

Scaling Symmetries, Contact Reduction and Poincaré's dream

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A symplectic Hamiltonian system admitting a scaling symmetry can be reduced to an equivalent contact Hamiltonian system in which some physically-irrelevant degree of freedom has been removed. As a consequence, one obtains an equivalent description for the same physical phenomenon, but with fewer inputs needed, thus realizing ‘Poincaré’s dream’ of a scale-invariant description of the universe.

This work is devoted to a thorough analysis of the mathematical framework behind such reductions. We show that generically such reduction is possible and the reduced (fundamental) system is a contact Hamiltonian system. The price to pay for this level of generality is that one is compelled to consider the coupling constants appearing in the original Hamiltonian as part of the dynamical variables of a lifted system. This however has the added advantage of removing the hypothesis of the existence of a scaling symmetry for the original system at all, without breaking the sought-for reduction in the number of inputs needed. Therefore a large class of Hamiltonian (resp. Lagrangian) theories can be reduced to scale-invariant contact Hamiltonian (resp. Herglotz variational) theories.

I. INTRODUCTION

Symmetry plays an important role in physics. When considering observations of measurable quantities, the action of symmetry should be carefully considered in its effect not only on the quantity being measured but also upon the apparatus through which the measurement is made. This is common when considering most physical symmetries. On applying a Galilean transformation to a system – for example transforming reference frames – we argue invariance of observation based on applying the transformation to both the observer and the observed quantity. Absolute motion should not be observable as it is not invariant under Galilean transformations. Physically meaningful quantities and their equations of motion are those invariant under such symmetry transformations.

It has been argued, at least since Poincaré (see eg. [27] for a detailed discussion), that the same considerations should be applied to scaling symmetries and the absolute scale of a system. In [36], pg. 94, Poincaré imagines that overnight all dimensions in the universe have been scaled, and argues that there will be no detectable change due to both the observed system (the universe) and any possible measuring apparatus (eg one’s height), having undergone a scaling by the same factor. Accordingly, we will refer to writing the equations of motion of a system in terms of scale-invariant quantities as *Poincaré’s dream*.

The counter to this argument was long held to be the fact that physical constants can be used to give an overall scale based on, for example, the Planck length. However in recent work it has been suggested that if these are included not as given facts about the universe, but rather as physical observations of quantities which must be measured, such a scaling symmetry can be restored [40].

It should be clear that in any scale-invariant description of the physical world one would need one less piece of information with respect to the standard descriptions that include reference to some (unmeasurable) absolute scale, the datum being removed being precisely the one corresponding to the scale. Moreover, such scale reductions contain valuable information on certain singularities, eg collision singularities, of the original systems and this has been exploited in several works, [28, 31, 33, 38], to find that the scale-invariant description is freed from the singularities of the original dynamics (although care must be taken, since this is not always necessarily the case [32]). As well, one finds interesting dynamical features such as a dissipative-like behavior, which can provide a natural origin for the observed arrow of time [3, 26, 37]. Therefore Poincaré’s dream is well worth pursuing after all.

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Despite the great amount of work in this direction, still the treatment of scaling symmetries and of their corresponding reductions so far has been based on a case-by-case study (with the exception of [37]) and with some of the methods being quite ad-hoc; for instance, in most situations the choice of scaling function and selection of scale invariant quantities has been given without justification. While in simple systems such as the Kepler problem the choice of r as scaling may indeed appear natural, as r represents the physical distance separation between the two bodies in the problem, when we move beyond this even simply to the n -body problem the choice is not so apparent – one could use the mean separation of particles, or the largest separation, or some combination of any variables that scale in a similar way. Further, even in the simple Kepler problem, one could have opted to use p_r , the momentum conjugate to r , instead of r . Would the results ensuing from these two different choices be related somehow? Although it is not difficult to show directly that analogous results hold in this particular case, this highlights the arbitrary nature of such choices.

In this work we will address for the first time the study of scaling symmetries in Hamiltonian systems and their related contact reductions in full generality. Our main results (Theorems 1 and 2) will establish that any Hamiltonian (resp. variational) theory admitting a scaling symmetry can be reduced to a dynamically equivalent scale-invariant contact Hamiltonian (resp. Herglotz variational) theory, thus illustrating that Poincaré’s dream is indeed feasible. Surprisingly, we will see that this is possible even in the absence of an a-priori evident scaling symmetry, provided some less stringent conditions are satisfied. Moreover, we will consider explicitly the case in which one has chosen a certain scaling function and show how to construct the resulting equations of motion in terms of scale invariant quantities (Corollary 2). Finally, a collection of physically-relevant examples will be provided throughout the manuscript in order to illustrate the main points and constructions.

The paper is laid out as follows: in Section II we give a precise definition of a scaling symmetry and prove that reducing a symplectic system with a scaling symmetry in general yields two contact systems on different regions, which coincide if and only if the degree of the scaling symmetry is one (Theorem 1 and Corollary 1). Besides, in Corollary 2 we prove an invariance (up to time reparametrizations) of the reduced system with respect to the choice of the scaling function. Finally, in Section III we show that Hamiltonian systems which do not exhibit scaling symmetries can be lifted into a broader space of models in which the symmetry is present, by promoting the physical constants (couplings) to observable variables within the system. In this way one may always reduce the description to an equivalent theory based on a contact Hamiltonian (or Herglotz variational) theory on a reduced space. Despite the dynamical equivalence among all such theories, the contact Hamiltonian one, involving less elements for its complete description, has more explanatory power (Proposition 6 and Theorem 2). We conclude in Section IV with an outlook on future work and applications. To have the paper as self-contained as possible and in order to further motivate our constructions, three appendices are also included: in Appendix A we introduce the symplectic and contact Hamiltonian systems that we consider, and briefly summarize some known relationship between them. In Appendix B we review the variational approach of Herglotz to contact systems and re-state our main results in terms of this formulation. In Appendix C we provide some examples of well-known constructions in celestial mechanics for which our formalism provides a unifying principle.

II. SCALING SYMMETRIES, CONTACT REDUCTION AND SYMPLECTIFICATION

In this section we describe how a special type of symmetry, called ‘scaling symmetry’, provides a new link between symplectic mechanics and contact mechanics (see Appendix A for the relevant definitions and our sign conventions). Such symmetries have been shown to be present in cosmology, the Kepler problem and a host of other physical contexts [27, 37–40].

A. Scaling symmetries

We start with a general definition, comprising all those transformations that reparametrize the dynamics without altering the (unparametrized) orbits.

Definition 1. A *dynamical similarity* of a vector field $X \in \mathfrak{X}(M)$ is a vector field $Y \in \mathfrak{X}(M)$ such that $[Y, X] = fX$, for some (in general non-constant) function $f : M \rightarrow \mathbb{R}$ (Y is also called a *conformal symmetry*).

In other words, a dynamical similarity is a symmetry of the one-dimensional distribution, or line field, generated by X . When $Y \neq 0$, such symmetries give rise to a ‘local reduction of order’ [2], pg. 7, in describing integral curves of X .

Among dynamical similarities, there is a special class appearing in Hamiltonian systems that is often found in physical applications, the so-called scaling symmetries.

Definition 2. $D \in \mathfrak{X}(M)$ is a *scaling symmetry* of degree $\Lambda \in \mathbb{R}$ for the Hamiltonian system (M, ω, H) if

- i) $L_{\mathbf{D}}\omega = \omega$
- ii) $L_{\mathbf{D}}H = \Lambda H$.

Note that with this definition

$$[\mathbf{D}, X_H] = (\Lambda - 1)X_H \quad (1)$$

and therefore scaling symmetries of symplectic Hamiltonian systems are a very special case of dynamical similarities. Further, note that for any conserved quantity generated by the vector field J , we have $[X_H, J] = 0$ hence from (1) and the Jacobi identity

$$[X_H, [\mathbf{D}, J]] = [J, [\mathbf{D}, X_H]] + [\mathbf{D}, [X_H, J]] = 0, \quad (2)$$

we see that dynamical similarities in general (and scaling symmetries in particular) map conserved quantities onto conserved quantities.

Example 1 (Kepler scalings). Our archetype motivating this definition is the map between solutions to the (planar) Kepler problem, with

$$M = T^*(\mathbb{C} \setminus 0) = (\mathbb{C} \setminus 0) \times \mathbb{C} \ni (q, p), \quad \omega = dp \wedge dq = d(p \cdot dq), \quad \text{and} \quad H_K = \frac{|p|^2}{2} - \frac{1}{|q|}. \quad (3)$$

In this case it is well-known that the *Kepler scalings*

$$q \rightarrow \lambda^2 q \quad \text{and} \quad p \rightarrow \lambda^{-1} p,$$

for $\lambda \in \mathbb{R} \setminus 0$, send orbits into orbits. Considering the generator of these scalings in the phase space,

$$\mathbf{D}_K = 2q \cdot \partial_q - p \cdot \partial_p, \quad (4)$$

one has $L_{\mathbf{D}_K}\omega = \omega$ and $L_{\mathbf{D}_K}H_K = -2H_K$, that is, \mathbf{D}_K is a scaling symmetry of degree -2 . By equation (1), it follows that the time parameter should be rescaled as $t \rightarrow \lambda^3 t$ in order to match the corresponding parametrizations ('Kepler's third law').

Before exploring further the role of scaling symmetries in Hamiltonian systems, the following two related remarks are at order:

Remark 1. The set of vector fields satisfying $L_{\mathbf{D}}\omega = \omega$ (the *Liouville vector fields*) form an affine space, directed by the *symplectic vector fields*: $Y \in \mathfrak{X}(M)$ such that $L_Y\omega = 0$ (equivalently $i_Y\omega$ is closed).

Remark 2. A scaling symmetry need not be unique, in fact for any two scaling symmetries, \mathbf{D}, \mathbf{D}' , of H one has $i_{\mathbf{D}'}\omega = i_{\mathbf{D}}\omega + \alpha$ for some closed 1-form α satisfying $i_{X_H}\alpha = (\Lambda' - \Lambda)H$, where Λ', Λ are the degrees of \mathbf{D}', \mathbf{D} . Conversely, any closed 1-form with $i_{X_H}\alpha = (\Lambda' - \Lambda)H$ determines another scaling symmetry of degree Λ' through $i_{\mathbf{D}'}\omega = i_{\mathbf{D}}\omega + \alpha$. For example, given a first integral F of H , and scaling symmetry \mathbf{D} , we have that $\mathbf{D} + X_F$ is also a scaling symmetry of the same degree as \mathbf{D} . In particular, one may always add X_H to \mathbf{D} .

Let us conclude this section by introducing a re-characterization of scaling symmetries that is equivalent to our definition above but will at times be more useful in derivations of explicit formulas.

Proposition 1. (M, ω, H) admits a scaling symmetry of degree Λ if and only if there exists a primitive, λ , of ω ($d\lambda = \omega$), satisfying

$$i_{X_H}\lambda = \Lambda H. \quad (5)$$

Proof. One takes $\lambda = i_{\mathbf{D}}\omega$ so that $d\lambda = \omega$ is equivalent to $L_{\mathbf{D}}\omega = \omega$. Moreover, one then has the relation $i_{X_H}\lambda = L_{\mathbf{D}}H$ yielding the equivalence of the condition in Eq. (5) to (ii) of Definition 1. \square

Now we are ready to ask ourselves the main question: given that many symplectic Hamiltonian systems admit a scaling symmetry, how may one use it in order to *reduce* the system to one defined on a lower-dimensional manifold? Furthermore, can we do so in such a way that the reduced system has a Hamiltonian (resp. variational) structure? Note that if we can prove such equivalence, then we consider Poincaré's dream of an equivalent description of the physical reality in terms of scale-invariant quantities to be fulfilled.

In the next sections we will address these questions by steps.

B. Scaling symmetries: from symplectic mechanics to contact mechanics

Once a scaling symmetry \mathbf{D} is in hand, the idea is to quotient by its action and obtain a reduced dynamics on a lower-dimensional manifold.

More precisely, we consider the projections of trajectories of the Hamiltonian system to the quotient of M by \mathbf{D} : $M/\mathbf{D} = M/\sim$ where $m \sim m'$ when m and m' lie on a common integral curve of \mathbf{D} . In general this quotient space need not be a manifold. Here we will simply assume that M/\mathbf{D} is a manifold, e.g. that the \mathbf{D} 's flow acts freely (\mathbf{D} has no zeroes) and properly, with submersion $M \rightarrow M/\mathbf{D}$.

The first step is to describe the additional structure on this reduced space which may be used to characterize these projected trajectories. This is the content of the next results.

Proposition 2. Suppose the flow of \mathbf{D} is complete, acting freely and properly on M so that $C := M/\mathbf{D}$ is a smooth manifold with submersion $\pi : M \rightarrow C$. Set $\lambda := i_{\mathbf{D}}\omega$. Then

- (i) C is a contact manifold with contact distribution $\mathcal{D} = \pi_*(\ker \lambda)$,
- (ii) on \mathcal{D} there is a conformal symplectic structure,
- (iii) there is a local (exact) symplectomorphism from $(M, d\lambda)$ to the symplectification $(\tilde{C}, d\tilde{\alpha})$ of C .

Proof. (i): see e.g. [19], pg. 36. Since there are no zeroes of \mathbf{D} , we have local coordinates on C by taking a local transverse slice, Σ , to \mathbf{D} . Then with $\lambda = i_{\mathbf{D}}\omega$, $L_{\mathbf{D}}\omega = \omega$ we have $\lambda \wedge (d\lambda)^n = i_{\mathbf{D}}\omega^n$ is non-degenerate on Σ determining a contact distribution $\ker \lambda|_{\Sigma}$ on Σ , i.e. $\pi_* \ker \lambda$ on C .

(ii): this is a familiar property of contact manifolds. Namely when a contact distribution, \mathcal{D} , is given by the kernel of a 1-form η then $d\eta|_{\mathcal{D}}$ is a symplectic form on \mathcal{D} . Changing η to $f\eta$, with $f \neq 0$, modifies $d(f\eta)|_{\mathcal{D}}$ to $f d\eta|_{\mathcal{D}}$, and therefore there is a conformal symplectic structure on \mathcal{D} . In our situation, this conformal symplectic structure is represented by $[\omega(\tilde{u}_1, \tilde{u}_2)]$ where $\pi_* \tilde{u}_j = u_j \in \mathcal{D}$.

(iii): recall (see Definition 9) that $\tilde{C} = \text{Ann}(\mathcal{D}) \setminus C \subset T^*C$, the symplectification of C , is an \mathbb{R}^\times principal bundle over C with exact symplectic structure, $d\tilde{\alpha}$, given by restriction of the standard symplectic structure on T^*C . Consider $\varphi : M \rightarrow \tilde{C}$ defined by

$$(\varphi(m), \pi_* v) := \lambda_m(v)$$

for all $v \in T_m M$ and with (\cdot, \cdot) the natural pairing of a vector space and its dual. One verifies that

$$((\varphi^* \tilde{\alpha})_m, v) \stackrel{\text{def of } \varphi^*}{=} (\tilde{\alpha}_{\varphi(m)}, \varphi_* v) \stackrel{\text{def of } \tilde{\alpha}}{=} (\varphi(m), (\tilde{\pi}_C)_* \varphi_* v) = (\varphi(m), \pi_* v) \stackrel{\text{def of } \varphi}{=} \lambda_m(v) \quad \forall v \in T_m M,$$

that is, $\varphi^* \tilde{\alpha} = \lambda$, and so φ is a (local) exact symplectomorphism. \square

By item (ii) in Proposition 2 there is sense in speaking of orthogonal complements with respect to the conformal symplectic structure in \mathcal{D} . We will denote these complements by $^\perp$. As for the projected orbits in C , we have

Proposition 3. The line field $\text{span}(X_H)$ on M determines a line field $\ell := \pi_* \text{span}(X_H)$ on C . Integral curves of ℓ are projections of orbits of the Hamiltonian system to C . Moreover, letting $\Sigma_0 := \pi(H = 0)$, we have

$$\ell|_{\Sigma_0} = (T\Sigma_0 \cap \mathcal{D})^\perp \subset T\Sigma_0 \cap \mathcal{D}.$$

Proof. ℓ is well defined from the scale invariance of $\text{span}(X_H)$: $L_{\mathbf{D}}X_H = (\Lambda - 1)X_H$. Since the conformal symplectic structure of Proposition 2 (ii) is given by $[\omega(\tilde{u}_1, \tilde{u}_2)]$ with $\pi_* \tilde{u}_j = u_j \in \mathcal{D}$, we will show that $\omega(\tilde{u}, X_H) = 0$ for any $\pi_* \tilde{u} \in T\Sigma_0 \cap \mathcal{D}$. First, note that \mathbf{D} is tangent to $H = 0$ and by $i_{X_H} i_{\mathbf{D}}\omega = \Lambda H$, that $X_H|_{H=0} \in \ker \lambda$. Hence $\ell|_{\Sigma_0} \subset \mathcal{D}$. Let $u \in T\Sigma_0 \cap \mathcal{D}$. Any lift, $\pi_* \tilde{u} = u$, of u lies in $T(H^{-1}(0)) \cap \ker \lambda$ and so $\omega(\tilde{u}, X_H) \sim dH(\tilde{u}) = 0$, since \tilde{u} is tangent to $H = 0$. \square

Note that for degree zero scaling symmetries, $\Lambda = 0$, then \mathbf{D} is tangent to each energy level, and the last proposition applies to arbitrary projections of energy levels to determine the line field ℓ on C . From here on we will consider the case $\Lambda \neq 0$. As for describing this line field more explicitly in the general case, we first define:

Definition 3. A *scaling function* for \mathbf{D} is a function $\rho : M \rightarrow \mathbb{R}$ such that $L_{\mathbf{D}}\rho = \rho$.

And we may now establish the equivalence

Proposition 4. Given \mathbf{D} , the following are equivalent:

- (i) the existence of a *global scaling function*, $\rho : M \rightarrow \mathbb{R}^\times$, for \mathbf{D}

- (ii) the existence of a global contact 1-form ($\ker \eta = \mathcal{D}$) on C ,
- (iii) an embedding $\iota : C \rightarrow M$ as a slice of the \mathbf{D} -action: $\pi \circ \iota = id$.

Proof. The equivalence of (i) and (ii) is through the relation $\pi^* \eta = \lambda/\rho$. (ii) and (iii) is through $\iota^* \lambda = \eta$. (iii) and (i) is through $\iota(x) = m$ s.t. $\rho(m) = 1$. \square

Example 2 (Kepler scaling functions). Consider the Kepler problem from Example 1. Here there are a number of natural choices for scaling functions available, for example any of:

$$|q|^{1/2}, \quad \frac{1}{|p|}, \quad p \cdot iq, \quad p \cdot q$$

are scaling functions of $\mathbf{D}_K = 2q \cdot \partial_q - p \cdot \partial_p$ (a degree -2 scaling symmetry of the Kepler Hamiltonian, $H_K = \frac{|p|^2}{2} - \frac{1}{|q|}$). Observe that $p \cdot iq$, is the angular momentum and $p \cdot q$ is (half) the rate of change of the moment of inertia, $q \cdot q$. The quotient, $C = M/\mathbf{D}$, may be identified with $S^1 \times \mathbb{C}$.

The items of Proposition 4 may fail to exist globally. Indeed they are equivalent to the symplectification of C being a trivial \mathbb{R}^\times bundle, with connected components, C_\pm , symplectomorphic to M . In what follows, we will simply assume that there is a global – nowhere vanishing on M – scaling function, or if the reader prefers that we are working locally over the set $\rho \neq 0$, where the scaling function is non-vanishing. Essentially, the *choice* of such a ρ leads to explicit coordinate expressions, see Eqs. (8)–(10), as well as the following relation to contact Hamiltonian flows, generalizing Arnold’s description of contact Hamiltonian vector fields (see Proposition 7 and Corollary 1 below).

Theorem 1 (Contact reduction by scaling symmetries: general case). Let $\rho : M \rightarrow \mathbb{R}^\times$ be a global scaling function, with corresponding contact form $\pi^* \eta = \lambda/\rho$ on C . With $\Sigma_0 := \pi(H = 0)$,

- (i) the contact Hamiltonian $\pi^* \mathcal{H}_0 := -H/\rho^\Lambda$ has contact Hamiltonian vector field spanning ℓ on Σ_0 ,
- (ii) the contact Hamiltonian $\mathcal{H} := -|\mathcal{H}_0|^{1/\Lambda}$ has contact Hamiltonian vector field spanning ℓ on $C \setminus \Sigma_0$.

We call this reduction a *contact reduction by scaling symmetries*, or simply a *contact reduction*.

Proof. Recall from Proposition 7 that Hamiltonian flows of degree one functions F on M ($L_{\mathbf{D}} F = F$) commute with the scaling action of \mathbf{D} and so induce contact Hamiltonian flows associated with η . Item (i) follows by considering the degree one function $H' := \rho^{1-\Lambda} H$ and item (ii) with the degree one function $|H|^{1/\Lambda}$. Note that the Hamiltonian vector fields, X_H and $X_{H'}$, are proportional over $H = 0$, while those of X_H and $X_{|H|^{1/\Lambda}}$ are proportional over $H \neq 0$. \square

Before proceeding, some further comments are in order:

Remark 3. Case (i) in Theorem 1 is exactly the case considered in [37]. In many relevant physical examples the scaling function ρ can be so chosen so that it exists globally on M and is physically interpreted as the overall scale of the system, thus being an irrelevant (unmeasurable) degree of freedom for intrinsic observers. Therefore the system can be reduced by eliminating this overall scale and considering only the relational (shape) degrees of freedom (see also [27, 40]).

Case (ii) in Theorem 1 is actually somewhat well-known. Namely, there is always over $C \setminus \Sigma_0$ a contact 1-form and an embedding into M associated to the scaling function $\rho = |H|^{1/\Lambda}$ (the associated embedding being given by restricting to the energy surfaces $H = \pm 1$). Then $\eta = \lambda|_{H=\pm 1}$ and ℓ is defined by $i_\ell d\eta = 0$ (i.e. the dynamics may be parametrized as the Reeb flow of $\lambda|_{H=\pm 1}$), see [9, 20].

Remark 4. In practice, the reduction may be carried out by embedding C as a hypersurface $\Sigma := \{\rho = 1\}$ transverse to \mathbf{D} . Then $\eta = \lambda|_\Sigma$ and $\mathcal{H}_0 = -H|_\Sigma$. An equivalent way to state Theorem 1 (without cases), is using Definition 11 below. Then the Λ -Hamiltonian vector field of \mathcal{H}_0 spans ℓ on C (see Corollary 2 below).

By Theorem 1, one may treat the general reduction of the symplectic Hamiltonian system to a *contact system* locally as two subcases: with the degree one $|H|^{1/\Lambda}$ over $C \setminus \Sigma_0$, or $\rho^{1-\Lambda} H$ over Σ_0 , for some scaling function ρ . It is important to stress that for $\Lambda \neq 1$ none of these two cases can be extended as contact vector fields (whose flows preserve \mathcal{D}) to the other region. While this is obvious for $|H|^{1/\Lambda}$ over Σ_0 , it is not so immediate for $\rho^{1-\Lambda} H$ over $C \setminus \Sigma_0$, and this will be explained in more detail shortly (see Corollary 2). Clearly there is a very important special situation in which the two subcases coincide, i.e. when $\Lambda = 1$. In this situation, the two contact Hamiltonian flows in Theorem 1 coincide and we obtain the following reduction:

Corollary 1 (Contact reduction by scaling symmetries: degree one case). Let (M, ω, H) be a Hamiltonian system admitting a scaling symmetry \mathbf{D} of degree one. Then:

- i) X_H defines a vector field $X = \pi_* X_H$ on C ,
- ii) for a scaling function ρ , and $\pi^* \eta = \lambda/\rho$, X is the contact Hamiltonian vector field of $\pi^* \mathcal{H} = -H/\rho$.

By virtue of the above Corollary, in the case $\Lambda = 1$ one obtains a ‘full’ contact reduction, meaning that the reduced contact system is completely determined by the original symplectic one (this is precisely the content of Proposition 7, which we have recovered as a particular case). Note also that in this case H itself may be chosen as a scaling function giving the reduced dynamics as a Reeb flow of $\pi^* \eta = \lambda/H$ on $C \setminus \Sigma_0$.

In the general case, the line field we have been considering may in fact be described more explicitly with the use of a scaling function, as detailed in the following

Corollary 2 (Dependence on the scaling function). Let (M, ω, H) be a Hamiltonian system admitting a scaling symmetry \mathbf{D} of degree Λ and ρ a scaling function for \mathbf{D} . Then the line field ℓ on M/\mathbf{D} is spanned by

$$X := X_{\mathcal{H}_0} + (1 - \Lambda) \mathcal{H}_0 \mathcal{R}, \quad (6)$$

where $X_{\mathcal{H}_0}$ is the contact Hamiltonian vector field of $\pi^* \mathcal{H}_0 = -H/\rho^\Lambda$ with respect to $\pi^* \eta = \lambda/\rho$ and \mathcal{R} is the Reeb vector field of η on C . Moreover, for any other choice of scaling function, $\tilde{\rho}$, we have

$$\tilde{X} = \left(\frac{\tilde{\rho}}{\rho} \right)^{1-\Lambda} X.$$

Proof. Given a scaling function, ρ , then by Eq. (1), the vector field $\rho^{1-\Lambda} X_H$ is scale invariant and projects to a vector field,

$$X = \pi_* \rho^{1-\Lambda} X_H,$$

on C spanning ℓ . Set $H' := \rho^{1-\Lambda} H$, a degree one Hamiltonian on M with $\pi_* X_{H'} = X_{\mathcal{H}_0}$ the contact Hamiltonian vector field of $\pi^* \mathcal{H}_0 = -H/\rho^\Lambda$ with respect to $\pi^* \eta = \lambda/\rho$. Then

$$dH' = (1 - \Lambda) \rho^{-\Lambda} H \, d\rho + \rho^{1-\Lambda} dH$$

so that

$$X_{H'} = (1 - \Lambda) \rho^{-\Lambda} H \, X_\rho + \rho^{1-\Lambda} X_H. \quad (7)$$

Note that by Corollary 1, the Hamiltonian vector field X_ρ of ρ projects to the Reeb vector field $\mathcal{R} = \pi_* X_\rho$ of η on C . Applying π_* to the last equation we have Eq. (6).

For the last relation, note that for two scaling functions, ρ and $\tilde{\rho}$, we have that $\tilde{\rho}/\rho$ is scale invariant, so there is a function σ on C through $\pi^* \sigma = \tilde{\rho}/\rho$. Then

$$\tilde{X} = \pi_* \tilde{\rho}^{1-\Lambda} X_H = \sigma^{1-\Lambda} X.$$

□

Remark 5. In Darboux coordinates $\eta = dS - p_a dq^a$ on C , Eq. (6) reads

$$(q^a)' = \partial_{p_a} \mathcal{H}_0, \quad (p_a)' = -\partial_{q^a} \mathcal{H}_0 - p_a \partial_S \mathcal{H}_0, \quad S' = p_a \partial_{p_a} \mathcal{H}_0 - \Lambda \mathcal{H}_0. \quad (8)$$

These equations may also be derived directly from Hamilton’s equations for H by following Remark 4 or the comment after Proposition 7. Namely, given a scaling function ρ , we take $\Sigma = \{\rho = 1\}$. Then a system of contact coordinates, $\eta = dS - p_a dq^a$ on C , may be extended to symplectic coordinates $P_0 = \rho, Q_0 = S, P = -\rho p, Q = q$ on M with $\lambda = \rho \eta = P_0 dQ_0 + P \cdot dQ$. Then, we have that

$$H(Q_0, Q, P_0, P) = -P_0^\Lambda \mathcal{H}_0(Q_0, Q, -P/P_0)$$

since $H/\rho^\Lambda = -\mathcal{H}_0$. By chain rule, we may write Hamilton’s equations of motion as:

$$\rho^{1-\Lambda} \dot{S} = p_a \partial_{p_a} \mathcal{H}_0 - \Lambda \mathcal{H}_0, \quad \rho^{1-\Lambda} \dot{\rho} = \rho \partial_S \mathcal{H}_0, \quad (9)$$

$$\rho^{1-\Lambda} \dot{q}^a = \partial_{p_a} \mathcal{H}_0, \quad \rho^{1-\Lambda} \dot{p}_a = -\partial_{q^a} \mathcal{H}_0 - p_a \partial_S \mathcal{H}_0, \quad (10)$$

which project onto $\Sigma = \{\rho = 1\}$ as Eqs. (8). It is also evident from Eqs. (9), (10) of this last computation that, under the reparametrization $\rho^{1-\Lambda} d\tau = dt$, the scale-reduced equations of motion (8) contain a ‘blow up’, of Hamilton’s equations of motion to $\rho = 0$, as one would expect, compare with [35] for n -body problems.

Remark 6. The flow of the vector field X in Eq. (6), induced by a scaling function ρ , does not in general preserve \mathcal{D} . However restricted to the invariant set $\Sigma_0 = \{\mathcal{H}_0 = 0\}$ it does, being a contact Hamiltonian vector field (case (i) of Theorem 1). In general, one may rescale X away from this set so that its flow does preserve \mathcal{D} . Namely, for the scaling function $\tilde{\rho} = |H|^{1/\Lambda}$, we have $\tilde{\mathcal{H}} = -1$, and $\tilde{X} = \Lambda \tilde{\mathcal{H}}$ is proportional to the Reeb vector field of $\tilde{\eta}$ (case (ii) of Theorem 1) so that $|\mathcal{H}_0|^{1-1/\Lambda} X = \tilde{X}$ preserves \mathcal{D} .

Another interesting aspect about item (i) in Theorem 1 is the fact that it provides a direct connection with the physics of the problem. Indeed, the function ρ is usually connected with a global scale within the physical description (hence the ‘scaling function’ name), and therefore by using it in order to reduce the dynamics we automatically obtain a description of the same physical problem in terms of scale-invariant functions only.

While it may seem that the ‘ambiguity’ in Corollary 2 due to one’s choice of scaling function is a defect of the general contact reduction, we posit instead that this ‘freedom’ to choose a scaling function to describe the projected trajectories is in fact an asset of the theory, allowing one to highlight certain aspects of the original systems dynamics by various choices of scaling function. We illustrate the reduction procedure with the Kepler problem in the following example.

Example 3 (Contact-reduced Kepler). Consider the Kepler problem from Example 1. To illustrate the contact reduction process, one may first consider polar coordinates:

$$q = r e^{i\theta}, \quad p = \left(p_r + i \frac{p_\theta}{r} \right) e^{i\theta},$$

with scaling functions (Example 2) that we denote:

$$\rho := |q|^{1/2} = r^{1/2}, \quad J := p \cdot q = r p_r, \quad G := p \cdot i q = p_\theta.$$

In the coordinates (ρ, θ, J, G) on $M = T^*(\mathbb{C} \setminus 0)$, we have

$$\omega = \frac{2dJ \wedge d\rho}{\rho} + dG \wedge d\theta, \quad (11)$$

and

$$\mathbf{D}_K = \rho \partial_\rho + J \partial_J + G \partial_G, \quad \lambda = i_{\mathbf{D}_K} \omega = G d\theta - 2J d \log \frac{J}{\rho}, \quad H_K = \frac{J^2 + G^2}{2\rho^4} - \frac{1}{\rho^2}. \quad (12)$$

The functions J/ρ , G/ρ and θ are \mathbf{D}_K invariant, determining coordinates $(\tilde{J}, \tilde{G}, \theta)$ on the quotient M/\mathbf{D}_K by:

$$\pi^* \tilde{J} = J/\rho, \quad \pi^* \tilde{G} = G/\rho, \quad \pi^* \theta = \theta$$

where $\pi : M \rightarrow M/\mathbf{D}_K = C$. According to Theorem 1, the scale-reduced orbits may be described on C using:

$$\eta = \tilde{G} d\theta - 2d\tilde{J}, \quad \mathcal{H}_0 = 1 - \frac{\tilde{J}^2 + \tilde{G}^2}{2} \quad (13)$$

with $\pi^* \eta = \lambda/\rho$ and $\pi^* \mathcal{H}_0 = -\rho^2 H_K$. More explicitly, by Corollary 2 or Remark 5 with $\Lambda = -2$, the scale-reduced equations of motion may be written:

$$\tilde{J}' = \frac{\tilde{G}^2}{2} - \mathcal{H}_0, \quad \tilde{G}' = -\frac{\tilde{G}\tilde{J}}{2}, \quad \theta' = \tilde{G}. \quad (14)$$

Remark 7. The scaling symmetry, \mathbf{D}_K , of the Kepler problem and its corresponding scale-invariant functions, e.g. $\tilde{J}, \tilde{G}, \mathcal{H}_0$, have long been known and used in celestial mechanics, in particular their analogues for n -body problems. See for example Section 2.3 of Chenciner’s [11], where relations of the scale reduction to McGehee blow-up are explained. On the other hand, the contact structure associated to these contact reductions has been emphasized only more recently [37]. Observe that the contact reduction here is closely related to the structure of ‘ b -manifolds’ in [4]. In particular compare eq. (11) here to the Darboux normal form (Theorem 2 of [4]) of a ‘ b -symplectic form’ and Example 8.2 of [34] to our construction here of a scale-reduced contact form.

An important property of the reduction by Theorem 1 is its relation to certain blow-ups as mentioned at the end of Remark 5. We illustrate this relation with the scale-reduced Kepler problem.

Example 4 (Kepler blow-up). We continue from the previous example with the scale-reduced Kepler problem using the global scaling function $\rho = |q|^{1/2}$. According to Eqs. (9) and (10) with $\Lambda = -2$, the scale-reduced equations of motion on $\pi^* \mathcal{H}_0 = -\rho^2 H_K = 0$, represent a blow-up of the dynamics at $\rho = 0$ (collision).

One may apply Proposition 3 to determine the scale-reduced orbits on $\mathcal{H}_0 = 0$. From Eq. (13), we have:

$$\mathcal{D} = \ker \eta = \text{span}\{2\partial_\theta + \tilde{G}\partial_{\tilde{J}}, \partial_{\tilde{G}}\}$$

while $\Sigma_0 = \{\mathcal{H}_0 = 0\}$ is a torus, parametrized by the angle $\theta \in S^1$ and the circle $\tilde{J}^2 + \tilde{G}^2 = 2$. So:

$$T\Sigma_0 = \text{span}\{\partial_\theta, \tilde{G}\partial_{\tilde{J}} - \tilde{J}\partial_{\tilde{G}}\}.$$

The collision orbits of the Kepler problem are the homothetic motions. They tend in forwards or backwards time to the fixed points of eqs. (14), $\tilde{J} = \pm\sqrt{2}$, $\tilde{G} = 0$. Observe that \mathcal{D} is tangent to Σ_0 exactly at these fixed points. Away from these points, by Proposition 3, the projected Kepler orbits on Σ_0 are integral curves of:

$$\ell|_{\Sigma_0} = \mathcal{D} \cap T\Sigma_0 = \text{span}\{2\partial_\theta + \tilde{G}\partial_{\tilde{J}} - \tilde{J}\partial_{\tilde{G}}\}.$$

Taking the (\mathbf{D}_K -invariant) angle φ by $p = |p|e^{i\varphi}$, we have the line field:

$$\ell|_{\Sigma_0} = \text{span}\{\partial_\varphi + 2\partial_\theta\}.$$

The scale-reduced dynamics on this torus, Σ_0 , are a blow-up at collision $\rho = |q|^{1/2} = 0$ of the equations of motion for the Kepler problem. We recover the well-known ‘elastic bounce’ regularization and collision torus (see §1.3 of [16]), although our path here has been rather different.

Remark 8. In Appendix C we outline some more cases of contact reductions by scaling symmetries in celestial mechanics.

We close this section with an interesting comment about the relationship between Corollary 1 and the contact reduction in [41], which in turn gives yet another proof of Arnold’s Proposition 7 in terms of a contact version of symplectic reduction.

Remark 9. Corollary 1 may be alternatively deduced as a special case of the contact reduction in [41]. The relevant case is for an \mathbb{R} action on a contact manifold (N, η) by *exact* contactomorphisms: $L_X \eta = 0$, where $X \in \mathfrak{X}(N)$ generates the \mathbb{R} action and where \mathcal{H} is an \mathbb{R} -invariant contact Hamiltonian on N . Let $J_0 := (i_X \eta)^{-1}(0)$. Then $X, X_{\mathcal{H}}$ are tangent to J_0 and the quotient $P_0 := J_0/\mathbb{R}$ is a contact manifold with reduced contact dynamics defined through:

$$\begin{array}{ccc} & & N \\ & \nearrow \iota_0 & \\ J_0 & & \\ & \searrow \pi_0 & \\ & & P_0 \end{array}$$

$$\pi_{0*} X_{\mathcal{H}} = X_{\mathcal{H}} \quad \text{for} \quad \pi_0^* \bar{\mathcal{H}} = \iota_0^* \mathcal{H}, \quad \pi_0^* \bar{\eta} = \iota_0^* \eta.$$

Applying this construction, we obtain

Proposition 5. Consider a Hamiltonian system (M, ω, H) admitting \mathbf{D} as a degree one scaling symmetry. For $\lambda = i_{\mathbf{D}} \omega$ and a global scaling function $\rho : M \rightarrow \mathbb{R}^\times$, let $N := \mathbb{R} \times M$ with $\eta := e^{-t}(\lambda + d\rho) - dt$ where t is the coordinate on \mathbb{R} . Then

- (i) N is a contact manifold and $\tilde{\mathbf{D}} = \partial_t + \mathbf{D}$ generates an \mathbb{R} -action on N by exact contactomorphisms of η ,
- (ii) the contact Hamiltonian $e^{-t}H$ is invariant under the flow of $\tilde{\mathbf{D}}$.
- (iii) The contact manifolds $C = M/\mathbf{D}$ and $P_0 = J_0/\tilde{\mathbf{D}}$ are contactomorphic and moreover the reduced contact dynamics (of [41]) on P_0 corresponds to the contact Hamiltonian dynamics on C (of Theorem 1).

Proof. According to Remark 9, the corresponding moment map is $i_{\tilde{\mathbf{D}}} \eta = e^{-t} \rho - 1$ and so J_0 is the graph $\{e^t = \rho\} \subset N$, identified with M via $\iota : m \mapsto (\log \rho(m), m)$. Under this identification, we have $\iota_* \mathbf{D} = \partial_t + \mathbf{D} = \tilde{\mathbf{D}}|_{J_0}$ so that $C = M/\mathbf{D} \cong P_0 = J_0/\tilde{\mathbf{D}}|_{J_0}$. Moreover, $\iota^* \eta = \lambda/\rho$ and $\iota_*(e^{-t}H) = H/\rho$. Hence, the contact-reduced flow corresponds to the contact Hamiltonian flow of $\pi^* \mathcal{H} = H/\rho$ with respect to $\pi^* \eta = \lambda/\rho$. \square

III. CAN ONE ALWAYS FIND A CONTACT REDUCTION?

Given the above discussion on the general existence of a contact reduction ensuing from Theorem 1 and Corollary 1, we are now interested in the following fundamental question:

given a symplectic Hamiltonian system (M, ω, H) admitting a scaling symmetry \mathbf{D} of degree Λ , can one transform \mathbf{D} to $\tilde{\mathbf{D}}$ such that the new vector field is a scaling symmetry with $\tilde{\Lambda} = 1$?

This will be the content of this section. We will first prove that $\tilde{\mathbf{D}}$ always exists locally and find an explicit algorithm to construct it, and then we will use an example in order to make clear that such a symmetry cannot be found globally in general on M . This will lead us to consider an extension of (M, ω, H) to a ‘lifted system’, where the parameters are considered as dynamical variables (together with their conjugate momenta). Rather surprisingly, in this space we can prove the existence of a scaling symmetry of degree one, and therefore obtain its corresponding contact reduction, even for cases in which the original Hamiltonian system on M had no (evident) scaling symmetry at all.

A. Locally on M : yes

We begin with the following observation: for any Hamiltonian system in a neighborhood of m with $X_H(m) \neq 0$, there exist Darboux coordinates such that $H = p_1$ and $\omega = dp_a \wedge dq^a$. Then $\tilde{\mathbf{D}} = p_a \partial_{p_a}$ is a degree one scaling symmetry with $\lambda = p_a dq^a = i_{\tilde{\mathbf{D}}} \omega$. We conclude that locally, away from critical points of H , it is always possible to find $\tilde{\mathbf{D}}$.

This observation may be described in the following algorithm to find $\tilde{\mathbf{D}}$ locally:

1. fix *some* primitive, λ , of ω and seek a closed 1-form α such that

$$i_{X_H}(\lambda - \alpha) = H. \quad (15)$$

2. By the discussion in Section II A (see e.g. Proposition 1), the vector field $\tilde{\mathbf{D}}$ corresponding to $\lambda - \alpha$ via

$$i_{\tilde{\mathbf{D}}} \omega = \lambda - \alpha \quad (16)$$

is the generator of a degree one scaling symmetry.

We conclude that (15) is the equation to be solved (for α) in order to find $\tilde{\mathbf{D}}$.

Notice that, locally at least, $\alpha = dS$ for some function $S : M \rightarrow \mathbb{R}$ and then by (16) its associated Hamiltonian vector field X_S is such that $\tilde{\mathbf{D}} = X + X_S$, with X the Liouville vector field satisfying $i_X \omega = \lambda$. Locally therefore, we can make (15) even more explicit. Indeed, we obtain

$$X_H S = i_{X_H} \lambda - H. \quad (17)$$

In particular, in local Darboux coordinates with $\lambda = p_a dq^a$, this gives

$$\dot{S} = p_a \dot{q}^a - H = L, \quad (18)$$

and therefore S is the action. Remarkably, it is always possible to solve (18) locally away from critical points of H by the choice of a transverse slice Σ to X_H identifying $M \cong \Sigma \times (-\varepsilon, \varepsilon) \ni (\mathbf{s}, t)$ and taking $S(\mathbf{s}, t) := \int_0^t L(\varphi_\tau(\mathbf{s})) d\tau$. However, it is clear that globally there may be obstructions to solving (17) – resp. (15) – in general (see e.g. the Kepler case in Section III B). Therefore an alternative route should be put forward, as we will do in Section III C.

In order to illustrate the above procedure, we consider now a very simple case in which the scaling symmetry $\tilde{\mathbf{D}}$ can be found explicitly:

Example 5 (The 2d harmonic oscillator). We have $H = \frac{|p|^2 + k|q|^2}{2}$, $\lambda = p \cdot dq$ and $\omega = d\lambda$. Imposing $i_X \omega = \lambda$ we get $X = p \cdot \partial_p$ which is by definition a Liouville vector field but not a scaling symmetry of H .

Now we seek a function S such that $\tilde{\mathbf{D}} = X + X_S$ is a scaling symmetry with $\Lambda = 1$. Solving (17) we get

$$S = \frac{p \cdot q}{2} + cp \cdot iq \quad (19)$$

where c is a constant of integration that we will fix to 0 (this freedom in fixing $\tilde{\mathbf{D}}$ comes from the fact that $p \cdot iq$, the angular momentum, is an integral of motion, cf. the discussion in Remark 2). This degree one scaling symmetry with $c = 0$ is just $\tilde{\mathbf{D}} = X + X_S = \frac{p \cdot \partial_p + q \cdot \partial_q}{2}$.

Note that in the harmonic oscillator example S (and thus $\tilde{\mathbf{D}}$) is globally defined on M . In the next section we will see an example in which a global solution to eq. (15) on M cannot exist.

B. Globally on M : no

Let us consider the Kepler problem of Example 1. This admits the ‘standard scaling symmetry’ $\mathbf{D}_K = 2q \cdot \partial_q - p \cdot \partial_p$, which has degree -2 . May we apply our algorithm to find a different ‘hidden scaling symmetry’ of degree 1?

By the discussion in the previous section, this question is equivalent to finding a primitive λ' of ω_K with $i_{X_{H_K}} \lambda' = H_K$. The degree one scaling symmetry is then defined by $i_{\mathbf{D}} \omega_K = \lambda'$. So, let $\lambda = p_a dq^a$ be the canonical 1-form, then all of our options for primitives are given by $\lambda - \alpha$ with α a closed 1-form.

It is instructive to consider first the case in which $\alpha = dS$ is exact. Then we look for a solution of (17). As an immediate observation, we see that for the Kepler problem a solution S fails to exist globally because there are periodic orbits with positive action. Namely, for an elliptic orbit, $\gamma \subset T^*(\mathbb{R}^2 \setminus \{0\})$, with period T and (negative) energy E , a solution S to (17) would satisfy:

$$0 \stackrel{!}{=} \oint_{\gamma} dS = S(\mathbf{s}, T) - S(\mathbf{s}, 0) = \int_0^T L(\varphi_{\tau}(\mathbf{s})) d\tau = 3\pi \left(\frac{T}{2\pi} \right)^{1/3} = 3\pi \sqrt{\frac{-1}{2E}} > 0. \quad (20)$$

In general, for any closed α as above, consider two periodic elliptic trajectories $\gamma_1, \gamma_2 \subset T^*(\mathbb{R}^2 \setminus \{0\})$ with different values of (negative) energies, $E_1 \neq E_2$ resp. Such trajectories bound some surface, $\Sigma \subset T^*(\mathbb{R}^2 \setminus \{0\})$, in the phase space. Then, since α is closed, $d\alpha = 0$, we have by Stokes theorem:

$$\oint_{\gamma_1} \alpha = \oint_{\gamma_2} \alpha \quad (21)$$

As we have fixed $\lambda = p_a dq^a$ as the canonical one-form, eq. (15) reads

$$i_{X_H} \alpha = p_a \frac{\partial H}{\partial p_a} - H = L. \quad (22)$$

Therefore, since γ_1, γ_2 are integral curves of X_H , (21) now reads

$$\int_0^{T_1} L(\gamma_1(t)) dt = \int_0^{T_2} L(\gamma_2(t)) dt \quad (23)$$

where $T_1 \neq T_2$ are the periods of γ_1, γ_2 . This is impossible, because for the Kepler problem, the total action around an elliptic orbit is a non-constant function of its energy (and period). We conclude that such a α cannot exist in the Kepler case.

In Example 5 for the 2d harmonic oscillator we have used precisely this route to find $\alpha = dS$. Note that in that case we have that the condition (23) is identically satisfied (both sides always being zero).

C. Globally on \hat{M} : yes. The evolution of couplings

Now let us consider a symplectic Hamiltonian system, (M, ω, H_a) , with the Hamiltonian depending on some parameters $a \in \mathbb{R}^k$ (e.g. the parameters might be masses, gravitational constants, etc.). We will define its extension to a ‘lifted system’, where the parameters are considered as dynamical variables (together with their conjugated momenta) and show that for such system a global (degree one) contact reduction always exists.

Definition 4. Given a (parametrized) symplectic Hamiltonian system (M, ω, H_a) , the associated *lifted system* is the symplectic Hamiltonian system $(\hat{M}, \hat{\omega}, \hat{H})$ with

$$\hat{M} := M \times \mathbb{R}^{2k} \ni (m, a, b), \quad \hat{\omega} := \omega + da_i \wedge db^i, \quad \text{and} \quad \hat{H} : \hat{M} \rightarrow \mathbb{R}, (m, a, b) \mapsto H_a(m).$$

Clearly, as a consequence of the definition, we have

$$X_{\hat{H}} = X_{H_a} + \partial_a H_a \cdot \partial_b \quad (24)$$

from which we see that $\dot{a} = 0$ (the parameters a are first integrals of the lifted system) and thus, after a choice of a has been made once and for all, the lifted dynamics always projects down to the original one.

In most cases, one may simply redefine the parameters so that H_a is degree one in a , which includes for example the lift of *any* Hamiltonian system by taking $H_a := aH$. Similarly one may consider a system of *coupled* Hamiltonians:

$$H_a = a_1 H_1 + \dots + a_k H_k$$

with parameters $a_j \in \mathbb{R}$ the *couplings*.

With the above definitions we have the following important result:

Proposition 6 (Lifted degree one scaling symmetries). Consider a Hamiltonian system, (M, ω, H_a) depending on parameters, $a \in \mathbb{R}^k$, of the form:

$$H_a = a_1 H_1 + \dots + a_k H_k$$

with $H_j : M \rightarrow \mathbb{R}$, and admitting a Liouville vector field \mathbf{D} such that $L_{\mathbf{D}} H_j = \Lambda_j H_j$. Then

$$\hat{\mathbf{D}} := \mathbf{D} + \sum_{i=1}^k (1 - \Lambda_i) a_i \partial_{a_i} + \sum_{i=1}^k \Lambda_i b^i \partial_{b^i}$$

is a degree one scaling symmetry for the associated lifted system, $(\hat{M}, \hat{\omega}, \hat{H})$, that is: $L_{\hat{\mathbf{D}}} \hat{\omega} = \hat{\omega}$ and $L_{\hat{\mathbf{D}}} \hat{H} = \hat{H}$.

Proof. By a direct calculation we have that $L_{\hat{\mathbf{D}}} \hat{H} = \hat{H}$ and $i_{\hat{\mathbf{D}}} \hat{\omega} = \lambda + \sum_{i=1}^k a_i db^i - d\left(\sum_{i=1}^k \Lambda_i a_i b^i\right)$, where $\lambda = i_{\mathbf{D}} \omega$, and therefore $\hat{\mathbf{D}}$ is a scaling symmetry of degree one. \square

Remark 10. Note that the same construction above, upon replacing a_i with $a_i^{1-\Lambda_i}$ in the definition of the Hamiltonian H_a , results in a degree one scaling symmetry

$$\hat{\mathbf{D}} = \mathbf{D} + \sum_{i=1}^k a_i \partial_{a_i}.$$

This is often more useful in directly finding the scale-reduced contact Hamiltonian. Furthermore, when the terms $1 - \Lambda_i$ are not integers, one may avoid complex values by considering various cases of sign choices in the sum

$$H_a = |a_1|^{1-\Lambda_1} H_1 \pm \dots \pm |a_k|^{1-\Lambda_k} H_k. \quad (25)$$

According to Section II (Corollary 1), we may project the Hamiltonian dynamics of \hat{H} onto a contact Hamiltonian mechanics of the vector field \hat{X} on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$. Our main result here describes how the dynamics of each H_a is related to this reduced space.

Let us observe first that the scaling symmetry $\hat{\mathbf{D}}$, and the Hamiltonian vector field, $X_{\hat{H}}$, of \hat{H} on \hat{M} under

$$\pi_o : \hat{M} \rightarrow M \times \mathbb{R}^k \ni (m, a)$$

project to vector fields $\bar{\mathbf{D}} = \pi_{o*} \hat{\mathbf{D}}$ and $X = \pi_{o*} X_{\hat{H}}$ on this parameter space. Moreover,

$$X(m, a) = X_{H_a}(m)$$

is $\bar{\mathbf{D}}$ invariant, ie $[X, \bar{\mathbf{D}}] = 0$, so projects to a vector field \bar{X} on the quotient, $\bar{M} := (M \times \mathbb{R}^k)/\bar{\mathbf{D}}$.

Remark 11. The symmetry of $\bar{\mathbf{D}}$ on X is easily understood. Under the assumptions of Proposition 6, if ψ_s is the flow of \mathbf{D} on M then for $\gamma_a(t) \in M$ a trajectory of X_{H_a} , we have that $\psi_s(\gamma_a(t))$ is a trajectory of $H_{a'}$ where $a'_j = e^{(1-\Lambda_j)s} a_j$.

From this last remark, the flow of \bar{X} on \bar{M} represents the trajectories of H_a on M modulo equivalence in parameter values and the scaling symmetry $\bar{\mathbf{D}}$ on \bar{M} . It is easy to see that these trajectories are all obtained by a projection of the contact flow of \hat{X} on \hat{C} . Let us denote by

$$\bar{\pi}_o : \hat{C} \rightarrow \bar{M}$$

the induced projection. Then, with the diagram

$$\begin{array}{ccc} \hat{M} & \xrightarrow{\pi_o} & M \times \mathbb{R}^k \\ \downarrow \hat{\pi} & & \downarrow \pi \\ \hat{C} & \xrightarrow{\bar{\pi}_o} & \bar{M} \end{array}$$

$$\pi_{o*} X_{\hat{H}} = X, \quad \pi_{o*} \hat{\mathbf{D}} = \bar{\mathbf{D}}, \quad \hat{C} = \hat{M}/\hat{\mathbf{D}}, \quad \bar{M} = (M \times \mathbb{R}^k)/\bar{\mathbf{D}},$$

we have:

Theorem 2. Consider a Hamiltonian system depending on parameters as in Proposition 6. Let $\hat{X} := \hat{\pi}_* X_{\hat{H}}$ be the induced contact Hamiltonian vector field on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$. Then, for each fixed value a_o of the parameters, there is an \hat{X} -invariant set $\hat{C}_{a_o} \subset \hat{C}$, with:

$$\bar{M}_{a_o} := \bar{\pi}_o(\hat{C}_{a_o})$$

an \bar{X} -invariant set on \bar{M} , and moreover:

$$\bar{X}|_{\bar{M}_{a_o}} = \bar{\pi}_{o*} \hat{X}.$$

Proof. Take $\hat{C}_{a_o} = \hat{\pi}(M \times \{a_o\} \times \mathbb{R}^k)$. □

Remark 12. As a special case of the above Proposition 6 and Theorem 2, we may have all $\Lambda_j = \Lambda$, so that \mathbf{D} is a scaling symmetry, of degree Λ , for each H_a on M . In this case, there is a further projection from $\bar{\pi}_1 : \hat{C} \rightarrow C = M/\mathbf{D}$, induced by $\pi_1 : \hat{M} \rightarrow M$ so that the line fields ℓ_a on C induced by H_a are spanned by $\bar{\pi}_{1*} \text{span} \hat{X}|_{\hat{C}_{a_o}}$.

Remark 13. Some important observations here are in order:

The first point is the fact that the assumption on H_a in Proposition 6 is not restrictive at all. Indeed, by introducing such parameters, basically any physical Hamiltonian can be written as a sum of pieces that scale differently. This allows one to apply contact reductions to a much more general class of Hamiltonian systems than those which admit a scaling symmetry of some degree Λ , as the examples below shall show.

The second crucial point (which might have interesting consequences on the way we formulate physical theories) is the fact that while on the lifted symplectic system \hat{H} the dynamical variables a are constant, on the corresponding reduced contact system they become *dissipated quantities*, i.e. they dissipate at the same rate as the contact Hamiltonian \mathcal{H} , namely $L_{X_{\mathcal{H}}} a = -\mathcal{R}(\mathcal{H})a$ (see [6, 15, 18] to see that this is the equivalent of conserved quantities for the contact case). However, the global equivalence of the two dynamics is well established by the discussion in Section II. Therefore, we are ready to remark the significance of Proposition 6 and Theorem 2: as anticipated in the discussion on Poincaré's dream in the Introduction, since the parameters in a Hamiltonian description of the physical reality are *inferred* from measurements based on the observed trajectories and on the specified model, there is no reason a priori to think of them as fixed numbers. Indeed we see from Theorem 2 that if we allow the 'space of physical Hamiltonian (variational) theories' describing a given phenomenon to vary, meaning that the coupling constants of each theory are different, then what we have just proved means that there always exists in this space a way to reduce the description to an equivalent theory based on a contact Hamiltonian (or Herglotz variational) theory on a reduced space. Despite the dynamical equivalence among all such theories, the contact Hamiltonian one, involving less elements for its complete description, has more descriptive power, as pointed out already in [27, 40]. We call this a *reduction to a more descriptive theory*. We note that the universal dissipative nature of the reduced system provides support for the position that open systems are fundamental to physics [13].

A third point is that it is simple to extend Proposition 6 to construct a scaling symmetry of degree Λ by taking

$$\hat{\mathbf{D}} := \mathbf{D} + \sum_{i=1}^k (\Lambda - \Lambda_i) a_i \partial_{a_i} + \sum_{i=1}^k (1 - \Lambda + \Lambda_i) b^i \partial_{b^i}.$$

This construction may be of particular interest when investigating blow-ups of a particular degree; by choosing Λ such that the rescaling can tame the component of H which becomes infinite, we can then investigate the other components to see if a regularization can be achieved.

Now we consider some simple examples that illustrate the generality of Proposition 6 and Theorem 2.

Example 6 (Global reduction of Kepler). Let us consider again the Kepler example (see Examples 1 and 3). This time we reinsert the parameters m (mass) and k (Kepler coupling) that we had previously fixed to 1 to obtain

$$H_{m,k} = \underbrace{\frac{1}{m} \left(\frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} \right)}_{H_1} - \underbrace{\frac{k}{r}}_{H_2}.$$

Clearly, for $\mathbf{D} = \mathbf{D}_K$, we have

$$L_{\mathbf{D}} H_1 = -2H_1 \quad \text{and} \quad L_{\mathbf{D}} H_2 = -2H_2,$$

and therefore we can apply the procedure detailed in Proposition 6 to find the degree one scaling symmetry

$$\hat{\mathbf{D}} = \mathbf{D}_K + 3(\mu\partial_\mu + k\partial_k) - 2(b^\mu\partial_{b^\mu} + b^k\partial_{b^k}),$$

where $\mu := m^{-1}$ and, by Corollary 1, the scale-reduced contact Hamiltonian dynamics on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$. We stress that the final dynamics on \hat{C} carries no information about the scale used for the reduction, and therefore needs one less parameter with respect to the original Kepler formulation. Specifically, while the original problem needed $(r_0, \theta_0, p_r^0, p_\theta^0, m, k)$ as an input to completely specify a solution, the reduced problem only needs e.g. $(\theta_0, p_r^0, p_\theta^0, m_0, k_0)$ (supposing we are using r as the scaling function for the reduction).

Contrary to the Kepler example, in the next case we present a system that does not possess any evident scaling symmetry of degree Λ , but to which we can still apply Proposition 6.

Example 7 (Combined Kepler-Hooke System). With M and ω as before, consider now a Hamiltonian which combines the Kepler ($1/r$) and the Hooke (r^2) potential:

$$H_{KH} := \underbrace{\mu \left(\frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} \right)}_{H_1} - \underbrace{\frac{k_K}{r}}_{H_2} + \underbrace{k_H r^2}_{H_3}.$$

Clearly in this case we do not have a (evident) scaling symmetry of degree Λ . However, we may notice that for $\mathbf{D} = -p_r\partial_{p_r} + 2r\partial_r + p_\theta\partial_{p_\theta}$, we get

$$L_{\mathbf{D}}H_1 = -2H_1 \quad L_{\mathbf{D}}H_2 = -2H_2 \quad \text{and} \quad L_{\mathbf{D}}H_3 = 4H_3. \quad (26)$$

Therefore, by applying Proposition 6 we find the degree one scaling symmetry

$$\hat{\mathbf{D}} = D + 3\mu\partial_\mu + 3k_K\partial_{k_K} - 3k_H\partial_{k_H} - 2b^\mu\partial_{b^\mu} - 2b^{k_K}\partial_{b^{k_K}} + 4b^{k_H}\partial_{b^{k_H}} \quad (27)$$

and, by Corollary 1, the scale-reduced contact Hamiltonian dynamics on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$.

Indeed, we can generalize the last example at once to the case of any potential that can be written as a Laurent series of r as follows.

Example 8 (General potential as a Laurent series of r). Consider now the Laurent Hamiltonian

$$H_L := \underbrace{\mu \left(\frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} \right)}_{H_{\text{kin}}} + \underbrace{\sum_{j=-n_1}^{n_2} a_j r^j}_{\sum_{j=-n_1}^{n_2} a_j H_j}.$$

Clearly in this case we do not have a (evident) scaling symmetry of degree Λ . However, we may notice that for $\mathbf{D} = -p_r\partial_{p_r} + 2r\partial_r + p_\theta\partial_{p_\theta}$, we get

$$L_{\mathbf{D}}H_{\text{kin}} = -2H_{\text{kin}} \quad \text{and} \quad L_{\mathbf{D}}H_j = 2jH_j. \quad (28)$$

Therefore, by applying Proposition 6 we find the degree one scaling symmetry

$$\hat{\mathbf{D}} = D + 3\mu\partial_\mu + \sum_{j=-n_1}^{n_2} (1-2j)a_j\partial_{a_j} - 2b^\mu\partial_{b^\mu} + \sum_{j=-n_1}^{n_2} 2jb^j\partial_{b^j} \quad (29)$$

and, by Corollary 1, the scale-reduced contact Hamiltonian dynamics on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$.

The next example is somewhat simpler, but it has important physical ramifications in cosmology, and therefore it is worth presenting it at this point.

Example 9 (FLRW Cosmology). In the field of cosmology, homogeneous, isotropic solutions to Einstein's equations are given by the Friedmann-Lemaître-Robertson-Walker metrics. The symmetries of such space-times allow us to reduce a full field theory to a finite-dimensional 'particle' representation. In these solutions, the space-time manifold is $\mathbb{R} \times \Sigma_k$, in which Σ_k is the three-dimensional spatial slice, consisting of a three sphere ($S^3, k = 1$), Euclidean space ($\mathbb{R}^3, k = 0$) or hyperbolic space ($\mathbb{H}, k = -1$). The metric is determined by the topology of the spatial section:

$$ds^2 = -dt^2 + v^{\frac{2}{3}}(t) \left(\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right).$$

In the case of a non-compact topology, the volume, v is measured with respect to some fixed, fiducial cell. For minimally coupled matter, the dynamics of which are given in flat space by a Hamiltonian $H_m(p, q)$, we find the equations of motion for the gravitational sector, v, Π , from the Hamiltonian:

$$H = v \left(-\frac{3\Pi^2}{8\pi} + H_m\left(\frac{p}{v}, q\right) - \frac{k}{v^{\frac{2}{3}}} \right).$$

For flat ($k = 0$) spatial slices the Hamiltonian admits a degree one scaling symmetry $\mathbf{D} = v\partial_v + p\partial_p$ (summed over all matter momenta p if multiple are present), with $i_{\mathbf{D}}\omega = v(-d\Pi + pdq)$. Therefore in this case the reduction to the contact dynamics on $C = M/\mathbf{D}$ is immediate, and following Corollary 1 with the scaling function $\rho = v$, we have

$$\mathcal{H}_0 = \frac{3\Pi^2}{8\pi} - H_m(p, q) \quad \text{and} \quad \eta = -d\Pi + pdq.$$

To extend our results to the case in which $k \neq 0$, we follow the process outlined in Proposition 6 and Theorem 2. Indeed, the Hamiltonian in this case can be split as

$$H = \underbrace{v \left(-\frac{3\Pi^2}{8\pi} + H_m\left(\frac{p}{v}, q\right) \right)}_{a_1 H_1} \underbrace{- \frac{k v^{1/3}}{v^{\frac{2}{3}}}}_{a_2 H_2},$$

and for $\mathbf{D} = v\partial_v + p\partial_p$ as before we obtain

$$L_{\mathbf{D}}H_1 = H_1 \quad \text{and} \quad L_{\mathbf{D}}H_2 = \frac{1}{3}H_2,$$

which, by Proposition 6, leads to the degree one scaling symmetry

$$\hat{\mathbf{D}} = \mathbf{D} + \frac{2}{3}k\partial_k + b^v\partial_{b^v} + \frac{1}{3}b^k\partial_{b^k},$$

for the lifted system and, by Corollary 1, to a scale-reduced contact Hamiltonian dynamics on $\hat{C} = \hat{M}/\hat{\mathbf{D}}$.

Remark 14. There are two interesting facets of cosmological dynamics that are highlighted by this construction. The first is that the apparent ‘Hubble Friction’ arising due to the expansion of the universe can be directly observed since the equation of motion for our system are given:

$$\dot{q} = \partial_p H_m, \quad \dot{p} = -\partial_q H_m + \frac{3\Pi}{4\pi}p, \quad \dot{\Pi} = p\partial_p H_m - H_m + \frac{3\Pi^2}{8\pi},$$

with the effect of coupling H_m to gravity in the second term acting as apparent friction when compared to the behaviour of matter in a non-expanding background. The second is that both the dynamics of matter in an expanding space, and the expansion of space that results from the presence of matter are described by taking a symplectic matter system $(M, \omega = d\theta, H_m)$ and considering on its contactification $(\hat{M}, \hat{\theta}) = (M \times \mathbb{R}, \theta - d\Pi)$ the matter-gravity Hamiltonian $\mathcal{H} = \frac{3\Pi^2}{8\pi} - H_m$.

IV. CONCLUSIONS

Let us recapitulate the main points of our work. We have shown that a general class of scaling symmetries exist within symplectic systems, and how this symmetry can be used to reduce the system onto its invariants. In Theorem 1 and its Corollary 1 we have shown that the reduction by identifying orbits of the scaling vector field on the symplectic manifold yields a contact manifold, that the Hamiltonian vector field on the symplectic manifold projects onto a (contact) Hamiltonian vector field on the contact manifold, and that the contact Hamiltonian and contact form that generate this vector field are obtained by a simple procedure. We refer to this procedure as a *contact reduction by scaling symmetries*, or simply *contact reduction*. This process has been further generalized beyond simple systems admitting a scaling symmetry. Indeed, by lifting systems with multiple couplings we find that there is a general process under which these also admit a contact reduction to an equivalent scale-invariant description (Proposition 6 and Theorem 2), thus realizing Poincaré’s dream.

The resultant scale-reduced systems, being contact systems, have interesting features not apparent in the unreduced symplectic manifolds. The presence of a Reeb vector leads to measure focusing and dissipation of otherwise conserved quantities. Such features have important roles in describing physical phenomena such as the arrow of time [27].

An interesting application of the contact reduction is to examine the nature of ‘blow-ups’ (see Appendix C), for example, when the scale parameter in the symplectic system approaches zero, such as at the total collisions of the three-body problem. Many of the problems regarding the continuation of solutions beyond such points arise due to the ill-defined evolution of the scale parameter. Such a situation can be avoided in the reduced system as the evolution of the scale is immaterial to the behaviour of the contact system. One can envision situations in which the contact Hamiltonian vector field will remain well-defined on the contact manifold when the symplectic Hamiltonian vector field becomes ill defined, and thus a ‘natural’ continuation of the symplectic system may be considered as passing to the contact manifold for evolution beyond this point, and symplectifying the resultant evolution. In cosmological systems [28, 31, 38] this has been shown to allow continuation beyond the initial singularity, and in black holes beyond the central singularity [33].

Appendix A: Symplectic and contact Hamiltonian mechanics

In this appendix we review briefly some of the main definitions and properties of Hamiltonian mechanics on symplectic and contact manifolds, paying special attention to the relationships between the two that will be used in this work in order to have it self-contained. Most of the definitions and results presented here can be found in [1] and [20], to which we refer for more detailed discussions.

1. Symplectic and contact: definitions

We begin with some general definitions that establish the notation and sign conventions.

Definition 5. A *symplectic manifold* is a pair (M, ω) , where M is a $2n$ -dimensional manifold and ω is a 2-form on M which is closed ($d\omega = 0$) and non-degenerate ($\omega^n \neq 0$). A *symplectic vector field* on M is one whose flow preserves ω : $L_X \omega = 0$.

Dynamics on a symplectic manifold is usually given by Hamiltonian systems, which are defined as follows.

Definition 6. A (symplectic) *Hamiltonian system* is a triple (M, ω, H) , where (M, ω) is a symplectic manifold and $H : M \rightarrow \mathbb{R}$ is a differentiable function called the (symplectic) *Hamiltonian*. Given this structure, a unique symplectic vector field X_H , the (symplectic) *Hamiltonian vector field*, is defined on M by the condition

$$i_{X_H} \omega = -dH. \quad (\text{A1})$$

By a theorem of Darboux, locally we can always find coordinates (q^a, p_a) , $a = 1, \dots, n$, on a symplectic manifold such that

$$\omega = dp_a \wedge dq^a. \quad (\text{A2})$$

These coordinates are called *canonical* or *Darboux* coordinates. It follows directly from (A1) and (A2) that the *trajectories* of X_H – i.e. parametrized curves $\gamma : I \subset \mathbb{R} \rightarrow M$ satisfying $\dot{\gamma}(t) = X_H(\gamma(t))$ – are solutions to the *symplectic Hamiltonian equations*

$$\dot{q}^a = \frac{\partial H}{\partial p_a} \quad \text{and} \quad \dot{p}_a = -\frac{\partial H}{\partial q^a}. \quad (\text{A3})$$

On the contact side, we have the corresponding definitions

Definition 7. A *contact manifold* is a pair (C, \mathcal{D}) , where C is a $(2n + 1)$ -dimensional manifold and \mathcal{D} is a maximally non-integrable distribution of hyperplanes on C . Locally at least, there always exists a 1-form η on C such that $\mathcal{D} = \ker \eta$ and the maximal non-integrability condition is expressed as $\eta \wedge (d\eta)^n \neq 0$. Such η is called a *contact 1-form*. A *contact vector field* on C is one whose flow preserves \mathcal{D} : $L_X \eta \sim \eta$.

Note that, a contact 1-form η exists globally if and only if \mathcal{D} is co-orientable, meaning that the quotient line bundle TC/\mathcal{D} is trivial [20]. This will always be the case in this work. Furthermore, once we fix a contact 1-form η for \mathcal{D} , then any other 1-form of the type $\eta' = f\eta$ with $f : C \rightarrow \mathbb{R}^\times$ yields a different contact form for the same \mathcal{D} . That is, there is a freedom in choosing contact forms for the same \mathcal{D} , up to re-scaling by a nowhere-vanishing function. However, to define certain dynamics, the choice of η is relevant, as the next definition will show.

We recall that on a contact manifold with a fixed η there is a special contact vector field, the *Reeb vector field* \mathcal{R} , defined by

$$i_{\mathcal{R}} d\eta = 0 \quad \text{and} \quad i_{\mathcal{R}} \eta = 1. \quad (\text{A4})$$

Now we are ready to introduce the Hamiltonian dynamics on contact manifolds (see as well Proposition 7 and Remark 16 below for some context).

Definition 8. A (contact) *Hamiltonian system* is a triple (C, η, \mathcal{H}) , where (C, η) is a contact manifold with a fixed choice for the contact 1-form and $\mathcal{H} : C \rightarrow \mathbb{R}$ is a differentiable function called the (contact) *Hamiltonian*. Given this structure, a unique contact vector field $X_{\mathcal{H}}$, the *contact Hamiltonian vector field*, is defined on C , by the conditions

$$i_{X_{\mathcal{H}}} d\eta = d\mathcal{H} - \mathcal{R}(\mathcal{H})\eta \quad \text{and} \quad i_{X_{\mathcal{H}}} \eta = -\mathcal{H}. \quad (\text{A5})$$

By a theorem of Darboux, locally we can always find coordinates (q^a, p_a, S) on a contact manifold such that

$$\eta = dS - p_a dq^a, \quad \text{and} \quad \mathcal{R} = \frac{\partial}{\partial S}. \quad (\text{A6})$$

These coordinates are called *contact* or *Darboux* coordinates. It follows directly from (A5) and (A6) that the trajectories of $X_{\mathcal{H}}$ are solutions to the *contact Hamiltonian equations*

$$\dot{q}^a = \frac{\partial \mathcal{H}}{\partial p_a} \quad \dot{p}_a = -\frac{\partial \mathcal{H}}{\partial q^a} - p_a \frac{\partial \mathcal{H}}{\partial S} \quad \dot{S} = p_a \frac{\partial \mathcal{H}}{\partial p_a} - \mathcal{H}. \quad (\text{A7})$$

It should be stressed at this point that the dynamical vector fields X_H and $X_{\mathcal{H}}$ can be given variational definitions, provided the corresponding Lagrangians are (hyper-)regular, both in the symplectic and in the contact case. While in the symplectic case Hamilton's variational principle is well-known, its contact counterpart, known as Herglotz' variational principle, has been far less studied until recently (see [14, 21, 44]). In Appendix B we detail this correspondence and we re-write the main results of this work from the variational perspective.

2. Symplectic and contact: known relationships and new definitions

Now we review some known relationships between symplectic and contact manifolds, together with the respective Hamiltonian dynamics.

We start with a procedure by which there always exists a natural extension of a contact manifold to a symplectic one (see [1], Appendix 4).

Definition 9. Given a contact manifold, (C, \mathcal{D}) , the *symplectification*, \tilde{C} , of C is:

$$\tilde{C} := \{\beta_x \in T_x^*C : \ker \beta_x = \mathcal{D}_x\} \subset T^*C. \quad (\text{A8})$$

Note that \tilde{C} is an \mathbb{R}^\times principal bundle over C with (exact) symplectic structure $d\tilde{\alpha}$ defined by restriction of the canonical symplectic form on T^*C :

$$\tilde{\alpha}_{\tilde{x}}(\xi) := \tilde{x}((\pi_C)_*\xi), \quad (\text{A9})$$

where $\pi_C : \tilde{C} \rightarrow C$ is the projection (see also [10]).

Remark 15. The symplectification of C consists of the non-zero elements, $\text{Ann}(\mathcal{D}) \setminus C$, of the annihilator of \mathcal{D} : $\text{Ann}(\mathcal{D}) := \{\beta : \beta|_{\mathcal{D}} \equiv 0\}$ and so is naturally identified with the non-zero elements of $(TC/\mathcal{D})^*$.

When the contact distribution is co-orientable (TC/\mathcal{D} is a trivial line bundle), then so too is the symplectification a trivial principal bundle: $\tilde{C} \cong C \times \mathbb{R}^\times$, trivialized by a global choice of contact 1-form η . In this case \tilde{C} has two connected components, \tilde{C}_\pm and, when a contact 1-form η has been chosen, it is convenient to refer to the component $\tilde{C}_+ := \{e^s \eta\}$ consisting of positive multiples of η as the symplectification.

To understand further the relationship between \tilde{C} and C , we introduce the following definition.

Definition 10. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold, where a choice for the symplectic potential λ has been made. The vector field $\mathbf{X} \in \mathfrak{X}(M)$ such that

$$i_{\mathbf{X}}\omega = \lambda \quad (\text{A10})$$

is called the *Liouville vector field* of λ .

Then, for $(\tilde{C}, \tilde{\alpha})$ the symplectification of (C, η) , the generator, \mathbf{X} , of the \mathbb{R}^\times action on \tilde{C} is the Liouville vector field of $\tilde{\alpha}$. It turns out that there is a very special correspondence between contact vector fields on C and certain Hamiltonian vector fields on \tilde{C} . To see this, observe that symplectic flows of $\tilde{X} \in \mathfrak{X}(\tilde{C})$ commuting with the flow of \mathbf{X} induce flows of $X = (\pi_C)_*\tilde{X} \in \mathfrak{X}(C)$ preserving \mathcal{D} (contact flows). This implies the following (see [1], Appendix 4)

Proposition 7. Every contact vector field on C can be lifted to a symplectic Hamiltonian vector field on \tilde{C} (the symplectification of C) whose Hamiltonian is homogeneous of degree 1 with respect to \mathbf{X} , that is, $L_{\mathbf{X}}\tilde{H} = \tilde{H}$. Conversely, every Hamiltonian vector field on \tilde{C} such that \tilde{H} is homogeneous of degree 1 projects onto C as a contact vector field.

Remark 16. When a contact 1-form, η , of C has been chosen, any contact vector field X on C may be associated with a function $\mathcal{H} = -i_X\eta$. The vector field is recovered from the function (and contact 1-form), by the conditions $i_X\eta = -\mathcal{H}$, $L_X\eta \sim \eta$, which define a contact Hamiltonian vector field (eq. A5). According to Proposition 7, there is a Hamiltonian, \tilde{H} , on \tilde{C} determined by X , which we refer to as the *symplectic lift* of \mathcal{H} , related to \mathcal{H} by $-\iota^*\tilde{H} = \mathcal{H}$, for $\iota : C \rightarrow \tilde{C}$ the section associated to η .

See e.g. [1] and [42, 43] for applications of this correspondence to thermodynamics. Above (Theorem 1 and Corollary 2) we extend this relationship for Hamiltonians, $L_{\mathbf{X}}\tilde{H} = \Lambda\tilde{H}$ of degree Λ , on \tilde{C} . In fact, defining:

Definition 11. The Λ -Hamiltonian vector field, $X_{\mathcal{H}}^\Lambda$, of a function \mathcal{H} on the contact manifold C with respect to the contact 1-form η , is determined by the conditions:

$$i_{X_{\mathcal{H}}^\Lambda}\eta = -\Lambda\mathcal{H}, \quad i_{X_{\mathcal{H}}^\Lambda}d\eta = d\mathcal{H} - (\mathcal{R}\mathcal{H})\eta.$$

The results of Section IIB above may be stated in a more analogous manner to their degree one counterparts.

Remark 17. For $\Lambda = 1$, Definition 11 is exactly the definition of a contact Hamiltonian vector field. The trajectories of a degree Λ Hamiltonian, \tilde{H} , on \tilde{C} project to (reparametrized) trajectories of the Λ -Hamiltonian vector field on C determined by $\mathcal{H} = -\iota^*\tilde{H}$, where $\iota : C \rightarrow \tilde{C}$ is the section associated to η .

As a completely analogous construction to the symplectification, if (M, ω) is exact symplectic, there is also a natural procedure to extend it to a contact manifold (see [1], Appendix 4).

Definition 12. Given an exact symplectic manifold $(M, \omega = d\lambda)$, the *contactification* of M is

$$\tilde{M} := M \times \mathbb{R}, \quad \tilde{\lambda} := dS - \pi_M^*\lambda, \quad (\text{A11})$$

where $\pi_M : \tilde{M} \rightarrow M$ is the standard projection.

Again, there is an explicit relationship between Hamiltonian vector fields M and certain contact vector fields on \tilde{M} , given by

Proposition 8. Every symplectic Hamiltonian vector field on M can be lifted to a contact Hamiltonian vector field on \tilde{M} (the contactification of M) whose Hamiltonian is

$$\tilde{\mathcal{H}} = \pi_M^*H. \quad (\text{A12})$$

Conversely, every contact Hamiltonian vector field on \tilde{M} such that $\tilde{\mathcal{H}}\tilde{\mathcal{H}} = 0$ projects onto M as a symplectic Hamiltonian vector field X_H . In this case we call $\tilde{\mathcal{H}}$ the *contact lift* of H .

In Darboux coordinates (q^a, p_a) for M , one has $\tilde{\lambda} = dS - p_a dq^a$ and the integral curves of $X_{\tilde{\mathcal{H}}}$ are given by (A7) by setting $\tilde{\mathcal{H}}(q^a, p_a, S) := H(q^a, p_a)$ (note that in particular $\partial\tilde{\mathcal{H}}/\partial S = 0$).

Note that, by definition, the flow of X_H is just the projection of that of $X_{\tilde{\mathcal{H}}}$. However, on \tilde{M} there can be defined contact Hamiltonians that are not lifts of any symplectic Hamiltonian on M . This has been exploited largely in recent years to describe mechanical systems with dissipation (classical and quantum), thermostatted systems and thermomechanical phenomena in a Hamiltonian framework (see e.g. [5–8, 12, 17, 22–25]).

Dual to the above constructions, in which we *extended* either a contact or a symplectic manifold, under the appropriate hypotheses one can also *reduce* one of these structures to the other. In particular, we have

Definition 13. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold and $\mathbf{X} \in \mathfrak{X}(M)$ its *Liouville vector field*. Any hypersurface C transverse to \mathbf{X} is called *of contact type*.

Hypersurfaces of contact type on an exact symplectic manifold are naturally endowed with a contact structure as follows.

Proposition 9. Let $(M, \omega = d\lambda)$ be an exact symplectic manifold, \mathbf{X} the Liouville vector field and $\iota_C : C \rightarrow M$ a hypersurface of contact type. Then the 1-form $\eta_{\mathbf{X}} := \iota_C^*i_{\mathbf{X}}\omega = \iota_C^*\lambda$ is a contact form on C .

Example 10 (Transverse H levels). A typical example of a hypersurface of contact type is given by any regular energy level $C_E := H^{-1}(E)$ of a Hamiltonian system on the cotangent bundle $(T^*Q, d\alpha)$, where α is the canonical 1-form, which is transverse to the Liouville vector field. Interestingly, in this case the restriction of the Hamiltonian vector field X_H to C_E is just a reparametrization of the Reeb vector field of $\eta_{\mathbf{X}}$. Indeed, one has $\mathcal{R} = \frac{X_H|_{C_E}}{\mathbf{X}(H)|_{C_E}}$, and therefore the *orbits* – the unparametrized trajectories – of the two vector fields are the same.

Analogously, one can reduce a contact manifold to an exact symplectic one on a proper “energy hypersurface” of the contact Hamiltonian and find an equivalence between the orbits of the contact Hamiltonian vector field and the Liouville one as follows (see [9]).

Definition 14. Let (C, η) be a contact manifold and $\mathcal{R} \in \mathfrak{X}(C)$ its *Reeb vector field*. Any hypersurface S transverse to \mathcal{R} is called of *symplectic type*.

Hypersurfaces of symplectic type on a contact manifold are naturally endowed with a symplectic structure as follows.

Proposition 10. Let (C, η) be a contact manifold, \mathcal{R} the Reeb vector field and $\iota_S : S \rightarrow C$ a hypersurface of symplectic type. Then the 2-form $\Omega = d\theta$, with $\theta = \iota_S^* \eta$ is an exact symplectic form on S .

Example 11 (Transverse \mathcal{H} 0-level). Let $\mathcal{H} : C \rightarrow \mathbb{R}$ be a Hamiltonian function on a contact manifold (C, η) with Reeb vector field \mathcal{R} and assume that $S = \mathcal{H}^{-1}(0) \neq \emptyset$ and that $\mathcal{R}(\mathcal{H})(x) \neq 0$, for all $x \in S$. Then (S, Ω) is an exact symplectic manifold as above. Moreover, if Δ is the Liouville vector field of the exact symplectic manifold (S, Ω) , then $X_{\mathcal{H}|_S}$ is the reparametrization of Δ given by $X_{\mathcal{H}|_S} = -(\mathcal{R}(\mathcal{H}) \circ \iota_S) \Delta$.

Finally, let us also note that all the above relationships have motivated the definition of a *symplectic sandwich with contact bread*, see [9], where it was proved that the existence of an invariant measure for $X_{\mathcal{H}|_S}$ is equivalent to the existence of a symplectic sandwich with contact bread.

Appendix B: Herglotz’ Principle

In this appendix, we will consider certain contact reductions admitting variational descriptions (details will be left to the reader). We refer to eg [14, 21, 44] for the variational formulation of contact systems and to [29, 30, 45, 46] for the use of variational techniques in order to study dynamical properties and solutions of the related Hamilton-Jacobi equations.

Let us recall that certain Hamiltonian systems admit Lagrangian variational descriptions.

Definition 15. A *Lagrangian system* is a pair (Q, L) , where $L : TQ \rightarrow \mathbb{R}$. The *action* of a curve $q(t) \in Q$ is $S := \int L(q, \dot{q}) dt$.

The extremals of the action, with say fixed endpoints, satisfy the well-known Euler-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = \frac{\partial L}{\partial q} \quad (\text{B1})$$

The contact analogue of a Lagrangian system is

Definition 16. A *Herglotz Lagrangian system* (also known as a *contact Lagrangian system*) is a pair (\bar{Q}, \mathcal{L}) , where $\mathcal{L} : T\bar{Q} \times \mathbb{R} \rightarrow \mathbb{R}$. The *action* of a curve $q(t) \in \bar{Q}$ is $\Delta S := \int \mathcal{L}(q, \dot{q}, S) dt$, where $S(t)$ solves $\dot{S} = \mathcal{L}(q, \dot{q}, S)$.

This definition extends the scope of Lagrangian systems to allow for dependence on the action. The extremals of this action, with say fixed endpoints and initial condition $S(0) = S_0$, satisfy the Herglotz-Lagrange equations:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}} \right) = \frac{\partial \mathcal{L}}{\partial S} \frac{\partial \mathcal{L}}{\partial \dot{q}} + \frac{\partial \mathcal{L}}{\partial q}, \quad \dot{S} = \mathcal{L}. \quad (\text{B2})$$

The connection between Herglotz Lagrangian systems and contact Hamiltonian systems runs in parallel to the usual connection between Lagrangian and Hamiltonian systems. Under the *Legendre transform*,

$$\text{Leg} : T\bar{Q} \times \mathbb{R} \rightarrow T^*\bar{Q} \times \mathbb{R}, \quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}},$$

the Herglotz-Lagrangian extremals correspond to trajectories of the contact Hamiltonian system

$$C = T^*\bar{Q} \times \mathbb{R}, \quad \mathcal{H} = p \cdot \dot{q} - \mathcal{L}, \quad \eta = dS - p \cdot dq. \quad (\text{B3})$$

Remark 18. Here we consider systems whose Legendre transform is a diffeomorphism. One way to see the connection of a contact Hamiltonian system to its Herglotz variational principle is via the Poincaré-Cartan form: $\alpha := p \cdot dq - \mathcal{H} dt$ and $\hat{\eta} := dS - \alpha = \eta + \mathcal{H} dt$ on the extended space, $C \times \mathbb{R} \ni (c, t)$. The world lines of trajectories of the contact Hamiltonian system are characterized by lying tangent to the line field $\ker \hat{\eta} \cap \{X : i_X d\alpha \sim \eta\}$, or equivalently as extremals of $\gamma \mapsto \int_\gamma \alpha$ among curves satisfying the constraint $\hat{\eta}|_\gamma \equiv 0$.

One may obtain ‘Lagrangian analogues’ of our results above by applying the Legendre transform to Hamiltonian systems and their contact reductions. It is instructive to first consider a local coordinate analogue of Proposition 7, or Corollary 1 (the degree one case).

Example 12. Consider a Herglotz-Lagrangian system $\mathcal{L}(q, \dot{q}, S)$ corresponding under Legendre transform to the contact Hamiltonian system $\mathcal{H}(q, p, S)$, $\eta = dS - p \cdot dq$. The contact Hamiltonian system is the scale reduction of the Hamiltonian system:

$$H(\rho, q, S, P) = -\rho \mathcal{H}(q, -P/\rho, S)$$

on the symplectification, $\omega = d\tilde{\alpha} = d(\rho dS + P \cdot dq)$, by the degree one scaling symmetry $\mathbf{D} = \rho \partial_\rho + P \cdot \partial_P$. We identify this symplectification with the cotangent bundle of $Q \ni (\rho, q)$ by taking $S = -p_\rho$, with canonical 1-form:

$$\lambda_o = p_\rho d\rho + P \cdot dq = \tilde{\alpha} + d(\rho p_\rho).$$

For which $S = -\frac{i_{\mathbf{D}} \lambda_o}{\rho}$. Now, the Legendre transform of H is the Lagrangian system $L = -(\dot{\rho} S + \rho \mathcal{L})$, or

$$\mathcal{L} = -\frac{L + \dot{\rho} S}{\rho}$$

is our Herglotz-Lagrangian system on $q \in \bar{Q} = Q/\rho \partial_\rho$, for the scale reduction of the Lagrangian system L by the scaling symmetry $\mathbf{D}_L = \rho \partial_\rho + \dot{\rho} \partial_{\dot{\rho}}$ (corresponding to \mathbf{D} under Legendre transform). We note that we have as well the equivalent coordinate expressions:

$$\mathcal{L} = -\partial_\rho L, \quad S = -\partial_{\dot{\rho}} L.$$

Only certain Hamiltonian systems on cotangent bundles admit Lagrangian variational descriptions, and as well only certain contact Hamiltonian systems admit Herglotz variational descriptions. We will consider the following special case of scale reduction.

Proposition 11 (Cotangent contact reduction). Let H, T^*Q be a Hamiltonian system admitting a scaling symmetry, \mathbf{D} , of degree Λ such that

$$L_{\mathbf{D}} \lambda_o = \lambda_o$$

where λ_o is the canonical 1-form on T^*Q (\mathbf{D} being a *basic scaling symmetry*). Then \mathbf{D} projects to a vector field, $\bar{\mathbf{D}}$, on Q . Assume that the flow of $\bar{\mathbf{D}}$ acts freely and properly on Q , and set

$$\bar{Q} := Q/\bar{\mathbf{D}}.$$

Then, given a scaling function $\rho : Q \rightarrow \mathbb{R}^\times$ of $\bar{\mathbf{D}}$, one may identify:

$$C = (T^*Q)/\mathbf{D} \cong T^*\bar{Q} \times \mathbb{R} \ni (\bar{q}, \bar{p}, S)$$

with contact 1-form $\eta = dS - \bar{\lambda}_o$, where $\bar{\lambda}_o$ is the canonical 1-form on $T^*\bar{Q}$, and $S = -i_{\mathbf{D}} \lambda_o / \rho$.

The description of a scale reduction of a Lagrangian system may be obtained by applying the Legendre transform to a Hamiltonian system and its contact reduction of the form in Proposition 11. The relationship is laid out in the following commutative diagram:

$$\begin{array}{ccc} TQ, L & \xleftarrow{\text{Legendre}} & T^*Q, H \\ \text{Quotient} \downarrow & & \downarrow \text{Quotient} \\ T\bar{Q} \times \mathbb{R}, \mathcal{L} & \xleftarrow{\text{Legendre}} & T^*\bar{Q} \times \mathbb{R}, \mathcal{H} \end{array}$$

Thus, on the Lagrangian side, we define:

Definition 17. A *basic scaling symmetry* of degree Λ for a Lagrangian system (Q, L) is a vector field \mathbf{D} on TQ such that $\mathbf{D}(L) = \Lambda L$ and $L_{\mathbf{D}} \lambda_L = \lambda_L$, where $\lambda_L = \partial_{\dot{q}} L \cdot dq$ is the *Lagrangian 1-form* (the pullback of the canonical 1-form by the Legendre transform).

As we have remarked above (Remark 17), the trajectories of a degree Λ Hamiltonian system scale reduce to trajectories of a Λ -Hamiltonian vector field on C . Such trajectories may be described variationally.

Definition 18. A Λ -Herglotz system is a pair, (\bar{Q}, \mathcal{L}) , where $\mathcal{L} : T\bar{Q} \times \mathbb{R} \rightarrow \mathbb{R}$. The action of a curve $q(t) \in \bar{Q}$ is $S_\Lambda := \int \mathcal{L}(q, q', S) dt$, where S solves $S' = \mathcal{L} + (1 - \Lambda)\mathcal{E}$, for $\mathcal{E} := \text{Leg}^* \mathcal{H} = \partial_{q'} \mathcal{L} \cdot q' - \mathcal{L}$.

Remark 19. Under Legendre transform, $\mathcal{H} = p \cdot \dot{q} - \mathcal{L}$, extremals of the Λ -Herglotz system \mathcal{L} correspond to trajectories of the Λ -Hamiltonian vector field of \mathcal{H} (Definition 11). One way to see this connection is via the Poincaré-Cartan form: $\alpha := p \cdot dq - \mathcal{H} dt$ and $\hat{\eta}_\Lambda := \eta + \Lambda \mathcal{H} dt$ on the extended space, $C \times \mathbb{R} \ni (c, t)$. The world lines of trajectories of the Λ -Hamiltonian vector field are characterized by lying tangent to the line field $\ker \hat{\eta}_\Lambda \cap \{X : i_X d\alpha \sim \hat{\eta}_\Lambda\}$, or equivalently as extremals of $\gamma \mapsto \int_\gamma \alpha$ among curves satisfying the constraint $\hat{\eta}_\Lambda|_\gamma \equiv 0$. Alternately, one may verify the correspondence directly via the equations of motion (eqs. (8) above).

With this definition, we have:

Theorem 3 (Counterpart to Theorem 1). Let (Q, L) be a Lagrangian system, admitting a basic scaling symmetry, \mathbf{D} , of degree Λ and a scaling function $\rho : Q \rightarrow \mathbb{R}^\times$. Then one may identify

$$\pi : TQ \rightarrow (TQ)/\mathbf{D} \cong T\bar{Q} \times \mathbb{R} \ni (\bar{q}, \bar{q}', S)$$

where $\pi^* S = -i_{\mathbf{D}} \lambda_L / \rho$. The extremals of the Lagrangian system project to extremals of the Λ -Herglotz system:

$$\pi^* \mathcal{L} = -\frac{L + \dot{\rho} S}{\rho^\Lambda}.$$

Remark 20. As in the Hamiltonian case (see Remark 5 above), the scale-reduced trajectories are reparametrized according to

$$\rho^{1-\Lambda} d\tau = dt,$$

and we write $' = \frac{d}{d\tau}$. Note for $\Lambda = 1$, the Lagrangian system scale reduces to a Herglotz-Lagrangian system, as we saw in Example 12 above.

Example 13 (The 2d harmonic oscillator). Let us consider the two dimensional harmonic oscillator of Example 5, with Lagrangian

$$L = \frac{\dot{r}^2 + r^2 \dot{\theta}^2 - kr^2}{2}, \quad (\text{B4})$$

in polar coordinates. We have the basic scaling symmetry: $\mathbf{D} = \frac{1}{2}(r\partial_r + \dot{r}\partial_{\dot{r}})$, of degree one, and scaling function $\rho = r^2$. Following Theorem 3, we write

$$S = -\frac{i_{\mathbf{D}} \lambda_L}{\rho} = -\frac{\dot{\rho}}{4\rho}, \quad L = \frac{-\dot{\rho} S + \rho \dot{\theta}^2 - k\rho}{2}, \quad (\text{B5})$$

to obtain the scale-reduced Herglotz Lagrangian system:

$$\mathcal{L} = -\frac{L + \dot{\rho} S}{\rho} = 2S^2 + \frac{k - \dot{\theta}^2}{2} \quad (\text{B6})$$

on $TS^1 \times \mathbb{R} \ni (\theta, \dot{\theta}, S)$. Note that \mathcal{L} is the (contact) Legendre transform of $\mathcal{H} = -2S^2 - \frac{k + \dot{\theta}^2}{2}$ on $T^*S^1 \times \mathbb{R}$ obtained by contact reduction of $H = \frac{|p|^2 + k|q|^2}{2}$ by the scaling symmetry $\frac{q \cdot \partial_q + p \cdot \partial_p}{2}$ and $\rho = |q|^2$.

Example 14 (Contact-reduced Kepler). We consider the Kepler system of Example 2, with Lagrangian:

$$L = \frac{|\dot{q}|^2}{2} + \frac{1}{|q|}$$

and basic scaling symmetry $\mathbf{D}_K = 2q \cdot \partial_q - \dot{q} \cdot \partial_{\dot{q}}$, of degree $\Lambda = -2$, and scaling function $\rho = |q|^{1/2}$. Following Theorem 3, we take polar coordinates and write:

$$S = -\frac{i_{\mathbf{D}} \lambda_L}{\rho} = -4\rho^2 \dot{\rho}, \quad L = \frac{-\dot{\rho} S + \rho^4 \dot{\theta}^2}{2} + \frac{1}{\rho^2}$$

to obtain the scale-reduced -2 -Herglotz system:

$$\mathcal{L} = -\rho^2(L + \dot{\rho}S) = \frac{S^2}{8} - \frac{(\theta')^2}{2} - 1$$

on $TS^1 \times \mathbb{R} \ni (\theta, \theta', S)$ (using $'$ to denote $\frac{d}{d\tau}$ where $\rho^3 d\tau = dt$). Again, note that \mathcal{L} is related by Legendre transform to the -2 -Hamiltonian system, \mathcal{H} , obtained by an analogous contact reduction of the Legendre transform of L . We also observe that the reparametrization $\rho^3 d\tau = dt$ has:

$$L dt = \rho \left(\frac{2\rho'^2}{\rho^2} + \frac{\theta'^2}{2} + 1 \right) d\tau = \tilde{L} d\tau$$

where $\mathcal{L} = -\frac{\tilde{L} + \rho' S}{\rho}$ gives the scale-reduced -2 -Herglotz system ($S = -4\frac{\rho'}{\rho}$).

Finally, we consider a counterpart to Theorem 2 for certain Lagrangian systems depending on parameters, a_j , of the form:

$$L_a = T(q, \dot{q}) + \sum_{j=1}^k a_j L_j(q) \quad (\text{B7})$$

and for which we have a vector field \mathbf{D} on TQ with

$$\mathbf{D}(T) = T, \quad \mathbf{D}(L_j) = \Lambda_j L_j, \quad L_{\mathbf{D}} \lambda_L = \lambda_L. \quad (\text{B8})$$

Note that the first condition in (B8) fixes the parametrization of \mathbf{D} , being the precise analogue of the first condition in Definition 2.

Theorem 4 (Counterpart to Theorem 2). Consider a Lagrangian system depending on parameters as in eq. (B7), and admitting a vector field \mathbf{D} as in eq. (B8). Then the Lagrangian system

$$\hat{L} := T + \sum_{j=1}^k \dot{X}_j \left(\frac{L_j}{\dot{X}_j} \right)^{\frac{1}{\Lambda_j}}$$

on $\hat{Q} := Q \times \mathbb{R}^k \ni (q, X)$ admits \mathbf{D} as a basic degree one scaling symmetry. Moreover, the projection to Q of an extremal of \hat{L} is an extremal of L_a , with parameter values

$$\Lambda_j a_j = \left(\frac{\dot{X}_j}{L_j} \right)^{1 - \frac{1}{\Lambda_j}}$$

constant over the extremals of \hat{L} .

Remark 21. More generally, for $L = \sum a_j L_j$, one may consider $\hat{L} = \sum \dot{X}_j \left(\frac{L_j}{\dot{X}_j} \right)^{1/\Lambda_j}$. Under Legendre transform, this lifted system, \hat{L} , corresponds to the Hamiltonian system $\hat{H} = \sum a_j H_j$, where $\Lambda_j a_j = \left(\frac{\dot{X}_j}{L_j} \right)^{1 - \frac{1}{\Lambda_j}}$ and $H_j = \partial_{\dot{q}} L_j \cdot \dot{q} - L_j$ is the Legendre transform of L_j . A Legendre transform of $\hat{\mathbf{D}}$ from Remark 10, yields a degree one scaling symmetry for this Lagrangian system, \hat{L} (which in general is *not* \mathbf{D}).

Similar comments as those of Remark 10 apply here, in that in order to avoid potential complex numbers from the exponents $\frac{1}{\Lambda_j}$, one may consider various cases with absolute values to take care of the signs. In general, to avoid such exponents, one may consider eg:

$$\hat{L} := \sum e^{\dot{X}_j + L_j}$$

as the lift of the Lagrangian system, $L_a = \sum a_j L_j$, where $e^{\dot{X}_j + L_j} = a_j$ is constant over extremals of \hat{L} , with the degree one scaling symmetry $\hat{\mathbf{D}} = \sum \partial_{\dot{X}_j}$.

Example 15 (The Kepler-Hooke potential). Let us consider the system described in Example 13 and extend the system to include a Kepler potential:

$$L = \underbrace{\frac{\dot{\rho}^2}{8\rho} + \frac{\rho\dot{\theta}^2}{2} - \frac{k_H\rho}{2}}_{L_1} + \underbrace{\frac{k_K}{\sqrt{\rho}}}_{L_2} \quad \mathbf{D} = \rho\partial_\rho + \dot{\rho}\partial_{\dot{\rho}} \quad (\text{B9})$$

For this system, \mathbf{D} is not a scaling symmetry, as $\mathbf{D}(L_1) = L_1$ but $\mathbf{D}(L_2) = -\frac{1}{2}L_2$. However, following Theorem 4 we can consider

$$\hat{L} = \frac{\dot{\rho}^2}{8\rho} + \frac{\rho\dot{\theta}^2}{2} - \frac{k_H\rho}{2} + \frac{\dot{X}^3\rho}{k_K^2} \quad (\text{B10})$$

for which \mathbf{D} is a scaling symmetry. We then impose the initial condition $\dot{X}_0 = \frac{k_K}{(-2)^{\frac{1}{3}}r_0}$, where r_0 is the initial value of r . This gives rise to the same equations of motion as the original system. Since this is a Lagrangian scaling system of degree one, we can follow Example 13 and find the equivalent Herglotz Lagrangian system:

$$\mathcal{L} = -2S^2 + \frac{\dot{\theta}^2}{2} - \frac{k_H}{2} + \frac{\dot{X}^3}{k_K^2}. \quad (\text{B11})$$

Remark 22. Note that we could have dropped k_K entirely from the description of the system in equations (B10) and (B11) by appropriately choosing a boundary condition for \dot{X} , and hence we have in essence exchanged specifying a coupling constant, k_K , for an initial condition.

Appendix C: Blow-ups in celestial mechanics

In this appendix, we remark on the contact reduction by scaling symmetries presented here in relation to some well-known constructions in celestial mechanics (see for example [11, 34, 35]).

Let us first recall the scale-reduced Kepler problem (Examples 1–4). We denote the scaling functions of Example 2 by

$$\rho := |q|^{1/2}, \quad \kappa := \frac{1}{|p|}, \quad G := p \cdot iq, \quad J := p \cdot q.$$

where $\rho^4 = q \cdot q$ is the moment of inertia, and $\frac{1}{\kappa^2} = p \cdot p$ is (twice) the kinetic energy.

Of course, the contact reductions corresponding to each choice of scaling function represent the same curves in $C = M/\mathbf{D}$ obtained by projections of Kepler orbits. They are merely different choices of coordinates for C . However, each choice of coordinates highlights certain properties of these scale-reduced curves. For instance, the scaling function ρ , is naturally associated to a blow-up of the collision orbits (Example 4), while the scaling function κ to a blow-up of orbits with $|p| \rightarrow \infty$ (as well the collision orbits in the Kepler problem).

To determine some contact reductions (Table I below) for various choices of scaling function, we first note that the scaling functions above are not independent, satisfying the relation $\kappa^2(J^2 + G^2) = \rho^4$. One may use any three of these scaling functions, along with a scale invariant angle, as coordinates on the phase space. There are essentially three (dependent) natural angles, $\theta, \varphi, \delta = \varphi - \theta$, present, where

$$q = \rho^2 e^{i\theta}, \quad p = \frac{1}{\kappa} e^{i\varphi}, \quad \rho^2 e^{i\delta} = \kappa(J + iG) \quad (\text{C1})$$

for δ the angle between position and momentum.

To see the equations of motion obtained by these various choices of scaling functions, we will consider (ρ, θ, J, G) for coordinates on the phase space. Then we have (eq. (12) above):

$$\lambda = G d\theta + 2J d\log\left(\frac{\rho}{J}\right), \quad H_K = \frac{1}{2\kappa^2} - \frac{1}{\rho^2}, \quad \kappa^2(J^2 + G^2) = \rho^4.$$

One may describe the contact reduction for various choices of scaling functions according to Remark 4 by restriction of λ and $-H_K$ to a level set of the scaling function, obtaining the following representations of the scale-reduced Kepler problem: the contact reduction with ρ (item 1 of Table I) is Example 2 above. In this table we used, for example, in the κ case, the notation:

$$\pi^* J_\kappa = J/\kappa, \quad \pi^* G_\kappa = G/\kappa, \quad \pi^* \varphi = \varphi, \quad \pi^* \mathcal{H}_\kappa = -\kappa^2 H_K, \quad \pi^* \eta_\kappa = \lambda/\kappa,$$

the subscript notation in the other cases taking the analogous meaning.

Remark 23. We have already seen (Example 4), the relation of row 1 in Table I to the blow-up of the collision orbits. Likewise, row 2 yields a blow-up of $|p| \rightarrow \infty$ orbits. Here, for the Kepler problem, these are as well the collision orbits. In general n -body problems, choices of various mutual distances as scaling functions or analogues of κ will be interesting to examine.

Scaling function	Contact system	Scale invariant equations of motion
1. ρ	$\eta = \tilde{G} d\theta - 2 d\tilde{J}, \quad \mathcal{H}_0 = 1 - \frac{\tilde{J}^2 + \tilde{G}^2}{2}$	$\tilde{J}' = \frac{\tilde{G}^2}{2} - \mathcal{H}_0, \quad \tilde{G}' = -\frac{\tilde{J}\tilde{G}}{2}, \quad \theta' = \tilde{G}$
2. κ	$\eta_\kappa = G_\kappa d\varphi - dJ_\kappa, \quad \mathcal{H}_\kappa = \frac{1}{R} - \frac{1}{2}$	$J'_\kappa = -2\mathcal{H}_\kappa + \frac{G_\kappa^2}{R^3}, \quad G'_\kappa = -\frac{G_\kappa J_\kappa}{R^3}, \quad \varphi' = \frac{G_\kappa}{R^3}$
3. G	$\eta_G = d\theta - J_G d\log\left(\frac{J_G}{\rho_G}\right)^2, \quad \mathcal{H}_G = \frac{2\rho_G^2 - J_G^2 - 1}{2\rho_G^4}$	$J'_G = \frac{1}{\rho_G^2} - 2\mathcal{H}_G, \quad \rho'_G = \frac{J_G}{2\rho_G^3}, \quad \theta' = \frac{1}{\rho_G^4}$
4. J	$\eta_J = G_J d\theta + d\log\rho_J^2, \quad \mathcal{H}_J = \frac{2\rho_J^2 - G_J^2 - 1}{2\rho_J^4}$	$G'_J = G_J(2\mathcal{H}_J - \frac{1}{\rho_J^2}), \quad \rho'_J = \rho_J\mathcal{H}_J - \frac{G_J^2}{2\rho_J^3}, \quad \theta' = \frac{G_J}{\rho_J^4}$

TABLE I: Contact reductions of the Kepler problem with various scaling functions. In row 2 we set $R := \sqrt{J_\kappa^2 + G_\kappa^2}$.

We also remark that the use of a first integral as a scaling function, as in row 3, corresponds to a first integral for the scale-invariant curves. Here, this first integral, $\pi^*\mathcal{H}_G = -G^2 H_K$, is a well known scale invariant (sometimes referred to as the Dziobek constant). Also observe that the equations of motion for the scale invariant J are forms of the well-known Lagrange-Jacobi identity.

The reduction based on the scaling function J of row 4, is used in Barbour et. al's [3] (with n -body problems), where it is referred to as the dilational momentum and used, for instance, to describe long term 'clustering' behavior of solutions.

Remark 24. One has an analogous contact reduction for certain inverse power force laws:

$$H_\alpha = \frac{|p|^2}{2} - \frac{1}{\alpha|q|^\alpha}$$

with $\mathbf{D}_\alpha := \frac{2q \cdot \partial_q - \alpha p \cdot \partial_p}{2-\alpha}$ a scaling symmetry of degree $\Lambda = -\frac{2\alpha}{2-\alpha}$ (when $\alpha \neq 2$). Note that J, G above are still scaling functions, as is $\rho_\alpha := |q|^{1-\alpha/2}$. Carrying out the analogue of Example 4 here gives exactly the blown-up collision tori (see §1.3 of [16]) of these central force problems.

The contact reduction may be carried out similarly for n -body problems:

$$q = (q_1, \dots, q_n) \in \mathbb{R}^{nd}, \quad p = (p_1, \dots, p_n) \in \mathbb{R}^{nd}$$

with $q_j, p_j \in \mathbb{R}^d$ given as the Hamiltonian system:

$$\omega = dp \wedge dq = d(p \cdot dq), \quad H = \frac{\|p\|^2}{2} - U(q)$$

where $U(\lambda q) = \lambda^{-1}U(q)$ is homogeneous of degree -1 . For simplicity consider the masses are unit (otherwise, one may take, $\|\cdot\|$, as a suitable mass-weighted norm). The system admits the degree -2 scaling symmetry:

$$\mathbf{D} = 2q \cdot \partial_q - p \cdot \partial_p.$$

Taking $\rho := \|q\|^{1/2}$ as a scaling function, the quotient $\mathbb{R}^{2nd}/\mathbf{D} \cong S^{nd-1} \times \mathbb{R}^{nd}$, has scale-invariant coordinates:

$$s := q/\rho^2 \in S^{nd-1}, \quad y := \rho p \in \mathbb{R}^{nd}.$$

The contact reduction associated to the scaling function ρ gives the well-known McGehee blow-up. Taking $\rho\eta = i_{\mathbf{D}}\omega = -2q \cdot dp - p \cdot dq$ and $\mathcal{H} = -\rho^2 H$, one has:

$$\eta = d\nu + s \cdot dy, \quad \mathcal{H} = U(s) - \frac{\|y\|^2}{2}$$

where $\nu := s \cdot y$. The scale-invariant equations of motion (eqs. (8), with $\Lambda = -2$) are exactly the usual McGehee blow-up equations (eg §4.2 of [35]):

$$s' = y - \nu s, \quad y' = \frac{1}{2}\nu y + \nabla U(s),$$

the collision manifold being given by the invariant level $\mathcal{H} = 0$ (see as well [32]). According to Appendix B, these McGehee blow-up equations may be described variationally as a -2 -Herglotz system (see Def. 18):

$$\mathcal{L} = \frac{S^2}{8} - \frac{\|s'\|^2}{2} - U(s).$$

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