)\end{aligned}$$

Using asociativity with $(-1)^{n+m+n+a+b} = (-1)^{m+2n+a+b} = (-1)^{m+a+b}$ we have

$$\kappa_{(a,b)+(m,n)}(x) = \prod_{i=1}^{b+n} l((-1)^{i+m+n+a+b}x)\prod_{j=1}^{a+m} l((-1)^{j+n+a}x)$$

Now we get the second page of our spectral sequence

$$E_2^{p,q} \cong H^p(\mathbb{T}^2; K^q(S^2)) \Longrightarrow {}^P K^{p+q}(\mathbb{R}^2 \times_\rho M)$$

$$\begin{array}{r|cccc}
3 & H^0(\mathbb{Z}^2;K^3(S^2)) & H^1(\mathbb{Z}^2;K^3(S^2)) & H^2(\mathbb{Z}^2;K^3(S^2)) & H^3(\mathbb{Z}^2;K^3(S^2))\\
2 & H^0(\mathbb{Z}^2;K^2(S^2)) & H^1(\mathbb{Z}^2;K^2(S^2)) & H^2(\mathbb{Z}^2;K^2(S^2)) & H^3(\mathbb{Z}^2;K^2(S^2))\\
1 & H^0(\mathbb{Z}^2;K^1(S^2)) & H^1(\mathbb{Z}^2;K^1(S^2)) & H^2(\mathbb{Z}^2;K^1(S^2)) & H^3(\mathbb{Z}^2;K^1(S^2))\\
0 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & H^3(\mathbb{Z}^2;K^0(S^2))\\
-1 & H^0(\mathbb{Z}^2;K^{-1}(S^2)) & H^1(\mathbb{Z}^2;K^{-1}(S^2)) & H^2(\mathbb{Z}^2;K^{-1}(S^2)) & H^3(\mathbb{Z}^2;K^{-1}(S^2))\\
-2 & H^0(\mathbb{Z}^2;K^{-2}(S^2)) & H^1(\mathbb{Z}^2;K^{-2}(S^2)) & H^2(\mathbb{Z}^2;K^{-2}(S^2)) & H^3(\mathbb{Z}^2;K^{-2}(S^2))\\
-3 & H^0(\mathbb{Z}^2;K^{-3}(S^2)) & H^1(\mathbb{Z}^2;K^{-3}(S^2)) & H^2(\mathbb{Z}^2;K^{-3}(S^2)) & H^3(\mathbb{Z}^2;K^{-3}(S^2))
\end{array}$$

$$\begin{array}{r|cccc}
3 & 0 & 0 & 0 & 0\\
2 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & 0\\
1 & 0 & 0 & 0 & 0\\
0 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & 0\\
-1 & 0 & 0 & 0 & 0\\
-2 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & 0\\
-3 & 0 & 0 & 0 & 0
\end{array}$$

Note that the action of $\mathbb{Z}^2$ on $K^0(S^2) \cong \mathbb{Z}1 \oplus \mathbb{Z}l$ is given by

$$(1,0)(a1+bl) = \kappa_{(1,0)} L^*_{(1,0)^{-1}}(a1+bl) = al + b1$$

$$(0,1)(a1+bl) = \kappa_{(0,1)} L^*_{(0,1)^{-1}}(a1+bl) = al + b1$$

We refer to the calculations near Equation (4.13) for details. All differentials d_i with $i > 1$ are trivial, so we only need to describe the nonzero terms in this page. For this purpose, we use the Lyndon-Hochschild-Serre spectral sequence induced by the following exact sequence

$$0 \to A = \{(m,n) \mid m+n \text{ even}\} \to \mathbb{Z}^2 \to \mathbb{Z}/2 \to 0$$

whose E_2-page is given by

$$H^p(\mathbb{Z}/2; H^q(A; K^0(S^2))) \Longrightarrow H^{p+q}(\mathbb{Z}^2; K^0(S^2))$$

On the left hand side, since $\mathbb{Z}^2$ is abelian, the action of $\mathbb{Z}/2$ over A is trivial. Moreover, by the description of the action above, we see that the action of A on $K^0(S^2)$ is trivial. For the following values of q we have:

- $q = 0$, $H^0(A; K^0(S^2)) = K^0(S^2)^A = K^0(S^2)$
- $q = 1$, $H^1(A; K^0(S^2)) = \mathrm{Hom}(\mathbb{Z}^2, K^0(S^2)) = K^0(S^2) \oplus K^0(S^2)$
- $q = 2$, $H^2(A; K^0(S^2)) = K^0(S^2)$, because

$$0 \to \mathrm{Ext}(H_1(\mathbb{Z}^2), K^0(S^2)) \to H^2(A; K^0(S^2)) \to \mathrm{Hom}(H_2(\mathbb{Z}^2), K^0(S^2)) \to 0$$

$$0 \to 0 \to H^2(A; K^0(S^2)) \to K^0(S^2) \to 0$$

We can regard $K^0(S^2) \cong \mathbb{Z}[\mathbb{Z}/2]$ as $\mathbb{Z}/2$ module. Now we check the different values for p. To calculate the cases where $p \neq 0$ we use Example 2 from Section 3.1 in [11], where $\overline{N}\colon M_{\mathbb{Z}/2} \to M^{\mathbb{Z}/2}$ is the norm map, induced by $N\colon M \to M$ with $N(x) = x + \omega x$, here M is a $\mathbb{Z}/2$–module and $\mathbb{Z}/2 = \{1, \omega\}$.

- For $p = 0$
  $H^0(\mathbb{Z}/2; H^0(A; K^0(S^2))) = [K^0(S^2)]^{\mathbb{Z}/2} = \mathbb{Z}$
  $H^0(\mathbb{Z}/2; H^1(A; K^0(S^2))) = [K^0(S^2) \oplus K^0(S^2)]^{\mathbb{Z}/2} = \mathbb{Z}^2$
  $H^0(\mathbb{Z}/2; H^2(A; K^0(S^2))) = H^0(\mathbb{Z}/2, K^0(S^2)) = K^0(S^2)^{\mathbb{Z}/2} = \mathbb{Z}$
- For $p = 1$
  First we calculate $K^0(S^2)_{\mathbb{Z}/2} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{<1-l,l-1>} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{<l-1>} \cong \mathbb{Z}$.
  $H^1(\mathbb{Z}/2; H^0(A; K^0(S^2))) = \mathrm{Ker}[\overline{N}\colon K^0(S^2)_{\mathbb{Z}/2} \to K^0(S^2)^{\mathbb{Z}/2}] = 0$
  $H^1(\mathbb{Z}/2; H^1(A; K^0(S^2))) = \mathrm{Ker}[\overline{N}\colon (K^0(S^2)^{\oplus 2})_{\mathbb{Z}/2} \to (K^0(S^2)^{\oplus 2})^{\mathbb{Z}/2}] = 0$

- For $p = 2$
  $H^2(\mathbb{Z}/2; H^0(A; K^0(S^2))) = \mathrm{Coker}[\overline{N}\colon K^0(S^2)_{\mathbb{Z}/2} \to K^0(S^2)^{\mathbb{Z}/2}] = 0$

$$\begin{array}{c|ccc}
3 & H^0(\mathbb{Z}/2; H^3(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^3(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^3(A; K^0(S^2))) \\
2 & H^0(\mathbb{Z}/2; H^2(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^2(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^2(A; K^0(S^2))) \\
1 & H^0(\mathbb{Z}/2; H^1(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^1(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^1(A; K^0(S^2))) \\
0 & H^0(\mathbb{Z}/2; H^0(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^0(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^0(A; K^0(S^2)))
\end{array}$$

Replacing the groups we have just computed:

$$\begin{array}{c|ccc}
3 & 0 & 0 & 0 \\
2 & \mathbb{Z} & & \\
1 & \mathbb{Z}^2 & 0 & \\
0 & \mathbb{Z} & 0 & 0
\end{array}$$

Then

$$H^0(\mathbb{Z}^2; K^0(S^2)) \cong \mathbb{Z}$$
$$H^1(\mathbb{Z}^2; K^0(S^2)) \cong \mathbb{Z}^2$$
$$H^2(\mathbb{Z}^2; K^0(S^2)) \cong \mathbb{Z}$$

Replacing these groups in the original spectral sequence we have

$$\begin{array}{c|cccc}
3 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 \\
-1 & 0 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z} & 0 \\
-3 & 0 & 0 & 0 & 0
\end{array}$$

So we can conclude

$${}^P K^0(E\mathbb{Z}^2 \times_\rho S^2) \cong \mathbb{Z}^{\oplus 2},$$
$${}^P K^1(E\mathbb{Z}^2 \times_\rho S^2) \cong \mathbb{Z}^{\oplus 2}.$$

Remark 5.4.1. In the previous example we can calculate $H^0(\mathbb{Z}^2; K^0(S^2))$ and $H^1(\mathbb{Z}^2; K^0(S^2))$ in different ways. For $H^0(\mathbb{Z}^2; K^0(S^2)) \cong K^0(S^2)^{\mathbb{Z}^2}$ for the action described in the example, we obtain $K^0(S^2)^{\mathbb{Z}^2} \cong \mathbb{Z}$.

We continue with

$$H^1(\mathbb{Z}^2; K^0(S^2)) \cong \frac{\mathrm{Der}(\mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}l)}{\mathrm{PDer}(\mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}l)}$$

Let $a = (x, y)$ and $b = (z, w)$

$$d(1,1) = d(1,0) + (1,0)d(0,1) = a + (1,0)b = (x+w)1 + (y+z)l$$
$$d(1,1) = d(0,1) + (0,1)d(1,0) = b + (0,1)a = (y+z)1 + (x+w)l$$

So $\mathrm{Der}(\mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}l) \subseteq \{(a, b) \in (K^0(S^2))^2 \mid x + w = z + y\} \cong \mathbb{Z}^3$. On the other hand, we represent $\mathbb{Z}^2$ as the abelianization of the free group on two generators g, h. Given (a, b) satisfying $x + w = z + y$, we define a derivation d by $d(1, 0) = (x, y)$ and $d(0, 1) = (z, z + y - x)$. To see that this defines a derivation, we use the Exercise 4(a) in Section 4.2 of [11], which says that we need to check $d(ghg^{-1}h^{-1}) = 0$ to verify that d is a derivation.

$$\begin{aligned}
d(ghg^{-1}h^{-1}) &= d(g) + gd(h) + ghd(g^{-1}) + ghg^{-1}d(h^{-1}) \\
&= (x, y) + (1, 0)(z, z + y - x) + (1, 0)(0, 1)(-y, -x) + ghg^{-1}d(h^{-1}) \\
&= (z + y, y + z) + (-y, -x) + (1, 0)(0, 1)(-1, 0)(x - z - y, -z) \\
&= (z, y + z - x) + (-z, x - z - y) \\
&= 0
\end{aligned}$$

Now to describe the principal derivation

$$\begin{aligned}
&d(1, 0) = (1, 0)(x1 + yl) - (x1 + yl) = (y - x)1 + (x - y)l \\
&d(0, 1) = (y - x) + (x - y)l
\end{aligned}$$

$$\begin{tikzcd}
\mathbb{Z}^2 \arrow[r, "\cong"] & \mathrm{PDer}(\mathbb{Z}^2, K^0(S^2)) \arrow[r, hook] & \mathrm{Der}(\mathbb{Z}^2, K^0(S^2)) \arrow[d] \\
(x, y) \arrow[r, mapsto] & ((y - x)1 + (x - y)l, (y - x)1 + (x - y)l) \arrow[dr, mapsto] & \mathbb{Z}^3 \\
& & (y - x, x - y, y - x)
\end{tikzcd}$$

hence

$$\frac{\mathrm{Der}(\mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}l)}{\mathrm{PDer}(\mathbb{Z}^2, \mathbb{Z} \oplus \mathbb{Z}l)} \cong \frac{\mathbb{Z}^3}{\mathbb{Z}(-1, 1, -1) \oplus \mathbb{Z}(1, -1, 1)} = \frac{\mathbb{Z}^3}{\mathbb{Z}(1, -1, 1)} \cong \mathbb{Z}^2.$$

Remark 5.4.2. In this remark we perform the same computation as in the previous example, but when the twist P is trivial. Let $\rho\colon \pi_1(B\mathbb{Z}^2) \times S^2 \to S^2$ be the action given by $\rho((x, y), z) = z$ if $x + y$ is even and $\rho((x, y), z) = -z$ if $x + y$ is odd. We consider the fiber bundle $S^2 \hookrightarrow E\mathbb{Z}^2 \times_\rho S^2 \to B\mathbb{Z}^2$, which is a Borel construction.

Now we get the second page of our spectral sequence

$$E_2^{p,q} \cong H^p(\mathbb{T}^2; K^q(S^2)) \Longrightarrow K^{p+q}(\mathbb{R}^2 \times_\rho M)$$

$$\begin{array}{r|cccc}
3 & H^0(\mathbb{Z}^2; K^3(S^2)) & H^1(\mathbb{Z}^2; K^3(S^2)) & H^2(\mathbb{Z}^2; K^3(S^2)) & H^3(\mathbb{Z}^2; K^3(S^2)) \\
2 & H^0(\mathbb{Z}^2; K^2(S^2)) & H^1(\mathbb{Z}^2; K^2(S^2)) & H^2(\mathbb{Z}^2; K^2(S^2)) & H^3(\mathbb{Z}^2; K^2(S^2)) \\
1 & H^0(\mathbb{Z}^2; K^1(S^2)) & H^1(\mathbb{Z}^2; K^1(S^2)) & H^2(\mathbb{Z}^2; K^1(S^2)) & H^3(\mathbb{Z}^2; K^1(S^2)) \\
0 & H^0(\mathbb{Z}^2; K^0(S^2)) & H^1(\mathbb{Z}^2; K^0(S^2)) & H^2(\mathbb{Z}^2; K^0(S^2)) & H^3(\mathbb{Z}^2; K^0(S^2)) \\
-1 & H^0(\mathbb{Z}^2; K^{-1}(S^2)) & H^1(\mathbb{Z}^2; K^{-1}(S^2)) & H^2(\mathbb{Z}^2; K^{-1}(S^2)) & H^3(\mathbb{Z}^2; K^{-1}(S^2)) \\
-2 & H^0(\mathbb{Z}^2; K^{-2}(S^2)) & H^1(\mathbb{Z}^2; K^{-2}(S^2)) & H^2(\mathbb{Z}^2; K^{-2}(S^2)) & H^3(\mathbb{Z}^2; K^{-2}(S^2)) \\
-3 & H^0(\mathbb{Z}^2; K^{-3}(S^2)) & H^1(\mathbb{Z}^2; K^{-3}(S^2)) & H^2(\mathbb{Z}^2; K^{-3}(S^2)) & H^3(\mathbb{Z}^2; K^{-3}(S^2))
\end{array}$$

$$\begin{array}{r|cccc}
3 & 0 & 0 & 0 & 0 \\
2 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & 0 \\
-1 & 0 & 0 & 0 & 0 \\
-2 & H^0(\mathbb{Z}^2;K^0(S^2)) & H^1(\mathbb{Z}^2;K^0(S^2)) & H^2(\mathbb{Z}^2;K^0(S^2)) & 0 \\
-3 & 0 & 0 & 0 & 0
\end{array}$$

All differentials d_i with $i > 1$ vanish, so it only remains to describe the nonzero terms in this page. For this we use the Lyndon-Hochschild-Serre spectral sequence induced from the short exact sequence

$$0 \to A = \{(m,n) \mid m+n \text{ even}\} \to \mathbb{Z}^2 \to \mathbb{Z}/2 \to 0$$

which has E_2-page given by

$$H^p(\mathbb{Z}/2; H^q(A, K^0(S^2))) \Longrightarrow H^{p+q}(\mathbb{Z}^2; K^0(S^2))$$

On the left hand side, because $\mathbb{Z}^2$ is abelian the action of $\mathbb{Z}/2$ over A is trivial. In this case, it is clear that the action of A over $K^0(S^2)$ is trivial, hence for the following values of q we have

- $q = 0$, $H^0(A; K^0(S^2)) = K^0(S^2)^A = K^0(S^2)$
- $q = 1$, $H^1(A; K^0(S^2)) = \mathrm{Hom}(\mathbb{Z}^2, K^0(S^2)) = K^0(S^2) \oplus K^0(S^2)$
- $q = 2$, $H^2(A; K^0(S^2)) = K^0(S^2)$, because

$$0 \to \mathrm{Ext}(H_1(\mathbb{Z}^2), K^0(S^2)) \to H^2(A, K^0(S^2)) \to \mathrm{Hom}(H_2(\mathbb{Z}^2), K^0(S^2)) \to 0$$

$$0 \to 0 \to H^2(A; K^0(S^2)) \to K^0(S^2) \to 0$$

We can regard $K^0(S^2) \cong \mathbb{Z}[\mathbb{Z}/2]$ as a $\mathbb{Z}/2$–module. Now we check the different values for p. To calculate the cases when $p \neq 0$ we use Example 2 from Section 3.1 in [11], where $\overline{N}\colon M_{\mathbb{Z}/2} \to M^{\mathbb{Z}/2}$ is the norm map, induced by $N\colon M \to M$ with $N(x) = x + \omega x$, here M is a $\mathbb{Z}/2$–module and $\mathbb{Z}/2 = \{1, \omega\}$.

- For $p = 0$
  $H^0(\mathbb{Z}/2; H^0(A; K^0(S^2))) = [K^0(S^2)]^{\mathbb{Z}/2} = \mathbb{Z}$
  $H^0(\mathbb{Z}/2; H^1(A; K^0(S^2))) = [K^0(S^2) \oplus K^0(S^2)]^{\mathbb{Z}/2} = \mathbb{Z}^2$
  $H^0(\mathbb{Z}/2; H^2(A; K^0(S^2))) = H^0(\mathbb{Z}/2; K^0(S^2)) = K^0(S^2)^{\mathbb{Z}/2} = \mathbb{Z}$

- For $p = 1$
  First we calculate $K^0(S^2)_{\mathbb{Z}/2} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{<1-1, l-(2-l)>} = \frac{\mathbb{Z}\oplus\mathbb{Z}}{<2l-2>} = \mathbb{Z} \oplus \mathbb{Z}/2$.
  $H^1(\mathbb{Z}/2; H^0(A; K^0(S^2))) = \mathrm{Ker}[\overline{N}\colon K^0(S^2)_{\mathbb{Z}/2} \to K^0(S^2)^{\mathbb{Z}/2}] = \mathbb{Z}/2$
  $H^1(\mathbb{Z}/2; H^1(A; K^0(S^2))) = \mathrm{Ker}[\overline{N}\colon (K^0(S^2)^{\oplus 2})_{\mathbb{Z}/2} \to (K^0(S^2)^{\oplus 2})^{\mathbb{Z}/2}] = \mathbb{Z}/2 \oplus \mathbb{Z}/2$

- For $p = 2$
  $H^2(\mathbb{Z}/2; H^0(A; K^0(S^2))) = \mathrm{Coker}[\overline{N}\colon K^0(S^2)_{\mathbb{Z}/2} \to K^0(S^2)^{\mathbb{Z}/2}] = \mathbb{Z}/2$

$$\begin{array}{c|lll}
3 & H^0(\mathbb{Z}/2; H^3(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^3(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^3(A; K^0(S^2))) \\
2 & H^0(\mathbb{Z}/2; H^2(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^2(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^2(A; K^0(S^2))) \\
1 & H^0(\mathbb{Z}/2; H^1(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^1(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^1(A; K^0(S^2))) \\
0 & H^0(\mathbb{Z}/2; H^0(A; K^0(S^2))) & H^1(\mathbb{Z}/2; H^0(A; K^0(S^2))) & H^2(\mathbb{Z}/2; H^0(A; K^0(S^2)))
\end{array}$$

Replacing the computed groups, we obtain

$$\begin{array}{c|ccc}
3 & 0 & 0 & 0 \\
2 & \mathbb{Z} & & \\
1 & \mathbb{Z}^2 & (\mathbb{Z}/2)^2 & \\
0 & \mathbb{Z} & \mathbb{Z}/2 & \mathbb{Z}/2
\end{array}$$

Then for $p + q = 1$ in the E_2 of the spectral sequence

$$0 \to \mathbb{Z}/2 \to H^1(\mathbb{Z}^2; K^0(S^2)) \to \mathbb{Z}^2 \to 0$$

For $p + q = 2$

$$0 \to F^1H^2(\mathbb{Z}^2; K^0(S^2)) \to H^2(\mathbb{Z}^2; K^0(S^2)) \to \mathbb{Z} \to 0$$

$$0 \to \mathbb{Z}/2 \to F^1H^2(\mathbb{Z}^2; K^0(S^2)) \to (\mathbb{Z}/2)^2 \to 0$$

With this in mind and renaming $F^1H^2(\mathbb{Z}^2; K^0(S^2))$ as B, which must be an abelian group of order eight.

$$H^0(\mathbb{Z}^2; K^0(S^2)) \cong \mathbb{Z}$$

$$H^1(\mathbb{Z}^2; K^0(S^2)) \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2$$
$$H^2(\mathbb{Z}^2; K^0(S^2)) \cong \mathbb{Z} \oplus B$$

Replacing these groups in the original spectral sequence we have

$$\begin{array}{r|cccc}
3 & 0 & 0 & 0 & 0 \\
2 & \mathbb{Z} & \mathbb{Z}^2 \oplus \mathbb{Z}/2 & \mathbb{Z} \oplus B & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}^2 \oplus \mathbb{Z}/2 & \mathbb{Z} \oplus B & 0 \\
-1 & 0 & 0 & 0 & 0 \\
-2 & \mathbb{Z} & \mathbb{Z}^2 \oplus \mathbb{Z}/2 & \mathbb{Z} \oplus B & 0 \\
-3 & 0 & 0 & 0 & 0
\end{array}$$

Finally we have

$$K^0(E\mathbb{Z}^2 \times_\rho S^2) \cong \mathbb{Z}^{\oplus 2} \oplus B,$$
$$K^1(E\mathbb{Z}^2 \times_\rho S^2) \cong \mathbb{Z}^{\oplus 2} \oplus \mathbb{Z}/2.$$

Remark 5.4.3. From the main example of this section and Remark 5.4.2 we deduce that the projective bundles constructed in Section 5.1 are not necessarily trivial.

Chapter 6

Twisted differential K-theory

In this chapter we describe the Freed-Lott and Carey-Mickelsson-Wan models for differential extension for twisted K-theory, the first is based on the Grothendieck group of a suitable semi-group and twisted. The second one is related to sections and a suitable space. At first we make a topological identification of them and finally we indicate a path for a differential equivalence.

6.1 Models of twisted K-theory

We briefly review the construction of two relevant models of twisted K-theory in the topological framework. We start from the groups of degree 0. We will consider the extension to any degree in the last section.

Notation 6.1.1. We use the following notation.

- We denote by X a fixed compact topological space in which good covers exist and are cofinal (e.g. a finite CW-complex).
- We denote by $\mathfrak{U} = \{U_i\}_{i \in I}$ a fixed good cover of X.
- We denote by $\underline{\mathrm{U}}(1)$ the sheaf of $\mathrm{U}(1)$-valued continuous functions on X.
- We denote by $\check{Z}^\bullet(\mathfrak{U}, \underline{\mathrm{U}}(1))$ and by $\check{H}^\bullet(\mathfrak{U}, \underline{\mathrm{U}}(1))$ respectively the Cech cocycle and cohomology groups of the sheaf $\underline{\mathrm{U}}(1)$ with respect to $\mathfrak{U}$. Since $\mathfrak{U}$ is a good cover, $\check{H}^\bullet(\mathfrak{U}, \underline{\mathrm{U}}(1)) \simeq \check{H}^\bullet(X, \underline{\mathrm{U}}(1)) \simeq H^{\bullet+1}(X; \mathbb{Z})$ canonically.
- We denote by $\mathcal{H}$ a fixed separable infinite-dimensional Hilbert space and by $\mathrm{Fred}(\mathcal{H})$ the space of Fredholm operators in $\mathcal{H}$.

6.1.1 Model through Fredholm operators

Since $\mathrm{Fred}(\mathcal{H})$ is a classifying space for K-theory, a natural model for twisted K-theory consists of homotopy classes of sections of a suitable $\mathrm{Fred}(\mathcal{H})$-bundle, so that the twisting is encoded in the non-triviality of such bundle. Usually this model is constructed starting from a projective Hilbert bundle, but with this approach it is not canonical even fixing a twisting cocycle ζ (not only the cohomology class $[\zeta]$). On the contrary, if we start from a twisted Hilbert bundle (before projectivizing it), we get a canonical group $K_\zeta(X)$, as we are going to review.

Twisted Hilbert bundles.

Definition 6.1.2. Given a cocycle $\zeta := \{\zeta_{ijk}\} \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, a ζ*-twisted Hilbert bundle* with fibre $\mathcal{H}$ on X is a collection of (trivial) Hermitian vector bundles $\pi_i \colon E_i \to U_i$, with typical fibre $\mathcal{H}$, and of unitary vector bundle isomorphisms $\varphi_{ij} \colon E_i|_{U_{ij}} \to E_j|_{U_{ij}}$, such that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \zeta_{ijk} \cdot \mathrm{id}$.

Definition 6.1.3. Given two ζ-twisted Hilbert bundles $E := (\{E_i\}, \{\varphi_{ij}\})$ and $F := (\{F_i\}, \{\psi_{ij}\})$, for any fixed $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, a *morphism* from E to F is a collection of vector bundle morphisms $f_i \colon E_i \to F_i$ such that $f_j \circ \varphi_{ij} = \psi_{ij} \circ f_i$ for every $i, j \in I$. The morphism is called *unitary* if each f_i is.

Of course, an *isomorphism* is an invertible morphism, and this is equivalent to requiring that each f_i is an isomorphism. For every $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ there exists a ζ-twisted Hilbert bundle (see [4]) and, fixing ζ, any two ζ-twisted Hilbert bundles are isomorphic (see [31]).

Projective Hilbert bundles.

Given a twisted bundle $E = (\{E_i\}, \{\varphi_{ij}\})$, projecting each fibre $(E_i)_x \setminus \{0\}$ to the corresponding projective space, we get a well-defined (non-twisted) projective bundle with typical fibre $\mathbb{P}(\mathcal{H})$, that we denote by $\mathbb{P}(E)$. It follows from local triviality that every projective bundle can be obtained in this way up to isomorphism. Moreover, it is straightforward to verify that an isomorphism of ζ-twisted bundles induces an isomorphism of projective bundles, hence, fixing ζ, the unique isomorphism class of ζ-twisted Hilbert bundles induces a unique isomorphism class of projective bundles. Let us now fix ζ and ζ' cohomologous. We call $\mathrm{HB}_\zeta(X)$ the set of ζ-twisted bundles on X (*Notation* quotiented out up to isomorphism) and we set $\zeta' = \zeta \cdot \check{\delta}^1\eta$. We get the bijection

$$\begin{aligned} \Phi_\eta \colon \mathrm{HB}_\zeta(X) &\xrightarrow{\simeq} \mathrm{HB}_{\zeta'}(X) \\ E = (\{E_i\}, \{\varphi_{ij}\}) &\mapsto \Phi_\eta(E) := (\{E_i\}, \{\varphi_{ij}\eta_{ij}\}). \end{aligned} \tag{6.1}$$

Since $\mathbb{P}(E) = \mathbb{P}(\Phi_\eta(E))$, the isomorphism class of $\mathbb{P}(E)$ only depends on the cohomology class $[\zeta] \in \check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. Moreover, such isomorphism class is well-behaved with respect to cover refinements (see section 6.1.3 below for details, in particular diagrams (6.8) and (6.10)), hence it only depends on $[\zeta] \in$

$\check{H}^2(X, \underline{\mathrm{U}}(1)) \simeq H^3(X;\mathbb{Z})$. Conversely, it is easy to verify that, if $\mathbb{P}(E) \simeq \mathbb{P}(E')$, then $[\zeta] = [\zeta']$, therefore $H^3(X;\mathbb{Z})$ classifies projective Hilbert bundles on X. In fact, by local triviality, an isomorphism $\bar{f}\colon \mathbb{P}(E) \to \mathbb{P}(E')$ can be lifted to $f_i\colon E_i \to E'_i$ for each i. Since the family $\{f_i\}$ glues to $\bar{f}$, there exists a $\underline{\mathrm{U}}(1)$-cochain η_{ij} such that $f_j\varphi_{ij}\eta_{ij} = \psi_{ij}f_i$. It follows that

$$\begin{aligned}\zeta'_{ijk}f_i = \psi_{ki}\psi_{jk}\psi_{ij}f_i &= \eta_{ij}\psi_{ki}\psi_{jk}f_j\varphi_{ij} = \eta_{ij}\eta_{jk}\psi_{ki}f_k\varphi_{jk}\varphi_{ij}\\ &= \eta_{ij}\eta_{jk}\eta_{ki}f_i\varphi_{ki}\varphi_{jk}\varphi_{ij} = \eta_{ij}\eta_{jk}\eta_{ki}\zeta_{ijk}f_i,\end{aligned}$$

hence $\zeta' = \zeta \cdot \check{\delta}^1\eta$, therefore $[\zeta] = [\zeta']$.

If $\check{\delta}^1\eta = 1$ in (6.1), then both E and $\Phi_\eta(E)$ are ζ-twisted, hence there exists an isomorphism $f = \{f_i\}\colon \Phi_\eta(E) \to E$. This means that $f_i\colon E_i \to E_i$ and $\varphi_{ij}f_i = f_j\varphi_{ij}\eta_{ij}$, hence f induces an automorphism $\bar{f}\colon \mathbb{P}(E) \to \mathbb{P}(E)$. Any automorphism $\bar{f}$ can be realized in this way from suitable η and f. In fact, the computation of the previous paragraph shows that $\bar{f}$ can be lifted to $f_i\colon E_i \to E_i$ for each i, such that $f_j\varphi_{ij}\eta_{ij} = \varphi_{ij}f_i$ for a suitable cochain η_{ij} satisfying $\zeta = \zeta\cdot\check{\delta}\eta$, i.e. $\check{\delta}^1\eta = 1$. The only freedom we have in constructing the cocycle η is the choice of the lifts f_i. Any other choice is of the form $f_i\xi_i$, that replaces η by $\eta\cdot\check{\delta}^0\xi^{-1}$. Therefore, the following map is well-defined:

$$\begin{aligned}\Phi\colon \mathrm{Aut}(\mathbb{P}(E)) &\to H^2(X,\mathbb{Z})\\ \bar{f} &\mapsto [\{\eta_{ij}\}].\end{aligned} \tag{6.2}$$

It is easy to prove that it is a group homomorphism. Moreover, it follows from the previous construction that $\bar{f} \in \mathrm{Aut}(\mathbb{P}(E))$ lifts to an automorphisms of E if and only if $\Phi(\bar{f}) = 0$, therefore $\Phi(\bar{f})$ can be thought of as the obstruction to the existence such a lift. The following lemma is a consequence of the fact that $\mathrm{PU}(\mathcal{H})$ is an Eilenberg-MacLane space $K(\mathbb{Z},2)$ [4, Prop. 2.2].

Lemma 6.1.4. *The morphism* (6.2) *is surjective. Moreover, its kernel is the connected component of the identity of* $\mathrm{Aut}(\mathbb{P}(E))$*, therefore* Φ *induces a canonical bijection between the connected components of* $\mathrm{Aut}(\mathbb{P}(E))$ *and* $H^2(X,\mathbb{Z})$*.*

Definition of twisted K-theory.

We fix a cocycle $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ and a ζ-twisted Hilbert bundle $E = (\{E_i\}, \{\varphi_{ij}\})$, inducing the corresponding projective bundle $\mathbb{P}(E)$. We denote by $P_{\mathbb{P}(E)}$ the bundle of projective reference frames of $\mathbb{P}(E)$. We have a natural adjoint action of the projective unitary group $\mathrm{PU}(\mathcal{H})$ on the topological space $\mathrm{Fred}(\mathcal{H})$ by conjugation,[1] that we denote by $\rho\colon \mathrm{PU}(\mathcal{H}) \to C^0(\mathrm{Fred}(\mathcal{H}))$, $U \mapsto (A \mapsto UAU^{-1})$, hence we construct the associated $\mathrm{Fred}(\mathcal{H})$-bundle $F_{\mathbb{P}(E)} := P_{\mathbb{P}(E)} \times_\rho \mathrm{Fred}(\mathcal{H})$. Since the composition of two Fredholm operators is Fredholm too, the set $\mathrm{Fred}(\mathcal{H})$ has a natural structure of monoid. The action of $\mathrm{PU}(\mathcal{H})$ by conjugation respects composition, therefore $F_{\mathbb{P}(E)}$ is a bundle of monoids. It follows

[1] We consider the norm topology in the space of bounded linear operators in $\mathcal{H}$ and we restrict it to $\mathrm{Fred}(\mathcal{H})$.

that its set of global sections, that we denote by $\Gamma(F_{\mathbb{P}(E)})$, inherits a monoid structure as well. Quotienting $\Gamma(F_{\mathbb{P}(E)})$ up to homotopy of sections, we get an abelian group, the opposite of $[s]$ being $[s^*]$, where s^* is point-wise adjoint to s. We denote such a group by $\bar{\Gamma}(F_{\mathbb{P}(E)})$.

Definition 6.1.5. The *twisted K-theory group* $K_\zeta(X)$ is defined as the abelian group $\bar{\Gamma}(F_{\mathbb{P}(E)})$ for any ζ-twisted Hilbert bundle E.

Dependence on the cocycle.

Definition 6.1.5 seems to depend on E, not only on ζ. Nevertheless, fixing two ζ-twisted bundles E and E', an isomorphism $f\colon E \to E'$ is unique up to an automorphism of E. It follows from lemma 6.1.4 that the induced isomorphism $\bar{f}\colon \mathbb{P}(E) \to \mathbb{P}(E')$ is unique up to an automorphism of $\mathbb{P}(E)$ connected to the identity, the latter inducing the identity on $\bar{\Gamma}(F_{\mathbb{P}(E)})$. Hence, $K_\zeta(X)$ is canonically defined.

On the contrary, the definition is not canonical if we only fix the cohomology class $[\zeta]$. In fact, let us consider ζ and ζ' cohomologous. We fix $\eta \in \check{C}^1(\mathfrak{U}; \underline{\mathrm{U}}(1))$ such that $\zeta' = \zeta \cdot \check{\delta}^1\eta$. For any ζ-twisted bundle E, we have $\mathbb{P}(E) = \mathbb{P}(\Phi_\eta(E))$, the r.h.s. being ζ'-twisted. Hence, we get the isomorphism

$$\Phi_\eta\colon K_\zeta(X) \xrightarrow{\simeq} K_{\zeta'}(X) \tag{6.3}$$

defined as the identity between the representatives $\bar{\Gamma}(F_{\mathbb{P}(E)})$ and $\bar{\Gamma}(F_{\mathbb{P}(\Phi_\eta(E))})$ respectively.

The isomorphism (6.3) depends on η up to coboundaries. In fact, any other choice of η is of the form $\eta \cdot \nu$, where $\nu \in \check{Z}^1(\mathfrak{U}, \underline{\mathrm{U}}(1))$. We have $\Phi_{\eta\cdot\nu} = \Phi_\nu \circ \Phi_\eta$ and, because of lemma 6.1.4, Φ_ν is the identity if and only if ν is a coboundary. This implies that the set of isomorphisms of the form (6.3) is a torsor over $H^2(X;\mathbb{Z})$, hence, if $\zeta = \zeta'$, we get an action of $H^2(X;\mathbb{Z})$ on $K_\zeta(X)$. Only the quotient up to this action is canonically defined for a fixed class $[\zeta]$. Of course, if $H^2(X;\mathbb{Z}) = 0$, then the quotient is trivial, therefore the group $K_{[\zeta]}(X)$, with $[\zeta] \in \check{H}^2(\mathfrak{U}; \underline{\mathrm{U}}(1))$, is well-defined. In this case, $K_{[\zeta]}(X)$ does not depend on the cover even (see section 6.1.3 below for details), since we can take the direct limit with respect to $\mathfrak{U}$, hence the group $K_{[\zeta]}(X)$, with $[\zeta] \in \check{H}^2(X; \underline{\mathrm{U}}(1)) \simeq H^3(X;\mathbb{Z})$, is well-defined.

Remark 6.1.6. If $H^2(X,\mathbb{Z}) = 0$, we can show that $K_{[\zeta]}(X)$ is well-defined in the following equivalent way. Given a ζ-twisted bundle E and a ζ'-twisted bundle E', with ζ and ζ' cohomologous, we fix any isomorphism $\bar{f}\colon \mathbb{P}(E) \to \mathbb{P}(E')$. The latter is unique up to an automorphism of $\mathbb{P}(E)$, that is necessarily connected to the identity by lemma 6.1.4. Hence, the induced morphism $\bar{f}_*\colon \bar{\Gamma}(F_{\mathbb{P}(E)}) \to \bar{\Gamma}(F_{\mathbb{P}(E')})$ does not depend on $\bar{f}$, therefore $K_{[\zeta]}(X)$ is canonically defined.

6.1.2 Model through finite-dimensional twisted bundles

When the cohomology class $[\zeta]$ has finite order, the Grothendieck group of finite-dimensional ζ-twisted vector bundles provides another model for twisted K-theory. In particular, the following definition is analogous to 6.1.2.

Definition 6.1.7. Given a cocycle $\zeta := \{\zeta_{ijk}\} \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, a *$\zeta$-twisted vector bundle* of rank r on X is a collection of (trivial) Hermitian vector bundles $\pi_i \colon E_i \to U_i$ of rank r and of unitary vector bundle isomorphisms $\varphi_{ij} \colon E_i|_{U_{ij}} \to E_j|_{U_{ij}}$, such that $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \zeta_{ijk} \cdot \mathrm{id}$.

The definition of (iso)morphism is identical to 6.1.3. Given a twisted vector bundle $E := (\{E_i\}, \{\varphi_{ij}\})$ of rank r, for each $i \in I$ we can fix a set of r pointwise-independent local sections $s_{1,i}, \dots, s_{r,i} \colon U_i \to E_i$ of unit norm, determining vector bundle isomorphisms $\xi_i \colon E_i \to U_i \times \mathbb{C}^r$, $\lambda^k s_{i,k}(x) \mapsto (x, (\lambda^1, \dots, \lambda^r))$. The isomorphisms φ_{ij} determine local transition functions $g_{ij} \colon U_{ij} \to \mathrm{U}(r)$ such that $\varphi_{ij}(\xi_i^{-1}(x, \lambda)) = \xi_j^{-1}(x, g_{ij}(x) \cdot \lambda)$. Equivalently, $g_{ij}(x)$ is the change of basis in $(E_j)_x$ from $\{s_{j,1}(x), \dots, s_{j,r}(x)\}$ to $\{\varphi_{ij}(s_{i,1}(x)), \dots, \varphi_{ij}(s_{i,r}(x))\}$. The condition $\varphi_{ki}\varphi_{jk}\varphi_{ij} = \zeta_{ijk} \cdot \mathrm{id}$ is equivalent to $g_{ki}g_{jk}g_{ij} = \zeta_{ijk} \cdot I_n$. It is straightforward to verify, as for ordinary vector bundles, that the equivalence class $[\{g_{ij}\}]$, up to conjugation by $\{h_i \colon U_i \to \mathrm{U}(r)\}$, only depends on the isomorphism class of $E := (\{E_i\}, \{\varphi_{ij}\})$, in such a way that we get the natural bijection $[E] \mapsto [\{g_{ij}\}]$.

Remark 6.1.8. The class $[\zeta]$ is necessarily torsion. In fact, computing the determinants, we get $\det(g_{ki})\det(g_{jk})\det(g_{ij}) = \zeta_{ijk}^r$; since $\det(g_{ij})$ is a $\mathrm{U}(1)$-valued function, this shows that $\{\zeta_{ijk}^r\}$ is a trivial cocyle, hence $[\zeta]^r = 1$. Equivalently, $r[\zeta] = 0$ in $H^3(X; \mathbb{Z})$. In particular, the order of $[\zeta]$ divides the rank of the bundle. One can prove that, for any cocycle representing a torsion class, there exists a corresponding twisted bundle (see [4]).

We denote by $\mathrm{VB}^r_\zeta(X)$ the set of ζ-twisted vector bundles of rank r up to isomorphism. The direct sum is defined as $(\{E_i\}, \{\varphi_{ij}\}) \oplus (\{F_i\}, \{\psi_{ij}\}) := (\{E_i \oplus F_i\}, \{\varphi_{ij} \oplus \psi_{ij}\})$. The set $\mathrm{VB}_\zeta(X) := \bigoplus_{r \in \mathbb{N}} \mathrm{VB}^r_\zeta(X)$, endowed with this operation, is a commutative semi-group, hence we can consider the corresponding Grothendieck group, that we call *ζ-twisted K-theory group* of X and we denote by $K_\zeta(X)$.

Dependence on the cocycle.

If ζ and ζ' are cohomologous and we fix η, such that $\zeta' = \zeta \cdot \check{\delta}^1\eta$, the isomorphism (6.1) holds for finite-rank bundles too and it extends to the corresponding Grothendieck groups, defining $\Phi_\eta \colon K_\zeta(X) \xrightarrow{\simeq} K_{\zeta'}(X)$, analogous to (6.3). This shows that the isomorphism class of the group $K_\zeta(X)$ only depends on $[\zeta]$ in a non-canonical way. Moreover, since Φ_η depends on η only up to coboundaries, the set of isomorphisms of the form Φ_η is a torsor over $\check{H}^1(\mathfrak{U}, \underline{\mathrm{U}}(1)) \simeq H^2(X; \mathbb{Z})$. In particular, if $H^2(X; \mathbb{Z}) = 0$, then the group $K_{[\zeta]}(X)$ is canonically defined, independently of the cover too (see section 6.1.3 below for details), otherwise only the quotient up to the action of $H^2(X; \mathbb{Z})$ depends on $[\zeta]$ in a canonical way.

6.1.3 Good refinements

Let us suppose that $\mathfrak{V} = \{V_\alpha\}_{\alpha\in\Lambda}$ is a good cover and a refinement of the fixed good cover $\mathfrak{U} = \{U_i\}_{i\in I}$ through the function $\phi\colon \Lambda \to I$. This means that $V_\alpha \subset U_{\phi(\alpha)}$ for every $\alpha \in \Lambda$. We get the induced morphism of cochain complexes $\phi^*\colon \check{C}^\bullet(\mathfrak{U}, \underline{\mathrm{U}}(1)) \to \check{C}^\bullet(\mathfrak{V}, \underline{\mathrm{U}}(1))$ and, fixing $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, we set $\hat{\zeta} := \phi^*\zeta$. We get the function

$$\begin{aligned} \phi^*\colon \mathrm{HB}_\zeta(X) &\longrightarrow \mathrm{HB}_{\hat{\zeta}}(X) \\ E = (\{E_i\}, \{\varphi_{ij}\}) &\mapsto \phi^*E = (\{F_\alpha\}, \{\psi_{\alpha\beta}\}), \end{aligned} \tag{6.4}$$

where $F_\alpha := E_{\phi(\alpha)}|_{V_\alpha}$ and $\psi_{\alpha\beta} := \varphi_{\phi(\alpha)\phi(\beta)}|_{V_{\alpha\beta}}$. For every $E \in \mathrm{HB}_\zeta(X)$, we have the isomorphism

$$\overline{\phi^*}^E\colon \mathbb{P}(E) \xrightarrow{\simeq} \mathbb{P}(\phi^*E), \tag{6.5}$$

whose inverse identifies the projectivized fibre $\mathbb{P}(F_\alpha)_x$ with the corresponding one $\mathbb{P}(E_{\phi(\alpha)})_x$. Let us show that:

- the isomorphism (6.5) induces the (well-defined) isomorphism

$$\overline{\phi^*}\colon K_\zeta(X) \xrightarrow{\simeq} K_{\hat{\zeta}}(X); \tag{6.6}$$

- if $H^2(X;\mathbb{Z}) = 0$, then (6.6) induces the (well-defined) isomorphism

$$\overline{\overline{\phi^*}}\colon K_{[\zeta]}(X) \xrightarrow{\simeq} K_{[\hat{\zeta}]}(X), \tag{6.7}$$

  that does not depend on ϕ any more, where $[\zeta] \in \check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ and $[\hat{\zeta}] \in \check{H}^2(\mathfrak{V}, \underline{\mathrm{U}}(1))$. This implies that, if $H^2(X;\mathbb{Z}) = 0$, then $K_{[\zeta]}(X)$, with $[\zeta] \in \check{H}^2(X, \underline{\mathrm{U}}(1)) \simeq H^3(X;\mathbb{Z})$, is canonically defined.

Isomorphism (6.6)**.**

We fix any $E \in \mathrm{HB}_\zeta(X)$ and we represent $K_\zeta(X)$ by $\bar{\Gamma}(F_{\mathbb{P}(E)})$ and $K_{\hat{\zeta}}(X)$ by $\bar{\Gamma}(F_{\mathbb{P}(\phi^*E)})$. Fixing an isomorphism of ζ-twisted Hilbert bundles $f\colon E \xrightarrow{\simeq} E'$, where $E = (\{E_i\}, \{\varphi_{ij}\})$, $E' = (\{E'_i\}, \{\varphi'_{ij}\})$ and $f = \{f_i\}$, we get the induced isomorphism of $\hat{\zeta}$-twisted bundles $\phi^*f\colon \phi^*E \xrightarrow{\simeq} \phi^*E'$, where $\phi^*f = \{f'_\alpha := f_{\phi(\alpha)}|_{V_\alpha}\}$, in such a way that the following diagram commutes:

$$\begin{tikzcd} \mathbb{P}(E) \arrow[r, "\bar{f}"] \arrow[d, "\overline{\phi^*}^E"'] & \mathbb{P}(E') \arrow[d, "\overline{\phi^*}^{E'}"] \\ \mathbb{P}(\phi^*E) \arrow[r, "\overline{\phi^*f}"] & \mathbb{P}(\phi^*E'). \end{tikzcd} \tag{6.8}$$

The isomorphisms $(\bar{f})_*\colon \bar{\Gamma}(F_{\mathbb{P}(E)}) \to \bar{\Gamma}(F_{\mathbb{P}(E')})$ and $(\overline{\phi^*f})_*\colon \bar{\Gamma}(F_{\mathbb{P}(\phi^*E)}) \to \bar{\Gamma}(F_{\mathbb{P}(\phi^*E')})$ are the natural changes of representative respectively for $K_\zeta(X)$ and $K_{\hat{\zeta}}(X)$. Therefore, diagram (6.8) shows that $\big(\overline{\phi^*}^E\big)_*\colon \bar{\Gamma}(F_{\mathbb{P}(E)}) \to \bar{\Gamma}(F_{\mathbb{P}(\phi^*E)})$ is compatible with the change of representative from E to E', hence it induces (6.6).

Changing the refinement function.

Let us choose the same refinement $\mathfrak{V}$ of $\mathfrak{U}$, but another refinement function $\rho\colon \Lambda \to I$. We set $\hat{\hat{\zeta}} := \rho^*\zeta$. Given $E = (\{E_i\}, \{\varphi_{ij}\})$, we set $\phi^*E = (\{F_\alpha\}, \{\psi_{\alpha\beta}\})$ and $\rho^*E = (\{F'_\alpha\}, \{\psi'_{\alpha\beta}\})$. We have the natural isomorphisms $\xi_\alpha := \varphi_{\phi(\alpha)\rho(\alpha)}|_{V_\alpha}\colon F_\alpha \xrightarrow{\simeq} F'_\alpha$, that do not glue to a twisted bundle isomorphism from ϕ^*E to ρ^*E in general, but they do between the corresponding projective bundles (hence between the corresponding K-theory groups). In fact, we have:

$$\begin{aligned}
\psi'_{\alpha\beta} \circ \xi_\alpha &= \varphi_{\rho(\alpha)\rho(\beta)} \circ \varphi_{\phi(\alpha)\rho(\alpha)} = \varphi_{\phi(\alpha)\rho(\beta)} \cdot \zeta_{\phi(\alpha)\rho(\alpha)\rho(\beta)} \\
\xi_\beta \circ \psi_{\alpha\beta} &= \varphi_{\phi(\beta)\rho(\beta)} \circ \varphi_{\phi(\alpha)\phi(\beta)} = \varphi_{\phi(\alpha)\rho(\beta)} \cdot \zeta_{\phi(\alpha)\phi(\beta)\rho(\beta)}.
\end{aligned}$$

Hence, if we set $\hat{\eta}_{\alpha\beta} := \zeta_{\phi(\alpha)\rho(\alpha)\rho(\beta)} \cdot \zeta^{-1}_{\phi(\alpha)\phi(\beta)\rho(\beta)}$, the following diagram commutes:

$$\begin{tikzcd}
F_\alpha \arrow[r, "\psi_{\alpha\beta}\cdot\hat{\eta}_{\alpha\beta}"] \arrow[d, "\xi_\alpha"'] & F_\beta \arrow[d, "\xi_\beta"] \\
F'_\alpha \arrow[r, "\psi'_{\alpha\beta}"] & F'_\beta.
\end{tikzcd}$$

It follows that $\hat{\hat{\zeta}} = \hat{\zeta} \cdot \check{\delta}^1\hat{\eta}$, as the reader can verify by direct computation too. We obtain the isomorphism

$$\xi^E_{\phi,\rho} := \{\xi_\alpha\}\colon \Phi_{\hat{\eta}}(\phi^*E) \xrightarrow{\simeq} \rho^*E, \tag{6.9}$$

inducing $\bar{\xi}^E_{\phi,\rho}\colon \mathbb{P}(\phi^*E) \xrightarrow{\simeq} \mathbb{P}(\rho^*E)$, in such a way that the following diagram commutes:

$$\begin{tikzcd}
& \mathbb{P}(E) \arrow[dl, "\overline{\phi^*}^E"'] \arrow[dr, "\overline{\rho^*}^E"] & \\
\mathbb{P}(\phi^*E) \arrow[rr, "\bar{\xi}^E_{\phi,\rho}"] & & \mathbb{P}(\rho^*E).
\end{tikzcd} \tag{6.10}$$

The two morphisms $\overline{\phi^*}^E$ and $\overline{\rho^*}^E$ induce respectively $\overline{\phi^*}$ and $\overline{\rho^*}$ in K-theory, as we have seen above. Let us see that $\bar{\xi}^E_{\phi,\rho}$ induces the isomorphism (6.3) corresponding to $\hat{\eta}$, i.e. $\Phi_{\hat{\eta}}\colon K_{\hat{\zeta}}(X) \xrightarrow{\simeq} K_{\hat{\hat{\zeta}}}(X)$. In fact, the latter is by definition the identity between $\bar{\Gamma}(F_{\mathbb{P}(\phi^*E)})$ and $\bar{\Gamma}(F_{\mathbb{P}(\Phi_{\hat{\eta}}(\phi^*E))})$. Moreover, in $K_{\hat{\hat{\zeta}}}(X)$, we identify $\bar{\Gamma}(F_{\mathbb{P}(\Phi_{\hat{\eta}}(\phi^*E))})$ with $\bar{\Gamma}(F_{\mathbb{P}(\rho^*E)})$ through any twisted bundle isomorphism, like $\xi^E_{\phi,\rho}$, hence $(\bar{\xi}^E_{\phi,\rho})_*$ is the canonical identification. Therefore, (6.10) induces the following commutative diagram:

$$\begin{tikzcd}
& K_\zeta(X) \arrow[dl, "\overline{\phi^*}"'] \arrow[dr, "\overline{\rho^*}"] & \\
K_{\hat{\zeta}}(X) \arrow[rr, "\Phi_{\hat{\eta}}"] & & K_{\hat{\hat{\zeta}}}(X).
\end{tikzcd} \tag{6.11}$$

Isomorphism (6.7)**.**

The cochain $\hat{\eta}$ (hence the isomorphism $\Phi_{\hat{\eta}}$) is uniquely determined by the pair (ϕ, ρ). If $H^2(X;\mathbb{Z}) = 0$, then $\hat{\eta}$ is uniquely determined by $\hat{\zeta}$ and $\hat{\hat{\zeta}}$ up to coboundaries. In this case, as we have seen above, $\Phi_{\hat{\eta}}$ does not depend on $\hat{\eta}$ and it is the natural isomorphism between $K_{\hat{\zeta}}(X)$ and $K_{\hat{\hat{\zeta}}}(X)$.

Maintaining the hypothesis $H^2(X;\mathbb{Z}) = 0$, let us fix $\zeta, \zeta' \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ cohomologous and the unique (up to coboundaries) $\nu \in \check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(1))$ such that $\zeta' = \zeta \cdot \check{\delta}^1\nu$. Fixing a refinement $\mathfrak{V}$ of $\mathfrak{U}$, through the function $\phi\colon \Lambda \to I$, we set $\hat{\zeta} := \phi^*\zeta$, $\hat{\zeta}' := \phi^*\zeta'$ and $\hat{\nu} := \phi^*\nu$. The following diagram commutes:

$$\begin{tikzcd}
K_\zeta(X) \arrow[r, "\Phi_\nu"] \arrow[d, "\overline{\phi^*}"'] & K_{\zeta'}(X) \arrow[d, "\overline{\phi^*}"] \\
K_{\hat{\zeta}}(X) \arrow[r, "\Phi_{\hat{\nu}}"] & K_{\hat{\zeta}'}(X).
\end{tikzcd} \tag{6.12}$$

In fact, choosing a ζ-twisted bundle E to represent $K_\zeta(X)$ and $\Phi_\nu(E)$ to represent $K_{\zeta'}(X)$, we get the following commutative diagram:

$$\begin{tikzcd}
\mathbb{P}(E) \arrow[r, equal] \arrow[d, "\overline{\phi^*}^E"'] & \mathbb{P}(\Phi_\nu(E)) \arrow[d, "\overline{\phi^*}^{\Phi_\nu(E)}"] \\
\mathbb{P}(\phi^*E) \arrow[r, equal] & \mathbb{P}(\phi^*\Phi_\nu(E)),
\end{tikzcd} \tag{6.13}$$

where, in the bottom-right position, we have $\phi^*\Phi_\nu(E) = \Phi_{\hat{\nu}}(\phi^*E)$. This proves the commutativity of (6.12).

Joining (6.11) and (6.12), we get the following diagram:

$$\begin{tikzcd}
& & & & K_{\zeta'}(X) \arrow[dl, dotted, "\overline{\phi^*}"'] \arrow[dr, "\overline{\rho^*}"] & \\
& K_\zeta(X) \arrow[urrr, "\Phi_\nu"] \arrow[dl, "\overline{\phi^*}"'] \arrow[dr, "\overline{\rho^*}"] & & K_{\hat{\zeta}'}(X) \arrow[rr, dotted, "\Phi_{\hat{\eta}'}"] & & K_{\hat{\hat{\zeta}}'}(X) \\
K_{\hat{\zeta}}(X) \arrow[rr, "\Phi_{\hat{\eta}}"] \arrow[urrr, dotted, "\Phi_{\hat{\nu}}"] & & K_{\hat{\hat{\zeta}}}(X). \arrow[urrr, "\Phi_{\hat{\hat{\nu}}}"'] & & &
\end{tikzcd} \tag{6.14}$$

The isomorphisms Φ are the natural changes of representative in K-theory, hence the commutativity of (6.14) shows that the isomorphism $\overline{\phi^*}\colon K_\zeta(X) \xrightarrow{\simeq} K_{\hat{\zeta}}(X)$ is compatible both with change of representative and with change of refinement function (from ϕ to ρ), hence it induces (6.7). It follows that we can take the direct limit over $\mathfrak{U}$ and we get the well-defined group $K_{[\zeta]}(X)$, with $[\zeta] \in \check{H}^2(X;\underline{\mathrm{U}}(1)) \simeq H^3(X;\mathbb{Z})$, without fixing any (good) cover of X.

Finite order

If the order of $[\zeta]$ is finite, we get the function

$$\phi^*\colon \mathrm{VB}_{\zeta}(X) \xrightarrow{\simeq} \mathrm{VB}_{\hat{\zeta}}(X), \tag{6.15}$$

defined as (6.4) (up to isomorphism in the domain and in the codomain), that is a bijection (see [31, Theorem 3.6]). It easily follows that it induces the isomorphism (6.6) between the corresponding finite-dimensional models of twisted K-theory, i.e. between the corresponding Grothendieck groups. As above, let us choose the same refinement $\mathfrak{V}$ of $\mathfrak{U}$, but another refinement function $\rho\colon \Lambda \to I$. We set $\hat{\hat{\zeta}} := \rho^*\zeta$. Given $E = (\{E_i\}, \{\varphi_{ij}\})$, we set $\phi^*E = (\{F_\alpha\}, \{\psi_{\alpha\beta}\})$ and $\rho^*E = (\{F'_\alpha\}, \{\psi'_{\alpha\beta}\})$. We have the natural isomorphisms $\xi_\alpha := \varphi_{\phi(\alpha)\rho(\alpha)}|_{V_\alpha}\colon F_\alpha \xrightarrow{\simeq} F'_\alpha$, that glue to the isomorphism (6.9), in such a way that diagram (6.11) commutes. It is easy to verify that diagram (6.12) commutes too, hence we get diagram (6.14). If $H^2(X;\mathbb{Z}) = 0$, then the same argument of section 6.1.3 shows that (6.7) is well-defined in the finite-order setting too, hence we get $K_{[\zeta]}(X)$, with $[\zeta] \in \check{H}^2(X, \underline{\mathrm{U}}(1)) \simeq H^3(X;\mathbb{Z})$.

6.2 Isomorphism

We explicitly construct a natural isomorphism between the two models of twisted K-theory considered above. Starting from the fixed good cover $\mathfrak{U} = \{U_i\}_{i\in I}$, we choose a good finite refinement $\mathfrak{V} = \{V_1, \ldots, V_m\}$, through a refinement function $\phi\colon \{1, \ldots, m\} \to I$, such that $\bar{V}_k \subset U_{\phi(k)}$ for every k.[2] We fix a cocycle $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$, that represents a finite-order cohomology class, and we set $\hat{\zeta} := \phi^*\zeta$. Moreover, we fix a ζ-twisted rank-N vector bundle $\tilde{E}$, for any suitable $N \in \mathbb{N}$, endowed with a fixed set of local trivializations, so that we can naturally represent it in the form $\tilde{E} := (\{U_i \times \mathbb{C}^N\}, \{g_{ij}\})$, with $g_{ij}\colon U_{ij} \to \mathrm{U}(N)$. We consider the ζ-twisted Hilbert bundle $E := \tilde{E} \otimes \mathcal{H}$, where $\mathcal{H}$ actually denotes the trivial bundle $X \times \mathcal{H}$. This means that $E := (\{U_i \times (\mathbb{C}^N \otimes \mathcal{H})\}, \{g_{ij} \otimes 1\})$. Since $\mathbb{C}^N \otimes \mathcal{H} \simeq \mathcal{H}$, the bundle E satisfies definition 6.1.2. Applying the map (6.4), we get the $\hat{\zeta}$-twisted bundle $\hat{E} := \phi^*E = (\{V_k \times (\mathbb{C}^N \otimes \mathcal{H})\}, \{\hat{g}_{kh}\})$, where $\hat{g}_{kh} = g_{\phi(k)\phi(h)}|_{V_{kh}} \otimes 1$.

Let us consider a section $s \in \Gamma(F_{\mathbb{P}(E)})$, projecting to $[s] \in \bar{\Gamma}(F_{\mathbb{P}(E)})$, the latter group representing $K_\zeta(X)$ up to canonical identification. Since the local bundles $U_i \times (\mathbb{C}^N \otimes \mathcal{H})$ are already trivialized, the section s corresponds to a family of functions $s_i\colon U_i \to \mathrm{Fred}(\mathbb{C}^N \otimes \mathcal{H})$ such that $s_i = (g_{ij} \otimes 1) \cdot s_j \cdot (g_{ij}^{-1} \otimes 1)$.

---

[2] It is always possible to find such a refinement $\mathfrak{V}$ of $\mathfrak{U}$ under our hypotheses (see notation 6.1.1). In fact, since X is (para)compact, there exists a refinement $\mathfrak{W} = \{W_i\}_{i\in I}$ of $\mathfrak{U}$ such that $\bar{W}_i \subset U_i$ for every $i \in I$ (see [37, Lemma 41.6]). Since good covers are cofinal, we choose a good refinement $\mathfrak{V}' = \{V'_\alpha\}_{\alpha\in\Lambda}$ of $\mathfrak{W}$ and we extract a finite (necessarily good) sub-cover $\mathfrak{V} = \{V_1, \ldots, V_m\}$ of $\mathfrak{V}'$. It follows that $V_k = V'_{\alpha(k)} \subset W_{\phi(k)}$ for every $k = 1, \ldots, m$ and for a suitable function $\phi\colon \{1, \ldots m\} \to I$, hence $\bar{V}_k \subset \bar{W}_{\phi(k)} \subset U_{\phi(k)}$.

The isomorphism (6.5) induces

$$\phi^{\#} := \left(\overline{\phi^{*}}^{E}\right)_{*} \colon \Gamma(F_{\mathbb{P}(E)}) \xrightarrow{\simeq} \Gamma(F_{\mathbb{P}(\phi^{*}E)}), \tag{6.16}$$

hence we get $t := \phi^{\#}s \in \Gamma(F_{\mathbb{P}(\phi^{*}E)})$, represented by the family $t_k := s_{\phi(k)}|_{V_k} \colon V_k \to \mathrm{Fred}(\mathbb{C}^N \otimes \mathcal{H})$. We have the natural identification $\mathbb{C}^N \otimes \mathcal{H} \simeq \mathcal{H}^{\oplus N}$ and we denote by $\pi_1, \ldots, \pi_N \colon \mathcal{H}^{\oplus N} \to \mathcal{H}$ the canonical projections. By construction the functions t_k can be extended to $\bar{V}_k$ (the extension being $s_{\phi(k)}|_{\bar{V}_k}$), hence, for each $x \in \bar{V}_k$ and for every $k \in \{1, \ldots, m\}$, we consider the following space $\mathcal{V}_{x,k} \subset \mathcal{H}$:

$$\mathcal{V}_{x,k} := \left(\pi_1(\mathrm{Ker}\, t_k(x))\right)^{\perp} \cap \ldots \cap \left(\pi_N(\mathrm{Ker}\, t_k(x))\right)^{\perp}.$$

Such a space is closed and finite-codimensional. In fact, $\mathrm{Ker}\, t_k(x)$ is finite-dimensional, since $t_k(x)$ is Fredholm, hence each projection $\pi_h(\mathrm{Ker}\, t_k(x))$ is finite-dimensional too. It follows that the orthogonal complement is closed and finite-codimensional, hence the same holds about the finite intersection $\mathcal{V}_{x,k}$. Thus, $\mathcal{V}_{x,k}^{\oplus N}$ is closed and finite-codimensional in $\mathcal{H}^{\oplus N}$. Moreover, we have $(\mathcal{V}_{x,k}^{\oplus N}) \cap \mathrm{Ker}\, t_k(x) = \{0\}$, since $\mathcal{V}_{x,k}^{\oplus N} \subset (\mathrm{Ker}\, t_k(x))^{\perp}$. In fact, if $v := (v_1, \ldots, v_N) \in \mathcal{V}_{x,k}^{\oplus N}$ and $w := (w_1, \ldots, w_N) \in \mathrm{Ker}\, t_k(x)$, then, for every $h = 1, \ldots, N$, we have $v_h \in \left(\pi_h(\mathrm{Ker}\, t_k(x))\right)^{\perp}$ and $w_h \in \pi_h(\mathrm{Ker}\, t_k(x))$, hence $\langle v_h, w_h \rangle = 0$. This immediately implies $\langle v, w \rangle = 0$.

Following the proof of [2, Prop. A5], for each $x \in \bar{V}_k$ there exists a neighbourhood $W_{x,k} \subset \bar{V}_k$ such that $\mathcal{V}_{x,k}^{\oplus N} \cap \mathrm{Ker}\, t_k(y) = \{0\}$ for every $y \in W_{x,k}$. The family $\{W_{x,k}\}_{x \in \bar{V}_k}$ is an open cover of the compact space $\bar{V}_k$, hence we extract a finite sub-cover, that we denote by $\{W_{x_1,k}, \ldots, W_{x_{n_k},k}\}$, and we set $\mathcal{V}_k := \mathcal{V}_{x_1,k} \cap \ldots \cap \mathcal{V}_{x_{n_k},k}$ and $\mathcal{V} := \mathcal{V}_1 \cap \ldots \cap \mathcal{V}_m$. It follows that $\mathcal{V}^{\oplus N}$ is closed and finite-codimensional in $\mathcal{H}^{\oplus N}$ and that $\mathcal{V}^{\oplus N} \cap \mathrm{Ker}(t_k(x)) = \{0\}$ for every $x \in \bar{V}_k$ and for every $k \in \{1, \ldots, m\}$. Moreover, $(\hat{g}_{kh})_x(\mathcal{V}^{\oplus N}) = \mathcal{V}^{\oplus N}$ for every $x \in \bar{V}_{kh}$ and for every $k, h \in \{1, \ldots, m\}$, since the transition functions act as $N \times N$ invertible complex matrices on $\mathcal{H}^{\oplus N}$. Projecting to the quotient, we get the point-wise isomorphism $(\bar{g}_{kh})_x \colon \mathcal{H}^{\oplus N}/\mathcal{V}^{\oplus N} \to \mathcal{H}^{\oplus N}/\mathcal{V}^{\oplus N}$. Since $(\bar{g}_{kh})_x$ is defined for every $x \in \bar{V}_{kh}$, in particular it is defined for every $x \in V_{kh}$, hence we get the following $\hat{\zeta}$-twisted finite-dimensional vector bundle on X:

$$F_t := (\{V_k \times (\mathcal{H}^{\oplus N}/\mathcal{V}^{\oplus N})\}, \{\bar{g}_{kh}\}). \tag{6.17}$$

We set $\mathcal{H}^{\oplus N}/t_k(\mathcal{V}^{\oplus N}) := \bigsqcup_{x \in V_k} \mathcal{H}^{\oplus N}/(t_k)_x(\mathcal{V}^{\oplus N})$, as a quotient space of $V_k \times \mathcal{H}^{\oplus N} \simeq \bigsqcup_{x \in V_k} \mathcal{H}^{\oplus N}$. By [2, Prop. A3], the space $\mathcal{H}^{\oplus N}/t_k(\mathcal{V}^{\oplus N})$ is a vector bundle on V_k, hence, since V_k is contractible, it is a trivial vector bundle. Moreover, we get a well-defined isomorphism $\bar{\bar{g}}_{kh} \colon \mathcal{H}^{\oplus N}/t_k(\mathcal{V}^{\oplus N}) \to \mathcal{H}^{\oplus N}/t_h(\mathcal{V}^{\oplus N})$, since $(\hat{g}_{kh})_x((t_k)_x(\mathcal{V}^{\oplus N})) = (t_h)_x((\hat{g}_{kh})_x\ (\mathcal{V}^{\oplus N})) = (t_h)_x(\mathcal{V}^{\oplus N})$, therefore we get the following $\hat{\zeta}$-twisted finite-dimensional vector bundle on X:

$$G_t := (\{\mathcal{H}^{\oplus N}/t_k(\mathcal{V}^{\oplus N})\}, \{\bar{\bar{g}}_{kh}\}). \tag{6.18}$$

With these data, calling $K^{(f)}_{\zeta}(X)$ the finite-dimensional model of $K_{\zeta}(X)$, we get the following bijection:

$$\begin{aligned} \hat{\Theta}_{\phi^* E} \colon \bar{\Gamma}(F_{\mathbb{P}(\phi^* E)}) &\xrightarrow{\simeq} K^{(f)}_{\hat{\zeta}}(X) \\ [t] &\mapsto F_t - G_t. \end{aligned} \tag{6.19}$$

Denoting by $\overline{\phi^{\#}}$ the projection of (6.16) to the quotient up to homotopy, and applying (6.6) in the finite-order setting, we set

$$\Theta_E := \overline{\phi^*}^{-1} \circ \hat{\Theta}_{\phi^* E} \circ \overline{\phi^{\#}} : \bar{\Gamma}(F_{\mathbb{P}(E)}) \xrightarrow{\simeq} K^{(f)}_{\zeta}(X). \tag{6.20}$$

Calling $K^{(\infty)}_{\zeta}(X)$ the infinite-dimensional model of $K_{\zeta}(X)$, that we represent through $\bar{\Gamma}(F_{\mathbb{P}(E)})$, from (6.20) we get the following group isomorphism:

$$\Theta \colon K^{(\infty)}_{\zeta}(X) \xrightarrow{\simeq} K^{(f)}_{\zeta}(X). \tag{6.21}$$

Now we have to prove that (6.21) is well-defined. First we prove that it is well-defined as a group morphism, i.e. that it respects the group operations and it does not depend on:

- the representative $\mathcal{V}$ within the class of closed finite-codimensional vector subspaces of $\mathcal{H}$, such that $\mathcal{V}^{\oplus N} \cap \mathrm{Ker}(t_k(x)) = \{0\}$ for every $x \in \bar{V}_k$ and for every $k \in \{1, \ldots, m\}$;
- the representative section s in the class $[s]$;
- the refinement function ϕ;
- the refinement $\mathfrak{V}$ of $\mathfrak{U}$;
- the rank-N ζ-twisted bundle $\tilde{E}$, fixed at the beginning.

Afterwards, we prove that it is injective and surjective, following the same line of the appendix of [2], adapted to the twisted framework. Then we conclude by showing that, when $H^2(X; \mathbb{Z}) = 0$, the isomorphism (6.21) does not depend on the representative ζ of the cohomology class $[\zeta] \in \check{H}^2(X, \underline{\mathrm{U}}(1)) \simeq H^3(X; \mathbb{Z})$, hence it induces

$$\bar{\Theta} \colon K^{(\infty)}_{[\zeta]}(X) \xrightarrow{\simeq} K^{(f)}_{[\zeta]}(X). \tag{6.22}$$

6.2.1 The morphism Θ is well-defined

We prove the result following the steps we summarized above.

Independence of $\mathcal{V}$

The argument is essentially identical to the one in [2]. Fixing all of the other data, let $\mathfrak{W}$ be another closed and finite-codimensional vector subspace of $\mathcal{H}$ such that $\mathfrak{W}^{\oplus N} \cap \mathrm{Ker}(t_k(x)) = \{0\}$ for every $x \in \bar{V}_k$ and for every $k \in \{1, \ldots, m\}$. The same property is satisfied by $\mathcal{V} \cap \mathfrak{W}$, hence we can assume $\mathfrak{W} \subset \mathcal{V}$ without loss of generality. In this case, we get the following $\hat{\zeta}$-twisted vector bundles, the first two being respectively (6.17) and (6.18):

$$\begin{aligned} F_t &:= (\{V_k \times (\mathcal{H}^{\oplus N}/\mathcal{V}^{\oplus N})\}, \{\bar{g}_{kh}\}) \qquad & G_t &:= (\{\mathcal{H}^{\oplus N}/t_k(\mathcal{V}^{\oplus N})\}, \{\bar{\bar{g}}_{kh}\}) \\ F'_t &:= (\{V_k \times (\mathcal{H}^{\oplus N}/\mathfrak{W}^{\oplus N})\}, \{\bar{g}'_{kh}\}) \qquad & G'_t &:= (\{\mathcal{H}^{\oplus N}/t_k(\mathfrak{W}^{\oplus N})\}, \{\bar{\bar{g}}'_{kh}\}) \\ F''_t &:= (\{V_k \times (\mathcal{V}^{\oplus N}/\mathfrak{W}^{\oplus N})\}, \{\bar{g}''_{kh}\}) \qquad & G''_t &:= (\{t_k(\mathcal{V}^{\oplus N})/t_k(\mathfrak{W}^{\oplus N})\}, \{\bar{\bar{g}}''_{kh}\}). \end{aligned}$$

By identifying each quotient with the corresponding orthogonal complement, we get $F'_t \simeq F_t \oplus F''_t$ and $G'_t \simeq G_t \oplus G''_t$, hence in K-theory we have $F''_t = F'_t - F_t$ and $G''_t = G'_t - G_t$. Moreover, since each t_k is injective in $\mathcal{V}^{\oplus N}$ by construction, we have $F''_t \simeq G''_t$, thus $F'_t - F_t = G'_t - G_t$, that immediately implies $F_t - G_t = F'_t - G'_t$. It follows that (6.19) does not change, hence neither (6.20) and (6.21).

Independence of s within $[s]$

Also in this case the argument is essentially identical to the one in [2]. Fixing all of the other data, let us consider two homotopic sections $s, s' \in \Gamma(F_{\mathbb{P}(E)})$. We have the natural projection $\pi \colon X \times I \to X$, that induces the cocycle $\boldsymbol{\zeta} := \pi^*\zeta$, relative to the open cover $\boldsymbol{\mathfrak{U}} := \pi^*\mathfrak{U}$. We get the $\boldsymbol{\zeta}$-twisted bundle $\tilde{\boldsymbol{E}} := \pi^*\tilde{E}$, inducing $\boldsymbol{E} := \tilde{\boldsymbol{E}} \otimes \mathcal{H}$. Since $F_{\mathbb{P}(\boldsymbol{E})} \simeq \pi^* F_{\mathbb{P}(E)}$, a homotopy between s and s' can be thought of as a section $\boldsymbol{s} \in \Gamma(F_{\mathbb{P}(\boldsymbol{E})})$, where $\boldsymbol{E} := \tilde{\boldsymbol{E}} \otimes \mathcal{H}$, that restricts to s and s' respectively in $X \times \{0\}$ and $X \times \{1\}$. Considering the pull-back from $\boldsymbol{\mathfrak{U}}$ to its good refinement $\boldsymbol{\mathfrak{V}} := \pi^*\mathfrak{V}$, through the refinement function $\boldsymbol{\phi} := \pi^*\phi$, we apply (6.19) to $\boldsymbol{\phi}^*\boldsymbol{E}$ and we get the class $\boldsymbol{F_t} - \boldsymbol{G_t} \in K_{\boldsymbol{\zeta}}(X \times I)$, where $\boldsymbol{t} := \boldsymbol{\phi}^{\#}\boldsymbol{s}$, that restricts to $F_t - G_t$ in $X \times \{0\}$ and to $F_{t'} - G_{t'}$ in $X \times \{1\}$ (here $F_t = F_{t'}$, since the vector subspace $\mathcal{V}$ is the same, i.e. the one chosen for $\boldsymbol{t}$). Since the embeddings $i_0, i_1 \colon X \hookrightarrow X \times I$, defined by $i_\epsilon(x) := (x, \epsilon)$, induce the same pull-back $i_0^* = i_1^* \colon K_{\boldsymbol{\zeta}}(X \times I) \to K_\zeta(X)$, we get $F_t - G_t = F_{t'} - G_{t'}$. It follows that (6.19) does not change, hence neither (6.20) and (6.21).

Independence of ϕ

Let us choose the same refinement $\mathfrak{V}$, but another refinement function ρ. We set $\hat{\hat{\zeta}} := \rho^*\zeta$ and we call $\hat{\hat{\Theta}}_{\rho^*E}$ the isomorphism (6.19) with respect to the $\hat{\hat{\zeta}}$-twisted bundle ρ^*E. Moreover, we fix any cochain $\hat{\eta}$ such that $\hat{\hat{\zeta}} = \hat{\zeta} \cdot \check{\delta}^1\hat{\eta}$. In order to show that (6.20) does not depend on ϕ, we have to show that $\overline{\phi^*}^{-1} \circ \hat{\Theta}_{\phi^*E} \circ \overline{\phi^{\#}} = \overline{\rho^*}^{-1} \circ \hat{\hat{\Theta}}_{\rho^*E} \circ \overline{\rho^{\#}}$. This follows from the commutativity of the following diagram, that we are going to prove:

$$\begin{tikzcd}
\bar{\Gamma}(F_{\mathbb{P}(E)}) \arrow[rrr, dotted, "\Theta_E"] \arrow[dr, "\overline{\phi^{\#}}"] \arrow[ddr, "\overline{\rho^{\#}}"'] & & & K^{(f)}_{\zeta}(X) \arrow[dl, "\overline{\phi^*}"'] \arrow[ddl, "\overline{\rho^*}"] \\
& \bar{\Gamma}(F_{\mathbb{P}(\phi^*E)}) \arrow[r, "\hat{\Theta}_{\phi^*E}"] \arrow[d, "\overline{\xi_{\#}}"] & K^{(f)}_{\hat{\zeta}}(X) \arrow[d, "\Phi_{\hat{\eta}}"'] & \\
& \bar{\Gamma}(F_{\mathbb{P}(\rho^*E)}) \arrow[r, "\hat{\hat{\Theta}}_{\rho^*E}"] & K^{(f)}_{\hat{\hat{\zeta}}}(X). &
\end{tikzcd}$$

The vertical morphism $\overline{\xi_{\#}}$ is defined as follows: we start from (6.9), that we denote by ξ for simplicity, we consider the induced morphism $\xi_{\#}\colon \Gamma(F_{\mathbb{P}(\phi^*E)}) \to \Gamma(F_{\mathbb{P}(\rho^*E)})$ and we project it to the quotient up to homotopy of sections. The left and right triangles of the previous diagram are instances of diagram (6.11) (the left one fixing the representatives and the right one in the finite-order setting), hence we already know that they commute. Let us show that the rectangle commutes too. We fix $[t] \in \bar{\Gamma}(F_{\mathbb{P}(\phi^*E)})$, represented by the local functions $t_k\colon V_k \to \mathrm{Fred}(\mathbb{C}^N \otimes \mathcal{H})$. The latter satisfy the condition $t_k = \hat{g}_{kh} t_h \hat{g}_{kh}^{-1}$, where $\hat{g}_{kh} := g_{\phi(k)\phi(h)} \otimes 1$. By construction, $\hat{\Theta}_{\phi^*E}[t] = F_t - G_t$, where F_t and G_t are defined respectively by (6.17) and (6.18) with respect to a suitable vector subspace $\mathcal{V}$. We set $\hat{\hat{g}}_{kh} := g_{\rho(k)\rho(h)}|_{V_{kh}} \otimes 1$ and $\xi_k := g_{\phi(k)\rho(k)}|_{V_k} \otimes 1$. The class $\overline{\xi_{\#}}\,[t]$ is represented by $u_k := \xi_k t_k \xi_k^{-1}$ and $\hat{\hat{\Theta}}_{\rho^*E}(\overline{\xi_{\#}}\,[t]) = F'_t - G'_t$, where F'_t and G'_t are defined respectively by (6.17) and (6.18) with respect to a suitable vector subspace $\mathfrak{W}$, replacing the transition functions $\bar{g}_{kh}$ and $\bar{\bar{g}}_{kh}$, that are the projections of $\hat{g}_{kh}$, respectively by $\bar{g}'_{kh}$ and $\bar{\bar{g}}'_{kh}$, that are the projections of $\hat{\hat{g}}_{kh}$. The vector subspaces $\mathcal{V}$ and $\mathfrak{W}$ have to satisfy the following conditions:

- $\mathcal{V}$ and $\mathfrak{W}$ are closed and finite-codimensional in $\mathcal{H}$;
- $\mathcal{V}^{\oplus N} \cap \mathrm{Ker}(t_k(x)) = \{0\}$ and $\mathfrak{W}^{\oplus N} \cap \mathrm{Ker}(u_k(x)) = \{0\}$ for every $x \in \bar{V}_k$ and for every $k \in \{1, \ldots, m\}$.

It follows that $\mathcal{V} \cap \mathfrak{W}$ is a suitable (equivalent) choice in both cases. With this choice, the only difference between F_t and F'_t on one side, and between G_t and G'_t on the other side, is encoded in the transition functions. Let us show that $\Phi_{\hat{\eta}}(F_t) \simeq F'_t$ and $\Phi_{\hat{\eta}}(G_t) \simeq G'_t$. From these isomorphisms, it follows that $\Phi_{\hat{\eta}} \circ \hat{\Theta}_{\phi^*E}[t] = \Phi_{\hat{\eta}}(F_t - G_t) = F'_t - G'_t = \hat{\hat{\Theta}}_{\rho^*E} \circ \overline{\xi_{\#}}\,[t]$, as required.

We have the isomorphism (6.9), i.e. $\Phi_{\hat{\eta}}(\phi^*E) \simeq \rho^*E$, that in this framework coincides with the conjugation by $\{\xi_k\}$. We show the explicit computation, leaving the restrictions and '$\otimes 1$' implicit:

$$\begin{aligned}
\xi_k \hat{\hat{g}}_{kh} \xi_h^{-1} &= g_{\phi(k)\rho(k)} g_{\rho(k)\rho(h)} g_{\rho(h)\phi(h)} = \zeta_{\phi(k)\rho(k)\rho(h)} g_{\phi(k)\rho(h)} g_{\rho(h)\phi(h)} \\
&= \zeta_{\phi(k)\rho(k)\rho(h)} \zeta_{\phi(k)\rho(h)\phi(h)} g_{\phi(k)\phi(h)} = \hat{\eta}_{kh} \hat{g}_{kh}.
\end{aligned}$$

Projecting to the quotient in (6.17) and (6.18), we obtain the desired isomorphisms $\Phi_{\hat{\eta}}(F_t) \simeq F'_t$ and $\Phi_{\hat{\eta}}(G_t) \simeq G'_t$.

Independence of $\mathfrak{V}$

Let us replace $\mathfrak{V}$ by another good finite refinement $\mathfrak{W} = \{W_1, \dots, W_q\}$ of $\mathfrak{U}$, through a refinement function $\psi\colon \{1, \dots, q\} \to I$, such that $\bar{W}_k \subset U_{\psi(k)}$ for every k. We consider the common refinement formed by the corresponding intersections, i.e. $\mathfrak{X} := \{V_i \cap W_j\}_{(i,j)\in\{1,\dots,m\}\times\{1,\dots,q\}}$. We have $\overline{V_i \cap W_j} \subset \bar{V}_i \cap \bar{W}_j \subset U_{\phi(i)} \cap U_{\psi(j)}$, hence $\mathfrak{X}$ is a suitable refinement, through the function $\phi'(i,j) := \phi(i)$ or equivalently $\psi'(i,j) := \psi(j)$.

The argument above shows that, in order to prove the independence of $\mathfrak{V}$, it is not restrictive to choose another good finite refinement $\mathfrak{W} = \{W_1, \dots, W_q\}$, supposing that it is a refinement of $\mathfrak{V}$ through $\psi\colon \{1, \dots, q\} \to \{1, \dots, m\}$. The refinement function from $\mathfrak{U}$ to $\mathfrak{W}$ can be chosen to be $\eta := \phi \circ \psi$ without loss of generality. We set $\hat{\hat{\zeta}} := \psi^*\hat{\zeta} = \eta^*\zeta$ and we call $\hat{\hat{\Theta}}_{\eta^*E}$ the isomorphism (6.19) with respect to the $\hat{\hat{\zeta}}$-twisted bundle η^*E. In order to show that (6.20) does not depend on $\mathfrak{V}$, we have to show that $\overline{\phi^*}^{-1} \circ \hat{\Theta}_{\phi^*E} \circ \overline{\phi^\#} = \overline{\eta^*}^{-1} \circ \hat{\hat{\Theta}}_{\eta^*E} \circ \overline{\eta^\#}$. This follows from the commutativity of the following diagram, that we are going to prove:

$$\begin{tikzcd}
\bar{\Gamma}(F_{\mathbb{P}(E)}) \arrow[rrr, dotted, "\Theta_E"] \arrow[dr, "\overline{\phi^\#}"] \arrow[ddr, "\overline{\eta^\#}"'] & & & K^{(f)}_{\zeta}(X) \arrow[dl, "\overline{\phi^*}"'] \arrow[ddl, "\overline{\eta^*}"] \\
& \bar{\Gamma}(F_{\mathbb{P}(\phi^*E)}) \arrow[r, "\hat{\Theta}_{\psi^*E}"] \arrow[d, "\overline{\psi^\#}"] & K^{(f)}_{\hat{\zeta}}(X) \arrow[d, "\overline{\psi^*}"'] & \\
& \bar{\Gamma}(F_{\mathbb{P}(\eta^*E)}) \arrow[r, "\hat{\hat{\Theta}}_{\eta^*E}"] & K^{(f)}_{\hat{\hat{\zeta}}}(X). &
\end{tikzcd}$$

The commutativity of the triangles is straightforward from the definitions of $\overline{\phi^*}$ and $\overline{\phi^\#}$. Let us show that the rectangle commutes too. As above, we fix $[t] \in \bar{\Gamma}(F_{\mathbb{P}(\phi^*E)})$, represented by the local functions $t_k\colon V_k \to \mathrm{Fred}(\mathbb{C}^N \otimes \mathcal{H})$. The latter satisfy the condition $t_k = \hat{g}_{kh} t_h \hat{g}_{kh}^{-1}$, where $\hat{g}_{kh} := g_{\phi(k)\phi(h)} \otimes 1$. By construction, $\hat{\Theta}_{\psi^*E}[t] = F_t - G_t$, where F_t and G_t are defined respectively by (6.17) and (6.18) with respect to a suitable vector subspace $\mathcal{V}$. We set $\hat{\hat{g}}_{kh} := \hat{g}_{\psi(k)\psi(h)}|_{W_{kh}} = g_{\eta(k)\eta(h)}|_{W_{kh}} \otimes 1$. The class $\overline{\psi^\#}\,[t]$ is represented by $u_k := t_{\psi(k)}|_{W_k}$, hence the same vector subspace $\mathcal{V}$ is a suitable choice for $\overline{\psi^\#}\,[t]$ too. It follows that $\hat{\hat{\Theta}}_{\eta^*E}(\overline{\psi^\#}\,[t]) = F'_t - G'_t$, where F'_t and G'_t are defined respectively by (6.17) and (6.18), replacing V_k by W_k, t_k by u_k and the transition functions $\bar{g}_{kh}$ and $\bar{\bar{g}}_{kh}$ by their restrictions to W_{kh}. This easily implies that $F'_t = \psi^*(F_t)$ and $G'_t = \psi^*(G_t)$, so that $F'_t - G'_t = \overline{\psi^*}(F_t - G_t)$, i.e. $\hat{\hat{\Theta}}_{\eta^*E} \circ \overline{\psi^\#}\,[t] = \overline{\psi^*} \circ \hat{\Theta}_{\psi^*E}[t]$, as required.

Behaviour under direct sum

We consider two ζ-twisted bundles $\tilde{E}_1$ and $\tilde{E}_2$, each one endowed with a fixed set of local trivializations,[3] so that we can naturally represent them in the form $\tilde{E}_1 := (\{U_i \times \mathbb{C}^{N_1}\}, \{g_{ij}\})$ and $\tilde{E}_2 := (\{U_i \times \mathbb{C}^{N_2}\}, \{h_{ij}\})$. We set $E_1 := \tilde{E}_1 \otimes \mathcal{H}$ (as above) and $E_2 := \tilde{E}_2 \otimes \mathcal{H}$. We fix any isomorphism $\varphi\colon E_1 \xrightarrow{\simeq} E_2$, so that the two representatives $\bar{\Gamma}(F_{\mathbb{P}(E_1)})$ and $\bar{\Gamma}(F_{\mathbb{P}(E_2)})$ of $K_\zeta(X)$ are identified through the isomorphism $\varphi_*\colon \bar{\Gamma}(F_{\mathbb{P}(E_1)}) \xrightarrow{\simeq} \bar{\Gamma}(F_{\mathbb{P}(E_2)})$, the latter being independent of φ. Moreover, we set $\tilde{E}_3 := \tilde{E}_1 \oplus \tilde{E}_2$ and $E_3 := \tilde{E}_3 \otimes \mathcal{H}$, thus $E_3 \simeq E_1 \oplus E_2$. We denote by Θ_{E_1}, Θ_{E_2} and Θ_{E_3} the corresponding morphisms (6.20). It is easy to verify that, given $s_1 \in \Gamma(F_{\mathbb{P}(E_1)})$ and $s_2 \in \Gamma(F_{\mathbb{P}(E_2)})$, we have

$$\Theta_{E_3}[s_1 \oplus s_2] = \Theta_{E_1}[s_1] + \Theta_{E_2}[s_2], \tag{6.23}$$

where

$$s_1 \oplus s_2 = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix} \in \Gamma(F_{\mathbb{P}(E_3)}).$$

In fact, adapting in the straightforward way the notation of formulas (6.17)–(6.21), if $\mathcal{V}_1$ and $\mathcal{V}_2$ are suitable subspaces for t_1 and t_2, then the intersection $\mathcal{V} := \mathcal{V}_1 \cap \mathcal{V}_2$ is a common suitable choice. In this way, the bundles (6.17) and (6.18), induced by $[t_1 \oplus t_2]$, split as $F_{t_3} = F_{t_1} \oplus F_{t_2}$ and $G_{t_3} = G_{t_1} \oplus G_{t_2}$, since the transition functions split correspondingly, hence $\hat{\Theta}_{\phi^* E_3}[t_1 \oplus t_2] = (F_{t_1} \oplus F_{t_2}) - (G_{t_1} \oplus G_{t_2}) = F_{t_1} + F_{t_2} - G_{t_1} - G_{t_2} = \hat{\Theta}_{\phi^* E_1}[s_1] + \hat{\Theta}_{\phi^* E_2}[s_2]$. Formula (6.23) immediately follows.

Let us show that $s_1 \oplus s_2$ is homotopic to $(s_1 \cdot \varphi^{\#} s_2) \oplus 1$, where $\varphi^{\#}\colon \Gamma(F_{\mathbb{P}(E_2)}) \to \Gamma(F_{\mathbb{P}(E_1)})$ is naturally induced by φ. In fact, we have the following homotopy:

$$t \mapsto \begin{bmatrix} s_1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{2}t\right)1 & -\sin\left(\frac{\pi}{2}t\right)\varphi^{-1} \\ \sin\left(\frac{\pi}{2}t\right)\varphi & \cos\left(\frac{\pi}{2}t\right)1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & s_2 \end{bmatrix} \begin{bmatrix} \cos\left(\frac{\pi}{2}t\right)1 & -\sin\left(\frac{\pi}{2}t\right)\varphi^{-1} \\ \sin\left(\frac{\pi}{2}t\right)\varphi & \cos\left(\frac{\pi}{2}t\right)1 \end{bmatrix}^{-1}.$$

It follows that $\Theta_{E_3}[s_1 \oplus s_2] = \Theta_{E_3}[(s_1 \cdot \varphi^{\#} s_2) \oplus 1]$, therefore formula (6.23) implies

$$\Theta_{E_1}[s_1] + \Theta_{E_2}[s_2] = \Theta_{E_1}[s_1 \cdot \varphi^{\#} s_2]. \tag{6.24}$$

We applied the property $\Theta_E(1) = 0$ of the morphism (6.20), that easily follows by observing that, in this case, we can choose $\mathcal{V} = \mathcal{H}$, so that both bundles (6.17) and (6.18) vanish.[4]

Independence of $\tilde{E}$

We choose any two ζ-twisted bundles $\tilde{E}_1 := (\{U_i \times \mathbb{C}^{N_1}\}, \{g_{ij}\})$ and $\tilde{E}_2 := (\{U_i \times \mathbb{C}^{N_2}\}, \{h_{ij}\})$, and we set $E_1 := \tilde{E}_1 \otimes \mathcal{H}$ and $E_2 := \tilde{E}_2 \otimes \mathcal{H}$. The

[3] In particular, this includes the possibility of choosing the same bundle with two distinct sets of local trivializations.

[4] We remark that the zero-bundle is ζ-twisted for every ζ, since the identity of rank 0 coincides with the zero-map, hence the multiplication by ζ_{ijk} is immaterial.

two corresponding representatives of $K^{\infty}_{\zeta}(X)$ are identified through any isomorphism $\varphi\colon E_1 \xrightarrow{\simeq} E_2$. In particular, $[s_2] \in \bar{\Gamma}(F_{\mathbb{P}(E_2)})$ is identified with $[\varphi^{\#} s_2] \in \bar{\Gamma}(F_{\mathbb{P}(E_1)})$. Therefore, with the notation of section 6.2.1, in order to prove that (6.20) does not depend on $\tilde{E}$, we have to show that $\Theta_{E_2} = \Theta_{E_1} \circ \overline{\varphi^{\#}}$. Choosing $s_1 = 1$ in formula (6.24), we get $\Theta_{E_2}[s_2] = \Theta_{E_1}[\varphi^{\#} s_2] = \Theta_{E_1} \circ \overline{\varphi^{\#}}\,[s_2]$, as required.

Multiplicativity (or additivity)

By choosing $\tilde{E}_1 = \tilde{E}_2 = \tilde{E}$ (so that $E_1 = E_2 = E$) and $\varphi = 1$ in section 6.2.1, formula (6.24) becomes $\Theta_E[s_1] + \Theta_E[s_2] = \Theta_E[s_1 \cdot s_2]$, as required.

6.2.2 Injectivity

Let us suppose that $\Theta_E[s] = 0$. Since, in definition (6.20), $\overline{\phi^*}$ is an isomorphism, this is equivalent to $\hat{\Theta}_{\phi^* E}[t] = 0$, where $t = \phi^{\#} s$, hence to $F_t - G_t = 0$. This means that $F_t = G_t$ as K-theory classes, i.e. there exists a $\hat{\zeta}$-twisted bundle $\hat{P}$ such that $F_t \oplus \hat{P} \simeq G_t \oplus \hat{P}$. Since (6.15) is an isomorphism, there exists a unique ζ-twisted bundle $\tilde{P}$ such that $\hat{P} = \phi^* \tilde{P}$. We set $M := \dim(\mathcal{H}/\mathcal{V})$, where $\mathcal{V}$ is a suitable choice for t, and we deduce from (6.17) that

$$\begin{aligned} F_t &\simeq (\{V_k \times (\mathcal{H}/\mathcal{V})^{\oplus N}\}, \{\bar{g}_{kh}\}) \simeq (\{V_k \times (\mathbb{C}^N \otimes \mathcal{H}/\mathcal{V})\}, \{g_{\phi(k)\phi(h)} \otimes 1\}) \\ &\simeq (\{V_k \times (\mathbb{C}^N \otimes \mathbb{C}^M)\}, \{g_{\phi(k)\phi(h)} \otimes 1\}) \simeq \phi^* \tilde{E} \otimes \mathbb{C}^M, \end{aligned}$$

where, in the last term, $\mathbb{C}^M$ actually denotes the trivial bundle $X \times \mathbb{C}^M$. Hence, from $F_t \oplus \hat{P} \simeq G_t \oplus \hat{P}$ we deduce[5]

$$\begin{aligned} &F_t \oplus \hat{P} \oplus \overset{M \text{ times}}{\cdots} \oplus \hat{P} \simeq G_t \oplus \hat{P} \oplus \overset{M \text{ times}}{\cdots} \oplus \hat{P} \\ &F_t \oplus (\hat{P} \otimes \mathbb{C}^M) \simeq G_t \oplus (\hat{P} \otimes \mathbb{C}^M) \\ &\phi^*(\tilde{E} \oplus \tilde{P}) \otimes \mathbb{C}^M \simeq G_t \oplus \phi^*(\tilde{P} \otimes \mathbb{C}^M). \end{aligned} \tag{6.25}$$

Now we replace $\tilde{E}$ by $\tilde{E} \oplus \tilde{P}$ and s by $\varphi_{\#} s$, where $\varphi\colon \tilde{E} \otimes \mathcal{H} \xrightarrow{\simeq} (\tilde{E} \oplus \tilde{P}) \otimes \mathcal{H}$ is any isomorphism. It follows from section 6.2.1 that the corresponding class in the finite-dimensional model does not change. Moreover, identifying $(\tilde{E} \oplus \tilde{P}) \otimes \mathcal{H}$ with $(\tilde{E} \otimes \mathcal{H}) \oplus (\tilde{P} \otimes \mathcal{H})$, the section $\varphi_{\#} s$ is homotopic to $s \oplus 1$, as we will show below, therefore we consider the latter. Summarizing, the two following choices are equivalent:

I. $\tilde{E}$ as the initial bundle and $s \in \Gamma(F_{\mathbb{P}(E)})$, where $E := \tilde{E} \otimes \mathcal{H}$;

II. $\tilde{E} \oplus \tilde{P}$ as the initial bundle and $s \oplus 1 \in \Gamma(F_{\mathbb{P}(E \oplus P)})$, where $P := \tilde{P} \otimes \mathcal{H}$.

[5] It is not restrictive to suppose $M \geq 1$, since, if $M = 0$, we can replace $\mathcal{H} = \mathcal{V}$ by any subspace of codimension 1.

Moreover, if $\mathcal{V}$ is a suitable choice for s, then it is suitable for $s \oplus 1$ too. In this way, if F_t and G_t are the bundles (6.17) and (6.18) under choice I, then the corresponding bundles under choice II are respectively $F_t \oplus (\hat{P} \otimes \mathbb{C}^M)$ and $G_t \oplus (\hat{P} \otimes \mathbb{C}^M)$. In fact, under choice II, definition (6.17) becomes $(\{V_k \times ((\mathcal{H}/\mathcal{V})^{\oplus N} \oplus (\mathcal{H}/\mathcal{V})^{\oplus M})\}, \{\bar{g}_{kh} \oplus \bar{p}_{kh}\}) \simeq F_t \oplus (\hat{P} \otimes \mathbb{C}^M)$, where p_{kh} are the transition functions of $\hat{P}$. Similarly, definition (6.18) becomes $(\{V_k \times ((\mathcal{H}^{\oplus N}/t_k(\mathcal{V})) \oplus (\mathcal{H}/\mathcal{V})^{\oplus M})\}, \{\bar{\bar{g}}_{kh} \oplus \bar{p}_{kh}\}) \simeq G_t \oplus (\hat{P} \otimes \mathbb{C}^M)$.

We set $\hat{F}_t := F_t \oplus (\hat{P} \otimes \mathbb{C}^M)$ and $\hat{G}_t := G_t \oplus (\hat{P} \otimes \mathbb{C}^M)$. By (6.25), there exists an isomorphism $\Phi \colon \hat{F}_t \xrightarrow{\simeq} \hat{G}_t$. Through Φ, we can construct a section $u \in \Gamma(F_{\mathbb{P}(\phi^*(E \oplus P))})$, such that u_x is an invertible operator for every x, as follows. For every $x \in V_k$ and for every k, we split $\mathcal{H}^{\oplus N+M} \simeq \mathcal{V}^{\oplus N+M} \oplus (\mathcal{H}/\mathcal{V})^{\oplus N+M}$ in the domain and $\mathcal{H}^{\oplus N+M} \simeq (t_k \oplus 1)_x(\mathcal{V}^{\oplus N+M}) \oplus \mathcal{H}^{\oplus N+M}/(t_k \oplus 1)_x(\mathcal{V}^{\oplus N+M})$ in the codomain. We get the splitting $\phi^*(E \oplus P) \simeq \hat{F}'_t \oplus \hat{F}_t$ in the domain and $\phi^*(E \oplus P) \simeq \hat{G}'_t \oplus \hat{G}_t$, where the $\hat{\zeta}$-twisted bundles $\hat{F}'_t$ and $\hat{G}'_t$ are defined by restricting the transition functions $(g_{hk} \oplus p_{hk}) \otimes 1$ of $\phi^*(E \oplus P)$ to the corresponding subspaces of $\mathcal{H}^{\oplus N+M}$. We have the isomorphisms $t \oplus 1 \colon \hat{F}'_t \xrightarrow{\simeq} \hat{G}'_t$ and $\Phi \colon \hat{F}_t \xrightarrow{\simeq} \hat{G}_t$, hence we get $u := (t \oplus 1)|_{\hat{F}'_t} \oplus \Phi \colon \phi^*(E \oplus P) \xrightarrow{\simeq} \phi^*(E \oplus P)$.

Let us construct a homotopy from $t \oplus 1$ to u. We call $\tau \in I$ the variable of the homotopy. First we construct a homotopy from $t \oplus 1$ to $(t \oplus 1)|_{\hat{F}'_t} \oplus 0$, by setting $\tau \mapsto (t \oplus 1)|_{\hat{F}'_t} + (1 - \tau)(t \oplus 1)|_{\hat{F}_t}$. Then, we construct a homotopy from $(t \oplus 1)|_{\hat{F}'_t} \oplus 0$ to u, by setting $\tau \mapsto (t \oplus 1)|_{\hat{F}'_t} + \tau\Phi$. Since the space of invertible operators is contractible, it follows that u is homotopic to 1, hence $t \oplus 1 \simeq u \simeq 1$. By applying $\phi^{\#}$, we deduce that $s \oplus 1$ is homotopic to 1, therefore $\varphi_{\#} s \simeq s \oplus 1 \simeq 1$. By applying $\varphi^{\#}$, we conclude that $s \simeq 1$, as required.

It remains to show that $\varphi_{\#} s \simeq s \oplus 1$, as stated in the paragraph after formula (6.25). The choice of the isomorphism φ is immaterial, since, as we have seen above, any two twisted-bundle isomorphisms induce projective-bundle isomorphisms that belong to the same connected component. Hence, we choose φ as follows. We fix a Hilbert-space isomorphism $\psi \colon \mathcal{H} \xrightarrow{\simeq} \mathcal{H} \oplus \mathcal{H}$, inducing $\psi_* \colon \mathrm{Fred}(\mathcal{H}) \xrightarrow{\simeq} \mathrm{Fred}(\mathcal{H} \oplus \mathcal{H})$. We get $1 \otimes \psi \colon \tilde{E} \otimes \mathcal{H} \xrightarrow{\simeq} \tilde{E} \otimes (\mathcal{H} \oplus \mathcal{H}) \simeq (\tilde{E} \oplus \tilde{E}) \otimes \mathcal{H}$. Afterwards, we choose any isomorphism $\eta \colon \tilde{E} \otimes \mathcal{H} \xrightarrow{\simeq} \tilde{P} \otimes \mathcal{H}$ and we set $\varphi := (1 \oplus \eta) \circ (1 \otimes \psi) \colon \tilde{E} \otimes \mathcal{H} \xrightarrow{\simeq} (\tilde{E} \oplus \tilde{P}) \otimes \mathcal{H}$. It is enough to prove that $(1 \otimes \psi)_{\#} s \simeq s \oplus 1$, since $(1 \oplus \eta)_{\#}(s \oplus 1) = 1_{\#} s \oplus \eta_{\#} 1 = s \oplus 1$. This is equivalent to prove that $(1 \otimes \psi)_{\#} t \simeq t \oplus 1$, where $t = \phi^* s$ (we used the same notation $1 \otimes \psi$ for its pull-back on $\mathfrak{V}$). Let $\mathcal{V} \subset \mathcal{H}$ be a suitable choice for t in $\tilde{E} \otimes \mathcal{H}$. We consider the following splittings:

- $\mathcal{H} \simeq \mathcal{V} \oplus \mathcal{V}^{\perp}$, so that $\mathcal{H}^{\oplus N} \simeq (\mathcal{V}^{\oplus N}) \oplus (\mathcal{V}^{\oplus N})^{\perp}$, where $(\mathcal{V}^{\oplus N})^{\perp} \simeq (\mathcal{V}^{\perp})^{\oplus N}$. For every $x \in V_k$ and every k, the operator $(t_k)_x \oplus 1$, that we denote by $t_{k,x} \oplus 1 \colon \mathcal{H}^{\oplus N} \oplus \mathcal{H}^{\oplus N} \to \mathcal{H}^{\oplus N} \oplus \mathcal{H}^{\oplus N}$, sends $\mathcal{A} := \mathcal{V}^{\oplus N} \oplus \mathcal{H}^{\oplus N}$ isomorphically to $\mathcal{B}_{k,x} := t_{k,x}(\mathcal{V}^{\oplus N}) \oplus \mathcal{H}^{\oplus N}$, where $\mathcal{A}$ and $\mathcal{B}_{k,x}$ are closed and finite-codimensional, and its kernel has trivial intersection with $\mathcal{A}$.

- $\mathbb{C}^N \otimes (\mathcal{H} \oplus \mathcal{H}) \simeq (1 \otimes \psi)(\mathbb{C}^N \otimes \mathcal{V}) \oplus (1 \otimes \psi)(\mathbb{C}^N \otimes \mathcal{V}^{\perp})$, that is equivalent to $\mathcal{H}^{\oplus 2N} \simeq \psi(\mathcal{V})^{\oplus N} \oplus \psi(\mathcal{V}^{\perp})^{\oplus N}$. For every $x \in V_k$ and every k, the operator

$(1 \otimes \psi)_{\#}(t_{k,x})\colon \mathcal{H}^{\oplus 2N} \to \mathcal{H}^{\oplus 2N}$ sends $\mathcal{A}' := \psi(\mathcal{V})^{\oplus N}$ isomorphically to $\mathcal{B}'_{k,x} := (1 \otimes \psi)_{\#}(t_{k,x})(\psi(\mathcal{V})^{\oplus N})$. Since $(1 \otimes \psi)_{\#}(t_{k,x}) = (1 \otimes \psi) \circ t_{k,x} \circ (1 \otimes \psi)^{-1}$ and $\psi(\mathcal{V})^{\oplus N} = \mathbb{C}^N \otimes \psi(V) = (1 \otimes \psi)(\mathbb{C}^N \otimes \mathcal{V}) = (1 \otimes \psi)(\mathcal{V}^{\oplus N})$, it follows that $\mathcal{B}'_{k,x} = (1 \otimes \psi)(t_{k,x}(\mathcal{V}^{\oplus N}))$. Again $\mathcal{A}'$ and $\mathcal{B}'_{k,x}$ are closed and finite-codimensional, and the kernel of $(1 \otimes \psi)_{\#}(t_{k,x})$ has trivial intersection with $\mathcal{A}'$.

Hence, we have:

$$\begin{aligned} &t_{k,x} \oplus 1\colon \mathcal{A} \oplus \mathcal{A}^{\perp} \to \mathcal{B}_{k,x} \oplus \mathcal{B}_{k,x}^{\perp} \\ &(1 \otimes \psi)_{\#}(t_{k,x})\colon \mathcal{A}' \oplus {\mathcal{A}'}^{\perp} \to \mathcal{B}'_{k,x} \oplus {\mathcal{B}'}_{k,x}^{\perp}, \end{aligned}$$

where:

$$\begin{aligned} &\mathcal{A} = \mathcal{V}^{\oplus N} \oplus \mathcal{H}^{\oplus N} && \mathcal{B}_{k,x} := t_{k,x}(\mathcal{V}^{\oplus N}) \oplus \mathcal{H}^{\oplus N} \\ &\mathcal{A}^{\perp} = (\mathcal{V}^{\perp})^{\oplus N} \oplus 0 && \mathcal{B}_{k,x}^{\perp} := \big(t_{k,x}(\mathcal{V}^{\oplus N})\big)^{\perp} \oplus 0 \\ &\mathcal{A}' = \psi(\mathcal{V})^{\oplus N} && \mathcal{B}'_{k,x} = (1 \otimes \psi)\big(t_{k,x}(\mathcal{V}^{\oplus N})\big) \\ &{\mathcal{A}'}^{\perp} = \psi(\mathcal{V}^{\perp})^{\oplus N} && {\mathcal{B}'}_{k,x}^{\perp} = (1 \otimes \psi)\big((t_{k,x}(\mathcal{V}^{\oplus N}))^{\perp}\big). \end{aligned}$$

If we let k and x vary, we get

$$\begin{aligned} &t \oplus 1\colon \hat{F}'_t \oplus \hat{F}_t \longrightarrow \hat{G}'_t \oplus \hat{G}_t \\ &(1 \otimes \psi)_{\#} t\colon (1 \otimes \psi)(F'_t) \oplus (1 \otimes \psi)(F_t) \longrightarrow (1 \otimes \psi)(G'_t) \oplus (1 \otimes \psi)(G_t). \end{aligned}$$

We set $t_1 := t \oplus 1$ and $t_2 := (1 \otimes \psi)_{\#} t$. We have $t_1 = t_1|_{\hat{F}'_t} \oplus t_1|_{\hat{F}_t}$. Considering the homotopy $\tau \mapsto t_1|_{\hat{F}'_t} \oplus \tau t_1|_{\hat{F}_t}$, we see that $t_1 \simeq t_1|_{\hat{F}'_t} \oplus 0$. Similarly, $t_2 \simeq t_2|_{(1\otimes\psi)(F'_t)} \oplus 0$. Therefore, it is enough to prove that $t_1|_{\hat{F}'_t} \oplus 0 \simeq t_2|_{(1\otimes\psi)(F'_t)} \oplus 0$.

We can construct a section u_1 of invertible operators from $E \oplus E$ to $E \oplus E$, such that $u_1(\hat{F}'_t) = (1 \otimes \psi)(F'_t)$ and $u_1(\hat{F}_t) = (1 \otimes \psi)(F_t)$. This means that, fixing k and x, we require $(u_1)_{k,x}(\mathcal{A}) = \mathcal{A}'$ and $(u_1)_{k,x}(\mathcal{A}^{\perp}) = {\mathcal{A}'}^{\perp}$. For the latter condition, it is enough to set $(u_1)_{k,x}|_{\mathcal{A}'} := 1 \otimes \psi$. About the former condition, we fix an Hilbert-space isomorphism $\mu\colon \mathcal{V} \oplus \mathcal{H} \xrightarrow{\approx} \mathcal{V}$, and we set $(u_1)_{k,x}|_{\mathcal{A}} := (1 \otimes \psi) \circ (1 \otimes \mu)$. Since the transition functions of $\hat{F}'_t$ are of the form $g_{kh} \otimes 1\colon \mathbb{C}^N \otimes (\mathcal{V} \oplus \mathcal{H}) \to \mathbb{C}^N \otimes (\mathcal{V} \oplus \mathcal{H})$ and the ones of $(1 \otimes \psi)(F'_t)$ are of the form $g_{kh} \otimes 1\colon \mathbb{C}^N \otimes \psi(\mathcal{V}) \to \mathbb{C}^N \otimes \psi(\mathcal{V}) \otimes (\mathcal{V} \oplus \mathcal{H})$, the pointwise local isomorphisms $(u_1)_{k,x}$ glue to the section required section u_1. Similarly, we can construct a section u_2 of invertible operators from $E \oplus E$ to $E \oplus E$, such that $u_2(\hat{G}'_t) = (1 \otimes \psi)(G'_t)$ and $u_2(\hat{G}_t) = (1 \otimes \psi)(G_t)$. This means that, fixing k and x, we require $(u_2)_{k,x}(\mathcal{B}_{k,x}) = \mathcal{B}'_{k,x}$ and $(u_2)_{k,x}(\mathcal{B}_{k,x}^{\perp}) = {\mathcal{B}'}_{k,x}^{\perp}$. For the latter condition, it is enough to set $(u_2)_{k,x}|_{\mathcal{B}_{k,x}^{\perp}} := 1 \otimes \psi$. About the former condition, we set $(u_2)_{k,x}|_{\mathcal{B}_{k,x}} := (t_2)_{k,x} \circ (u_1)_{k,x} \circ (t_1)_{k,x}^{-1}$. The compatibility with the transition functions is guaranteed again by their form $g_{kh} \otimes 1$ in $\mathcal{B}_{k,x}^{\perp}$, while it is automatic in $\mathcal{B}_{k,x}$, since we composed three global sections. It easily follows from the construction of u_1 and u_2 that $t_2|_{(1\otimes\psi)(F'_t)} \oplus 0 = u_2 \circ (t_1|_{\hat{F}'_t} \oplus 0) \circ u_1^{-1}$. Since the space of bounded invertible operators is contractible, the sections u_1 and u_2 are homotopic to the identity, hence $t_2|_{(1\otimes\psi)(F'_t)} \simeq t_1|_{\hat{F}'_t} \oplus 0$, as required.

6.2.3 Surjectivity

Let us fix a K-theory class $F-G \in K_{\zeta}(X)$. We set $\hat{F} := \phi^*(F)$ and $\hat{G} := \phi^*(G)$, so that $\hat{F}-\hat{G} = \phi^*(F-G) \in K_{\hat{\zeta}}(X)$. Now we construct the isomorphism (6.20) starting from the bundle $\tilde{E} := G$, so that $E = G \otimes \mathcal{H}$. By choosing a set of local trivializations, we represent G in the form $G = (\{U_i \times \mathbb{C}^N\}, \{g_{ij}\})$, as above. Fixing an orthonormal set $\{e_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}$, we define the shift operator $T_1\colon \mathcal{H} \hookrightarrow \mathcal{H}$, $e_n \mapsto e_{n+1}$. It is straightforward to verify that T_1 is injective and $\mathrm{Coker}(T_1) \simeq \langle e_0 \rangle \simeq \mathbb{C}$. We consider the section $s := 1 \otimes T_1 \in \Gamma(F_{\mathbb{P}(E)})$, represented by $s_i := 1 \otimes T_1$. It follows that the section $t := \phi^* s$ is represented by $t_k = 1 \otimes T_1$, hence each t_k is injective. For this reason, we can choose $\mathcal{V} = \mathcal{H}$, so that the corresponding bundle (6.17) vanishes. Moreover, we have $t_k(\mathcal{V}^{\oplus N}) = t_k(\mathcal{H}^{\oplus N}) = t_k(\mathbb{C}^N \otimes \mathcal{H}) = \mathbb{C}^N \otimes \mathrm{Im}\, T_1 \simeq (\mathrm{Im}\, T_1)^{\oplus N}$, hence $\mathcal{H}^{\oplus N}/t_k(\mathcal{V}^{\oplus N}) \simeq \mathcal{H}^{\oplus N}/(\mathrm{Im}\, T_1)^{\oplus N} \simeq \mathbb{C}^N$. The transition functions $\bar{\bar{g}}_{kh}$ of (6.18) are the projections to the quotient of $g_{kh} \otimes 1$, hence they act as the pointwise $N \times N$ matrices g_{kh} on $(\mathcal{H}/\mathrm{Im}\, T_1)^{\oplus N} \simeq \mathbb{C}^N$, therefore they coincide with the functions g_{hk} themselves. It follows that the bundle (6.18) is $\hat{G}$, thus $\Theta[s] = 0 - G = -G$.

Now we repeat the same construction about F, starting from $\tilde{E} := F$. We get a section $s' \in \Gamma(F_{\mathbb{P}(E')})$, where $E' = F \otimes \mathcal{H}$, such that $\Theta[s'] = 0 - F = -F$. We fix any isomorphism $\varphi\colon E \xrightarrow{\simeq} E'$ and we set $s'' := (\varphi^* s')^* \cdot s \in \Gamma(F_{\mathbb{P}(E)})$. It follows that $\Theta[s''] = -\Theta[\varphi^* s'] + \Theta[s] = -\Theta[s'] + \Theta[s] = F - G$, as required.

6.2.4 Independence of the cocycle

Let us suppose that $H^2(X;\mathbb{Z}) = 0$. We have to prove that (6.22) is well-defined, with $[\zeta] \in \check{H}^2(X, \underline{\mathrm{U}}(1)) \simeq H^3(X;\mathbb{Z})$.

Fixed cover

First we fix the cover $\mathfrak{U}$ and we consider $[\zeta] \in \check{H}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$. We fix $\eta \in \check{C}^1(\mathfrak{U}, \underline{\mathrm{U}}(1))$ and we set $\zeta' := \zeta \cdot \check{\delta}^1 \eta$. We have to show that (6.21) is well-behaved with respect to change of representative, i.e. that the following diagram commutes:

$$\begin{tikzcd} K^{(\infty)}_{\zeta}(X) \arrow[r, "\Theta"] \arrow[d, "\Phi_\eta"'] & K^{(f)}_{\zeta}(X) \arrow[d, "\Phi_\eta"] \\ K^{(\infty)}_{\zeta'}(X) \arrow[r, "\Theta"] & K^{(f)}_{\zeta'}(X). \end{tikzcd} \tag{6.26}$$

This means that (6.20) makes the following diagram commutative:

$$\begin{tikzcd} \bar{\Gamma}(F_{\mathbb{P}(E)}) \arrow[r, "\Theta_E"] \arrow[d, equal] & K^{(f)}_{\zeta}(X) \arrow[d, "\Phi_\eta"] \\ \bar{\Gamma}(F_{\mathbb{P}(\Phi_\eta(E))}) \arrow[r, "\Theta_{\Phi_\eta(E)}"] & K^{(f)}_{\zeta'}(X), \end{tikzcd}$$

where $\Phi_\eta(E) \simeq \Phi_\eta(\tilde{E}) \otimes \mathcal{H}$, since $E := \tilde{E} \otimes \mathcal{H}$. The result follows from the commutativity of the following diagram, that we are going to show:

$$\begin{tikzcd}
\bar{\Gamma}(F_{\mathbb{P}(E)}) \arrow[rrr, dotted, "\Theta_E"] \arrow[dddd, equal] \arrow[dr, "\overline{\phi^{\#}}"] & & & K^{(f)}_{\zeta}(X) \arrow[dl, "\overline{\phi^*}"'] \arrow[dddd, "\Phi_\eta"] \\
& \bar{\Gamma}(F_{\mathbb{P}(\phi^* E)}) \arrow[r, "\hat{\Theta}_{\phi^* E}"] \arrow[d, equal] & K^{(f)}_{\hat{\zeta}}(X) \arrow[d, "\Phi_{\hat{\eta}}"] & \\
& \bar{\Gamma}(F_{\mathbb{P}(\phi^* \Phi_\eta(E))}) \arrow[r, "\hat{\Theta}_{\phi^* \Phi_\eta(E)}"] & K^{(f)}_{\hat{\zeta}'}(X) & \\
& & & \\
\bar{\Gamma}(F_{\mathbb{P}(\Phi_\eta(E))}) \arrow[uur, "\overline{\phi^{\#}}"'] \arrow[rrr, dotted, "\Theta_{\Phi_\eta(E)}"] & & & K^{(f)}_{\zeta'}(X), \arrow[uul, "\overline{\phi^*}"]
\end{tikzcd}$$

where $\hat{\eta}' := \phi^*\hat{\eta}$ and $\hat{\zeta}' := \phi^*\hat{\zeta}$, so that $\hat{\zeta}' = \hat{\zeta} \cdot \check{\delta}^1\hat{\eta}$. The "square" on the right coincides with diagram (6.12), therefore we already know that it commutes. Similarly, the "square" on the left follows from diagram (6.13) and definition (6.16), therefore we already know that it commutes as well. It remains to consider the central square. Since $\phi^*\Phi_\eta(E) = \Phi_{\hat{\eta}}(\phi^*E)$, in the upper line we start from $\phi^*E = \phi^*\tilde{E} \otimes \mathcal{H}$ and in the lower line we start from $\Phi_{\hat{\eta}}(\phi^*E) = \Phi_{\hat{\eta}}(\phi^*\tilde{E}) \otimes \mathcal{H}$. By definition, the transition functions of $\Phi_{\hat{\eta}}(\phi^*\tilde{E})$ are the ones of $\phi^*\tilde{E}$ multiplied by $\hat{\eta}$, hence, if F_t and G_t are the bundles (6.17) and (6.18) induced by $\phi^*\tilde{E}$, then the ones induced by $\Phi_{\hat{\eta}}(\phi^*\tilde{E})$ are respectively $\Phi_{\hat{\eta}}(F_t)$ and $\Phi_{\hat{\eta}}(G_t)$, thus commutativity follows.

Cover refinement

If we take the direct limit with respect to the good cover, we have to verify the compatibility with cover refinements. In particular, let us consider a refinement $\mathfrak{U}' = \{U'_\alpha\}_{\alpha \in \Lambda}$ of $\mathfrak{U} = \{U_i\}_{i \in I}$, through a refinement function $\mu\colon \Lambda \to I$. We fix $\zeta \in \check{Z}^2(\mathfrak{U}, \underline{\mathrm{U}}(1))$ and we set $\zeta' := \mu^*\zeta$. We have to verify that the following diagram commutes:

$$\begin{tikzcd}
K^{(\infty)}_{\zeta}(X) \arrow[r, "\Theta"] \arrow[d, "\overline{\mu^*}"'] & K^{(f)}_{\zeta}(X) \arrow[d, "\overline{\mu^*}"] \\
K^{(\infty)}_{\zeta'}(X) \arrow[r, "\Theta"] & K^{(f)}_{\zeta'}(X).
\end{tikzcd} \tag{6.27}$$

This means that, starting from a ζ-twisted bundle $\tilde{E}$ as above, the following diagram commutes:

$$\begin{tikzcd}
\bar{\Gamma}(F_{\mathbb{P}(E)}) \arrow[r, "\Theta_E"] \arrow[d, "\overline{\mu^{\#}}"'] & K^{(f)}_{\zeta}(X) \arrow[d, "\overline{\mu^{*}}"] \\
\bar{\Gamma}(F_{\mathbb{P}(\mu^* E)}) \arrow[r, "\Theta_{\mu^* E}"] & K^{(f)}_{\zeta'}(X).
\end{tikzcd}$$

We apply the construction of Θ starting from $\mathfrak{U}'$ and ζ'. In particular, we choose a refinement $\mathfrak{V} = \{V_1, \dots, V_m\}$ of $\mathfrak{U}'$ through $\phi\colon \{1, \dots, m\} \to \Lambda$, such that $\bar{V}_k \subset U'_{\phi(k)}$ for every k. It follows that $\mathfrak{V}$ is a suitable refinement of $\mathfrak{U}$ too through $\mu \circ \phi$, since $\bar{V}_k \subset U'_{\phi(k)} \subset U_{\mu(\phi(k))}$. Hence, considering definition (6.20), the result follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\bar{\Gamma}(F_{\mathbb{P}(E)}) \arrow[rrrr, dotted, "\Theta_E"] \arrow[dr, "\overline{\mu^{\#}}"] \arrow[ddr, "\overline{\phi^{\#}\mu^{\#}}"'] & & & & K^{(f)}_{\zeta}(X) \arrow[dl, "\overline{\mu^{*}}"'] \arrow[ddl, "\overline{\phi^{*}\mu^{*}}"] \\
& \bar{\Gamma}(F_{\mathbb{P}(\mu^* E)}) \arrow[rr, dotted, "\Theta_{\mu^* E}"] \arrow[d, "\overline{\phi^{\#}}"] & & K^{(f)}_{\zeta'}(X) \arrow[d, "\overline{\phi^{*}}"'] & \\
& \bar{\Gamma}(F_{\mathbb{P}(\phi^*\mu^* E)}) \arrow[rr, "\hat{\Theta}_{\phi^*\mu^* E}"] & & K^{(f)}_{\zeta'}(X). &
\end{tikzcd}$$

Only the commutativity of the triangles has to be verified, and it is straightforward.

Refinement function

Finally, if we change the refinement function from μ to μ', then, setting $\zeta'' := \mu'^{*}\zeta$ and $\zeta'' = \zeta' \cdot \check{\delta}^1\nu$, the following diagram commutes:

$$\begin{tikzcd}
& & & & K^{(f)}_{\zeta}(X) \arrow[dl, dotted, "\overline{\mu^{*}}"'] \arrow[dr, "\overline{\mu'^{*}}"] & \\
& K^{(\infty)}_{\zeta}(X) \arrow[urrr, "\Theta"] \arrow[dl, "\overline{\mu^{*}}"'] \arrow[dr, "\overline{\mu'^{*}}"] & & K^{(f)}_{\zeta'}(X) \arrow[rr, dotted, "\Phi_\nu"] & & K^{(f)}_{\zeta''}(X) \\
K^{(\infty)}_{\zeta'}(X) \arrow[rr, "\Phi_\nu"] \arrow[urrr, dotted, "\Theta"] & & K^{(\infty)}_{\zeta''}(X). \arrow[urrr, "\Theta"] & & &
\end{tikzcd}$$

In fact, the two triangles are instances of diagram (6.11), the rectangle at the base is an instance of diagram (6.26) and the other two rectangles are instances of diagram (6.27). It follows that (6.22) is well-defined.

6.3 Models of differential twisted K-theory

Now we consider the differential extensions of the two models considered in the previous section, and we explicitly show a natural isomorphism between them. Again, we consider the groups of degree 0. We will consider the extension to any degree in the last section.

6.3.1 Model through Schatten Grassmannians

We follow [13] to define the infinite-dimensional model of differential twisted K-theory. We keep on denoting by $\mathcal{H}$ a separable infinite-dimensional Hilbert space and we denote by $\mathfrak{B}(\mathcal{H})$ the Banach algebra of bounded linear operators in $\mathcal{H}$. Moreover, given two such Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, for any $p \in [1, +\infty)$ we denote by $\mathfrak{L}^p(\mathcal{H}_1, \mathcal{H}_2)$ the corresponding p-Schatten class, i.e. the space of compact linear operators $A\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\|A\|_{\mathfrak{L}^p} := \Big(\mathrm{Tr}\big((A^\dagger A)^{\frac{p}{2}}\big)\Big)^{\frac{1}{p}} < \infty, \tag{6.28}$$

the trace being the sum of the eigenvalues. We set $\mathfrak{L}^p(\mathcal{H}) := \mathfrak{L}^p(\mathcal{H}, \mathcal{H})$.

Schatten Grassmannians.

We use the following notation:

- $\hat{\mathcal{H}} := \mathcal{H} \oplus \mathcal{H}$;
- $\mathcal{H}_+ := \mathcal{H} \oplus 0 \simeq \mathcal{H}$;
- $\mathcal{H}_- := 0 \oplus \mathcal{H} \simeq \mathcal{H}$.

It follows that $\hat{\mathcal{H}}$ is $\mathbb{Z}_2$-graded, the corresponding self-adjoint involution being $\epsilon\colon \hat{\mathcal{H}} \to \hat{\mathcal{H}}$, defined by $\epsilon|_{\mathcal{H}_+} = 1$ and $\epsilon|_{\mathcal{H}_-} = -1$. Equivalently, $\mathcal{H}_\pm$ is the corresponding ± 1-eigenspace. More generally, given a closed subspace $\mathcal{V} \subset \hat{\mathcal{H}}$, we get the corresponding decomposition $\hat{\mathcal{H}} = \mathcal{V} \oplus \mathcal{V}^\perp$ and the corresponding self-adjoint involution $\epsilon_{\mathcal{V}}$, defined by $\epsilon|_{\mathcal{V}} = 1$ and $\epsilon|_{\mathcal{V}^\perp} = -1$ (hence, $\epsilon = \epsilon_{\mathcal{H}_+}$).

Definition 6.3.1. The Grassmannian of $\hat{\mathcal{H}}$, that we denote by $\mathbf{Gr}(\hat{\mathcal{H}})$, is the subset of $\mathfrak{B}(\mathcal{H})$ formed by self-adjoint involutions.

We get a natural bijection between the set of closed vector subspaces of $\hat{\mathcal{H}}$ and $\mathbf{Gr}(\hat{\mathcal{H}})$, defined by $\mathcal{V} \mapsto \epsilon_{\mathcal{V}}$.

Definition 6.3.2. The space $\mathbf{Gr}^p(\hat{\mathcal{H}}, \epsilon)$ is the subset of $\mathbf{Gr}(\hat{\mathcal{H}})$ formed by the self-adjoint involutions $\epsilon_{\mathcal{V}}$ such that $\epsilon_{\mathcal{V}} - \epsilon \in \mathfrak{L}^p(\hat{\mathcal{H}})$. Equivalently:

$$\mathbf{Gr}^p(\hat{\mathcal{H}}, \epsilon) := \mathbf{Gr}(\hat{\mathcal{H}}) \cap \big(\epsilon + \mathfrak{L}^p_{sa}(\hat{\mathcal{H}})\big),$$

where $\mathfrak{L}^p_{sa}(\hat{\mathcal{H}})$ denotes the subspace of $\mathfrak{L}^p(\hat{\mathcal{H}})$ formed by the self-adjoint elements.

Since $\mathfrak{L}^p(\hat{\mathcal{H}})$, with the norm (6.28), is a Banach space and $\mathfrak{L}^p_{sa}(\hat{\mathcal{H}})$ is a real vector subspace, the latter is trivially a Banach submanifold, hence the coset $\epsilon + \mathfrak{L}^p_{sa}(\hat{\mathcal{H}})$ is a Banach manifold too (we remark that we are not considering the topology induced by the embedding in $\mathbf{G}\mathrm{r}(\hat{\mathcal{H}})$, the latter being embedded in $\mathfrak{B}(\hat{\mathcal{H}})$, but the one induced by the Schatten norm).

Proposition 6.3.3. *The space* $\mathbf{G}\mathrm{r}^p(\hat{\mathcal{H}}, \epsilon)$ *is a Banach submanifold of* $\epsilon+\mathfrak{L}^p_{sa}(\hat{\mathcal{H}})$ *and it is smoothly homotopically equivalent to* $\mathrm{Fred}(\mathcal{H})$*. In particular, its homotopy type does not depend on* p*.*

In order to construct a homotopy equivalence between $\mathbf{G}\mathrm{r}^p(\hat{\mathcal{H}}, \epsilon)$ and $\mathrm{Fred}(\mathcal{H})$, we consider the group $\mathrm{GL}_p(\hat{\mathcal{H}}, \epsilon)$, formed by bounded invertible operators on $\hat{\mathcal{H}}$ of the form:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad b \in \mathfrak{L}^{2p}(\mathcal{H}_-, \mathcal{H}_+),\ c \in \mathfrak{L}^{2p}(\mathcal{H}_+, \mathcal{H}_-).$$

This definition implies that $a \in \mathrm{Fred}(\mathcal{H}_+)$. From now on we denote $\mathbf{G}\mathrm{r}^p(\hat{\mathcal{H}}, \epsilon)$ and $\mathrm{GL}_p(\hat{\mathcal{H}}, \epsilon)$ respectively by $\mathbf{G}\mathrm{r}^p$ and GL_p, and we identify $\epsilon_{\mathcal{V}} \in \mathbf{G}\mathrm{r}^p$ with $\mathcal{V} \subset \hat{\mathcal{H}}$ when necessary. We have a natural transitive action $\mathrm{GL}_p \times \mathbf{G}\mathrm{r}^p \to \mathbf{G}\mathrm{r}^p$, $(A, \mathcal{V}) \mapsto A(\mathcal{V})$. In particular, a closed subspace $\mathcal{V} \subset \hat{\mathcal{H}}$ belongs to $\mathbf{G}\mathrm{r}^p$ if and only if there exists $A \in \mathrm{GL}_p$ such that $A(\mathcal{H}_+) = \mathcal{V}$. We have the following natural maps, that turn out to be homotopy equivalences:

$$\begin{array}{ccccc} \mathrm{Fred}(\mathcal{H}) & \xleftarrow{\psi} & \mathrm{GL}_p & \xrightarrow{\varphi} & \mathbf{G}\mathrm{r}^p \\ a & \longleftarrow\!\shortmid & A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} & \mapsto & A(\mathcal{H}_+) = \big\{\big(a(x), c(x)\big) : x \in \mathcal{H}_+\big\}. \end{array} \tag{6.29}$$

We have a natural action of $\mathrm{PU}(\mathcal{H})$ by conjugation on each of the previous spaces, that we denote respectively by $\rho\colon \mathrm{PU}(\mathcal{H}) \to C^0(\mathrm{Fred}(\mathcal{H}))$, $\rho''\colon \mathrm{PU}(\mathcal{H}) \to C^0(\mathrm{GL}_p)$ and $\rho'\colon \mathrm{PU}(\mathcal{H}) \to C^0(\mathbf{G}\mathrm{r}^p)$. In ρ' and ρ'' we are applying the diagonal embedding $\mathrm{PU}(\mathcal{H}) \hookrightarrow \mathrm{PU}(\hat{\mathcal{H}})$. We get the three bundles

$$F_{\mathbb{P}(E)} := P_{\mathbb{P}(E)} \times_\rho \mathrm{Fred}(\mathcal{H}) \qquad F''_{\mathbb{P}(E)} := P_{\mathbb{P}(E)} \times_{\rho''} \mathrm{GL}_p \qquad F'_{\mathbb{P}(E)} := P_{\mathbb{P}(E)} \times_{\rho'} \mathbf{G}\mathrm{r}^p.$$

It is straightforward to verify that the homotopy equivalences φ and ψ in (6.29) commute with the actions of $\mathrm{PU}(\mathcal{H})$, therefore we get the induced maps:

$$\begin{array}{ccccc} F_{\mathbb{P}(E)} & \xleftarrow{\psi_*} & F''_{\mathbb{P}(E)} & \xrightarrow{\varphi_*} & F'_{\mathbb{P}(E)} \\ [p, \psi(A)] & \longleftarrow\!\shortmid & [p, A] & \mapsto & [p, \varphi(A)]. \end{array} \tag{6.30}$$

As above, we denote by $\Gamma(F_{\mathbb{P}(E)})$ the space of sections of $F_{\mathbb{P}(E)}$ and by $\bar{\Gamma}(F_{\mathbb{P}(E)})$ the corresponding quotient up to homotopy of sections. We use the same notation for $F'_{\mathbb{P}(E)}$ and $F''_{\mathbb{P}(E)}$. By definition $K_\zeta(X) := \bar{\Gamma}(F_{\mathbb{P}(E)})$ and we set $K'_\zeta(X) := \bar{\Gamma}(F'_{\mathbb{P}(E)})$. The maps (6.30) induces the corresponding maps between sections, that are isomorphisms up to homotopy, since the maps (6.29) are homotopy equivalences. We get the following canonical isomorphism:

$$\theta_\zeta := \varphi_* \circ \psi_*^{-1} \colon K_\zeta(X) \xrightarrow{\simeq} K'_\zeta(X). \tag{6.31}$$

The group $K'_\zeta(X)$ turns out to be more suitable than $K_\zeta(X)$ in order to define the corresponding differential extension.

Natural connection on $\mathrm{U}(\mathcal{H})$.

Since $\mathrm{PU}(\mathcal{H}) := \mathrm{U}(\mathcal{H})/\mathrm{U}(1)$, the projection $\pi\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$ naturally induces a principal $\mathrm{U}(1)$-bundle structure. A connection on this bundle is a suitable equivariant 1-form $\theta\colon T\mathrm{U}(\mathcal{H}) \to \mathbb{R}$, since $\mathbb{R}$ is the Lie algebra of $\mathrm{U}(1)$. Here we have a canonical connection, defined as follows. From the exact sequence

$$0 \longrightarrow \mathrm{U}(1) \xrightarrow{\ i\ } \mathrm{U}(\mathcal{H}) \xrightarrow{\ \pi\ } \mathrm{PU}(\mathcal{H}) \longrightarrow 0,$$

where i is the embedding of $\mathrm{U}(1)$ as the centre of $\mathrm{U}(\mathcal{H})$, we get the corresponding one among the Lie algebras, i.e.

$$0 \longrightarrow \mathbb{R} \xrightarrow{\ di_1\ } T_1\mathrm{U}(\mathcal{H}) \xrightarrow{\ d\pi_1\ } T_1\mathrm{PU}(\mathcal{H}) \longrightarrow 0.$$

We call $p\colon T_1\mathrm{U}(\mathcal{H}) \to \mathbb{R}$ the orthogonal projection to the image of di_1, composed with di_1^{-1}. Such a map splits the previous sequence. In order to define the connection θ, given $A \in \mathrm{U}(\mathcal{H})$ and $V \in T_A\mathrm{U}(\mathcal{H})$, we set $\theta_A(V) := \frac{1}{2\pi}\, p \circ dl_A^{-1}(V)$, where l_A is the left multiplication by A in $\mathrm{U}(\mathcal{H})$.[6] Fixing local trivializations of $\pi\colon \mathrm{U}(\mathcal{H}) \to \mathrm{PU}(\mathcal{H})$, we can represent the connection by local potentials A_θ on $\mathrm{PU}(\mathcal{H})$, that are the pull-backs of θ through the trivializations. The same can be done with respect to the curvature $d\theta$, but, in this case, we get a global integral 2-form F_θ on $\mathrm{PU}(\mathcal{H})$.

We recall some basic properties of θ that we are going to use. First of all, given two functions $f, g\colon Z \to \mathrm{U}(\mathcal{H})$, for any smooth manifold Z, we have

$$(fg)^*\theta = f^*\theta + g^*\theta, \tag{6.32}$$

where fg denotes the point-wise product. This can be proven by direct computation. Moreover, if $\zeta\colon Z \to \mathrm{U}(1)$, then, thinking of $\mathrm{U}(1) \subset \mathrm{U}(\mathcal{H})$ as the centre, we have:

$$\zeta^*\theta = \tfrac{1}{2\pi i}\, \zeta^{-1} d\zeta. \tag{6.33}$$

Such a formula easily follows from the fact that θ, restricted to the centre, coincides with the Maurier-Cartan 1-form of $\mathrm{U}(1)$. Moreover, for any 1-form $\Lambda\colon TZ \to \mathbb{R}$, there exists $h\colon Z \to \mathrm{U}(\mathcal{H})$ such that

$$h^*\theta = \Lambda. \tag{6.34}$$

In fact, since θ is the universal connection for line bundles and since Λ represents any connection ∇_Λ on $Z \times \mathbb{C}$, we can find a homotopically-trivial function $\bar{h}\colon Z \to \mathrm{PU}(\mathcal{H})$, covering $h'\colon \bar{h}^*\mathrm{U}(\mathcal{H}) \to \mathrm{U}(\mathcal{H})$, such that the connection ∇_Λ corresponds to $(h')^*\theta$. Choosing the trivialization $s\colon Z \to \bar{h}^*\mathrm{U}(\mathcal{H})$, inducing Λ as the potential representing ∇_Λ, we set $h := h' \circ i$ and we get that $h^*\theta = \Lambda$.

[6] The normalization constant $\frac{1}{2\pi}$ has been inserted only to make some easy computations more elegant.

Realizing a Deligne cocycle.

Given a ζ-twisted Hilbert bundle $E = (\{E_i\}, \{\varphi_{ij}\})$, let us fix a system of local sections $\{s_i^n : U_i \to E_i\}_{n\in\mathbb{N}}$, that locally trivializes E (i.e. that trivializes each E_i). Such sections determine a Deligne cocycle $(\zeta, \Lambda) \in \check{Z}^2(S^1_X)$ as follows. The first component is determined by E itself, that is ζ-twisted by definition. Moreover, the fixed sections determine the corresponding transition functions $g_{ij} : U_{ij} \to \mathrm{U}(\mathcal{H})$, therefore, from the universal connection θ on $\mathrm{U}(\mathcal{H})$, we set $\Lambda_{ij} := g_{ij}^*\theta$. It follows from (6.32) and (6.33) that (ζ, Λ) is a cocycle.

Let us show that any cocycle (ζ, Λ) can be reached in this way. In fact, we already know that for any ζ there exists a ζ-twisted bundle E. Fixing ζ and inducing any (ζ, Λ) through section $\{s_i^n\}$, any other representative of the same cohomology class is of the form $(\zeta, \Lambda) \cdot \check{D}^1(1, \lambda) = (\{\zeta_{ijk}\}, \{\Lambda_{ij} - \lambda_i + \lambda_j\})$. We can replace the sections $\{s_i^n\}$ through any change of basis $h_i : U_i \to \mathrm{U}(\mathcal{H})$, so that we get the transition functions $g'_{ij} := h_i g_{ij} h_j^{-1}$. We choose h_i such that $h_i^*\theta = -\lambda_i$, that is always possible, as we have shown in the remarks after formula (6.34). It follows from (6.32) that $(g'_{ij})^*\theta = \Lambda_{ij} - \lambda_i + \lambda_j$, as desired.

Differential twisted K-theory.

The spaces $\mathbf{G}\mathrm{r}^p$ and $\mathrm{Fred}(\mathcal{H})$ are Banach manifolds, smoothly homotopically equivalent among each other. In particular, they are all homotopic to the Hilbert manifold $\mathbf{G}\mathrm{r}^2$, therefore their de-Rham cohomology can be defined through smooth differential forms, as in the finite-dimensional setting, and it is canonically isomorphic to the real or complex singular one. On $\mathbf{G}\mathrm{r}^p$, using Quillen superconnections, we can fix smooth even-degree differential forms $\Phi_{2n} \in \Omega^{2n}(\mathbf{G}\mathrm{r}^p)$ such that $\Phi_{\mathrm{ev}} := \sum_{n=0}^{\infty} \Phi_{2n}$ represents the Chern character of the canonical K-theory class, the latter being the class represented by the identity.

Let us fix a ζ-twisted Hilbert bundle $E = (\{E_i\}, \{\varphi_{ij}\})$ and a section $\psi \in \Gamma(F'_{\mathbb{P}(E)})$. Given a Deligne cocycle (ζ, Λ, B), we fix a system of local sections $\{s_i^n : U_i \to E_i\}_{n\in\mathbb{N}}$, inducing the cocycle (ζ, Λ) as we have seen above. Such sections identify ψ with a family of local functions $\psi_i : U_i \to \mathbf{G}\mathrm{r}^p$, therefore we get the local pull-backs $\psi_i^*\Phi_{\mathrm{ev}} \in \Omega^{\mathrm{ev}}(U_i)$. This implies that the local sections determine at the same time the local forms $\psi_i^*\Phi_{\mathrm{ev}}$ and the local potentials B_i up to a global form $\tilde{B}$, therefore it is not surprising that these two data glue to the global form $\exp(B_i) \wedge \psi_i^*\Phi_{\mathrm{ev}}$, that is d_H-closed. This justifies the following definition.

Definition 6.3.4. Given a Deligne 2-cocycle (ζ, Λ, B) on X, with curvature H, we fix a ζ-twisted bundle $E = (\{E_i\}, \{\varphi_{ij}\})$ and a set of local trivializations $\{s_i^n\}$ inducing Λ as above. We define the group $\check{K}_{(\zeta,\Lambda,B)}(X)$ as follows. An element of this group is a homotopy class of pairs (ψ, η), where:

- ψ is a smooth section of $F'_{\mathbb{P}(E)}$;
- $\eta \in \Omega^{\mathrm{odd}}(X)/\mathrm{Im}(d_H)$;

- a homotopy between (ψ, η) and (ψ', η') is a homotopy of sections $\Psi\colon \psi \sim \psi'$ such that $\eta - \eta' \sim_{d_H} \int_I \big(\exp(B_i) \wedge \Psi_i^* \Phi_{\mathrm{ev}}\big)$, where '$\sim_{d_H}$' denotes that they are equal up to a d_H-exact form.[7]

We get the following functors:

- $I\colon \check{K}_{(\zeta,\Lambda,B)}(X) \to K_\zeta(X)$, $[\psi,\eta] \mapsto \theta_\zeta^{-1}[\psi]$, where $[\psi] \in K'_\zeta(X)$ and θ_ζ is the isomorphism (6.31);
- $R\colon \check{K}_{(\zeta,\Lambda,B)}(X) \to \Omega^{\mathrm{ev}}(X)$; $[\psi,\eta] \mapsto \exp(B_i) \wedge \psi_i^* \Phi_{\mathrm{ev}} - d\eta$;
- $a\colon \Omega^{\mathrm{odd}}(X)/\mathrm{Im}(d_H) \to \check{K}_{(\zeta,\Lambda,B)}(X)$, $\eta \mapsto [\psi,\eta] - [\psi, 0]$ for any section ψ.

Dependence on the cocycle.

Fixing a cocycle (ζ, Λ, B), we have to consider the dependence on the bundle E and on the sections s.

`I. Bundle.` Given a triple (E, s, B) and another ζ-twisted bundle E', we set $s' := f_* s$ and we consider the triple (E', s', B). We fix an isomorphism $f\colon E \to E'$, inducing $\bar{f}\colon \mathbb{P}(E) \to \mathbb{P}(E')$ and $\bar{\bar{f}}\colon F_{\mathbb{P}(E)} \to F_{\mathbb{P}(E')}$, and we get the isomorphism:

$$\begin{aligned} f_\#\colon K_{(E,s,B)}(X) &\to K_{(E',s',B)}(X) \\ [(\psi,\eta)] &\mapsto [(\bar{\bar{f}} \circ \psi, \eta)]. \end{aligned} \tag{6.35}$$

The reader can verify that it is well-defined, since, if $\Psi\colon (\psi_1, \eta_1) \sim (\psi_2, \eta_2)$, then $\bar{\bar{f}} \circ \Psi\colon (\bar{\bar{f}} \circ \psi_1, \eta_1) \sim (\bar{\bar{f}} \circ \psi_2, \eta_2)$. For this reason, we can fix E up to isomorphism.

`II. Sections.` If we consider two triples (E, s, B) and (E, s', B), in general there not exists an isomorphism sending s to s' (this happens if and only if the induced transition functions coincide, but the condition $g_{ij}^* \theta = \Lambda_{ij}$ does not fix g_{ij} completely). In this case, we fix a base change $\{h_i\colon U_i \to \mathrm{U}(\mathcal{H})\}$ from s to s' and we set $\alpha_i := h_i^* \theta$. Calling g and g' the transition functions induced by s and s' respectively, since $g_{ij}^* \theta = {g'}_{ij}^* \theta = \Lambda_{ij}$ and $g'_{ij} = h_i g_{ij} h_j^{-1}$, it follows that $h_i^* \alpha = h_j^* \alpha$, hence the local forms α_i glue to a global 1-form α on X. We get the following isomorphism:

$$\begin{aligned} h\colon K_{(E,s,B)}(X) &\to K_{(E,s',B)}(X) \\ [(\psi,\eta)] &\mapsto [(\psi, \eta \wedge e^{d\alpha})]. \end{aligned} \tag{6.36}$$

The reader can verify that, if $\Psi\colon (\psi_1, \eta_1) \sim (\psi_2, \eta_2)$ with respect to s, then $\Psi\colon (\psi_1, \eta_1 \wedge e^{d\alpha}) \sim (\psi_2, \eta_2 \wedge e^{d\alpha})$ with respect to s', hence h is well-defined.

If we change representative in the Deligne cohomology class $[(\zeta, \Lambda, B)]$, we consider separately coboundaries of the form $(\eta, 0)$ and of the form $(1, \lambda)$.

[7] Of course we are identifying B_i and H with $\pi^* B_i$ and $\pi^* H$, where $\pi\colon X \times I \to X$ is the natural projection. Moreover, Ψ_i comes from the obvious local trivialization of $\pi^* E$, induced by the one of E.

`Action of` $(\eta, 1)$. We consider a triple (E, s, B), relative to (ζ, Λ, B). Moreover, we set $(\zeta', \Lambda', B) := (\zeta, \Lambda, B) \cdot \check{D}^1(\eta, 1)$ and we fix a ζ'-twisted bundle E'. We have that $\zeta' = \zeta \cdot \check{\delta}^1\eta$ and $\Lambda' = \Lambda + \tilde{d}\eta$. We fix an isomorphism $\bar{f}\colon \mathbb{P}(E) \to \mathbb{P}(E')$ and we consider the triple (E', s', B), where s' is any lift of $\bar{f}_*\bar{s}$ such that ${g'}^*_{ij}\theta = \Lambda'_{ij}$.[8]

`Action of` $(1, \lambda)$. We consider a triple (E, s, B), relative to (ζ, Λ, B). Moreover, we set $(\zeta, \Lambda', B') := (\zeta, \Lambda, B) \cdot \check{D}^1(1, \lambda)$. We have that $\check{K}_{(E,s,B)}(X) = \check{K}_{(E,s',B')}(X)$, where s' is such that the change of basis $h_i\colon U_i \to \mathrm{U}(\mathcal{H})$ from s to s' verifies $h_i^*\theta = \lambda_i$.

`Global` B`-field variation.` If we reply B by $B + \tilde{B}$, where B is a global form (even changing the cohomology class), then we have the isomorphism $[(\psi, \eta)] \mapsto [(\psi, \eta \wedge e^{\tilde{B}}]$.

[8] The transition functions $\bar{g}_{ij}$ are fixed, since f is an isomorphism, hence $d\Lambda = d\Lambda'$ is fixed, since it is determined by $\bar{g}_{ij}$. Any variation by an exact form can be reach by choosing a suitable lift.

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