Variations on the Tait-Kneser theorem

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Abstract

The Tait-Kneser theorem, first demonstrated by Peter G. Tait in 1896, states that the osculating circles along a plane curve with monotone non-vanishing curvature are pairwise disjoint and nested. This note contains a proof of this theorem using the Lorentzian geometry of the space of circles. We show how a similar proof applies to two variations on the theorem, concerning the osculating Hooke and Kepler conics along a plane curve. We also prove a version of the 4-vertex theorem for Kepler conics.

1 The Euclidean case revisited

A smooth plane curve with non-vanishing curvature has at every point an *osculating circle* which is tangent to the curve at this point and shares the curvature with it. That is, the osculating circles are 2nd order tangent to the curve at every point. At some points the osculating circle may be tangent to higher order. Such points are called *vertices*, and these are the critical points of the curvature.

Theorem 1.1 (Tait-Kneser). *The osculating circles of a vertex-free plane curve with non-vanishing curvature are disjoint and pairwise nested, see Figure 1.*

This theorem is more than a century old [17]; it has numerous variations and ramifications, see the survey [9].

The osculating circles of a curve with monotone curvature form a foliation of the annulus bounded by the osculating circles at the end points of the the curve. We leave it to the reader to mull over the seemingly paradoxical property of this foliation: the curve is tangent to the leaves at every point, but it is not contained in a single leaf (isn't it similar to a non-constant function having everywhere vanishing derivative?!)

The original Tait's proof was very short, just two paragraphs long; it made use of the notions of evolute and involute of a curve. We present a different proof of Theorem 1.1 that will also work in the variations to follow.

A circle in $\mathbb{R}^2$ is given by an equation of the form $(x-a)^2+(y-b)^2=r^2$. Denote by $\mathbb{R}^{1,2}$ the three dimensional pseudo-Euclidean space with coordinates

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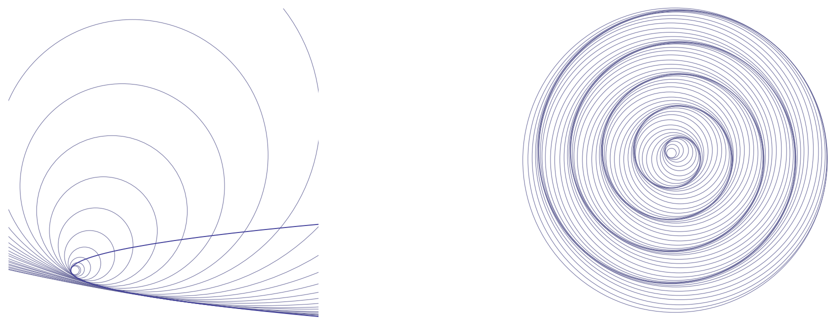

Figure 1: Osculating nested circles along a curve with monotone non-vanishing curvature: the lower part of a parabola (left) and an Archimedean spiral (right).

a, b, r equipped with the indefinite quadratic form $|(a,b,r)|^2 := -a^2 - b^2 + r^2$ (we use this notation even though the latter may happen to be a negative number!). The space of circles in $\mathbb{R}^2$ is parametrized by the upper half space $\mathbb{R}^{1,2}_+ := \{(a,b,r) \,|\, r > 0\}$.

The null cone with vertex $\mathbf{v}_0 \in \mathbb{R}^{1,2}$ is the set of points $\mathbf{v} \in \mathbb{R}^{1,2}$ such that $|\mathbf{v} - \mathbf{v}_0|^2 = 0$, and its interior consist of points $\mathbf{v}$ with $|\mathbf{v} - \mathbf{v}_0|^2 > 0$. We next describe when two circles are *nested*, that is, the interior of one of them is contained in that of the other.

Lemma 1.2. *Given two circles and the corresponding points $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{1,2}_+$, the circles are nested if and only if $|\mathbf{v}_1 - \mathbf{v}_2|^2 \geq 0$, with equality when the circles are nested and tangent.*

That is, the circles are nested when one of the corresponding points in $\mathbb{R}^{1,2}$ lies in the light cone whose vertex is the other point. See Figure 2.

Proof. Let the radii of the circles be $R \geq r$, and the distance between their centers be d. The nesting condition is $d + r \leq R$, or $|\mathbf{v}_1 - \mathbf{v}_2|^2 = -d^2 + (R-r)^2 \geq 0$, with equality if and only if $d + r = R$, that is, the circles are tangent. $\square$

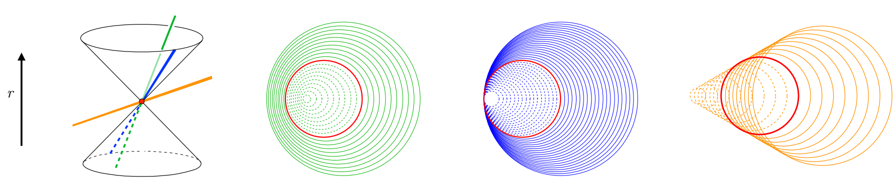

Figure 2: 'Lines' of circles. A timelike line (green): nested disjoint circles. A null line (blue): nested circles tangent at a point. A spacelike line (orange): intersecting circles, tangent to a pair of lines.

Let γ be a plane curve with non-vanishing curvature and let $\Gamma \subset \mathbb{R}^{1,2}$ be the curve of osculating circles of γ. The vertices of γ correspond to singular points of Γ, and Γ is regular if γ is vertex free (see Equation (1) in the proof of the next Lemma).

A curve in $\mathbb{R}^{1,2}$ is *null* if its tangent vector is tangent to the null cone at every point.

Lemma 1.3. *The curve Γ is a null curve.*

Proof. Let $\gamma(t) = (x(t), y(t))$ be an arc length parameterization of γ. Then $\kappa = x'y'' - y'x''$ is the curvature of γ, and the osculating circle at a point (x, y) of γ is given by the equation $(X - a)^2 + (Y - b)^2 = r^2$ with

$$(a, b) = (x, y) + r(-y', x'), \ r = \frac{1}{\kappa}.$$

Since $\Gamma(t) = (a(t), b(t), r(t))$, one has

$$\begin{aligned} \Gamma' &= -\frac{\kappa'}{\kappa^2}(-y', x', 1) + (x', y', 0) + \frac{1}{\kappa}(-y'', x'', 0) = \\ &= -\frac{\kappa'}{\kappa^2}(-y', x', 1), \end{aligned} \tag{1}$$

the last equality due to equation $(x'', y'') = \kappa(-y', x')$. Since $(x')^2 + (y')^2 = 1$, one has $|\Gamma'|^2 = 0$, as claimed. □

Here is the intuition behind it with a 'hand-waving' proof. An osculating circle C passes through three "consecutive" points of the curve, say $\gamma(t - \varepsilon), \gamma(t), \gamma(t+\varepsilon)$. The "next" osculating circle shares two of these points, $\gamma(t), \gamma(t+\varepsilon)$, with C. In the limit $\varepsilon \to 0$, this implies that the curve Γ is tangent to the cone whose vertex is the circle C and consists of the circles tangent to it.

The last ingredient of the proof of the Tait-Kneser theorem is the next lemma.

Lemma 1.4. *A regular null curve $\Gamma : [t_0, t_1] \to \mathbb{R}^{1,2}$ satisfies $|\Gamma(t_1) - \Gamma(t_0)|^2 \geq 0$, with equality if and only if Γ is the null line segment connecting its endpoints.*

Proof. Let $\Gamma(t) = (a(t), b(t), r(t))$ and $\bar{\Gamma}(t) = (a(t), b(t))$ be the projection on the horizontal plane. The nullity condition on Γ, $(r')^2 = (a')^2 + (b')^2$, implies that the length of $\bar{\Gamma}$ is $|r(t_1) - r(t_0)|$. This length is at least the distance between the endpoints $|\bar{\Gamma}(t_1) - \bar{\Gamma}(t_0)|$, with equality if and only if $\bar{\Gamma}$ is the straight line segment.

It follows that $|\Gamma(t_1) - \Gamma(t_0)|^2 = -|\bar{\Gamma}(t_1) - \bar{\Gamma}(t_0)|^2 + |r(t_1) - r(t_0)|^2 \geq 0$, with equality if and only $\bar{\Gamma}$ is the line segment. By the nullity condition, this is equivalent to Γ being a null line segment. □

Theorem 1.1 follows. Let γ be a plane curve with non-vanishing monotone curvature. Consider two osculating circles C_0, C_1 along γ and the regular null curve Γ connecting the corresponding points $\mathbf{v}_0, \mathbf{v}_1 \in \mathbb{R}^{1,2}$. By the above lemmas, either $|\mathbf{v}_1 - \mathbf{v}_0|^2 > 0$, in which case C_0, C_1 are nested, or $|\mathbf{v}_1 - \mathbf{v}_0|^2 = 0$, in which case Γ is a null segment connecting $\mathbf{v}_0$ to $\mathbf{v}_1$. The later case corresponds to a family of circles tangent at a point, which cannot occur as the family of osculating circles to a curve in $\mathbb{R}^2$.

Remark 1.5. The Lorentzian geometry of the space of circles and the relation between the osculating circles of a curve with null curves is investigated in [13].

2 Centroaffine geometry: Hooke orbits

A smooth plane curve γ is *star-shaped* with respect to the origin if $[\gamma, \gamma'] \neq 0$ (the bracket is the determinant made by a pair of vectors). Such a curve can be parameterized so that $[\gamma(t), \gamma'(t)] = 1$. Then $\gamma'' = -p\gamma$, where the function $p(t)$ is called the *centroaffine curvature* of γ. For example, the origin centered circle of radius r has $p = 1/r^4$.

A central conic $ax^2 + 2bxy + cy^2 = 1$ has $p = ac - b^2$, the determinant of the coefficient matrix $\left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. Central conics are the trajectories of mass points subject to the Hooke force law: the radial force is proportional to the distance to the origin. If the force is attractive the trajectory is a central ellipse with $p > 0$, and if it is repulsive the trajectory is a hyperbola with $p < 0$. Central conics play the role of circles in centroaffine geometry.

The *osculating central conic* of a star-shaped curve γ with non-vanishing centroaffine curvature is the central conic tangent to γ and sharing its centroaffine curvature at the tangency point. It coincides with γ to 2nd order at the point of tangency, and the order is higher if $p' = 0$ at this point.

Here is a centroaffine version of the Tait-Kneser theorem.

Theorem 2.1. *The osculating central conics of a star-shaped plane curve with monotone non-vanishing centroaffine curvature are pairwise disjoint and nested, see Figure 3.*

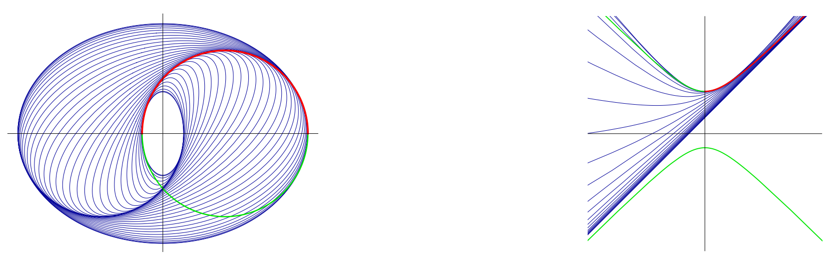

Figure 3: Osculating central conics. Left: along the upper part (red) of the circle $(x - .75)^2 + y^2 = 1$ (green). Right: along the upper right part (red) of the hyperbola $(y - .5)^2 - x^2 = 1$ (green).

The proof of Theorem 2.1 goes along the same lines as our proof of the Tait-Kneser theorem.

The space of central conics is 3-dimensional with coordinates (a, b, c). This space has a pseudo-Euclidean metric given by the determinant of the quadratic form, $ac - b^2$, and we identify it with $\mathbb{R}^{1,2}$.

An analog of Lemma 1.2 holds:

Lemma 2.2. *Let C_1, C_2 be two central conics of the same type (ellipses or hyperbolas). If $\det(C_2 - C_1) \geq 0$ then they are nested, and equality implies they are tangent. (We denote a conic and the quadratic form defining it by the same letter.)*

Proof. We use the well known fact from linear algebra that two real quadratic forms, one of which is definite (positive or negative), can be diagonalized simultaneously. In dimension 2, a quadratic form is definite if and only if its determinant is positive.

Now if C_1, C_2 are ellipses then the quadratic forms are both positive definite, so, without loss of generality, by the above fact, the ellipses are $x^2 + y^2 = 1$ and $ax^2 + by^2 = 1$, with $a, b > 0$. The condition $\det(C_2 - C_1) \geq 0$ is then $(a-1)(b-1) \geq 0$, which is equivalent to $a, b \geq 1$ or $a, b \leq 1$. In both cases the ellipses are nested, with equality when they are tangent.

If C_1, C_2 are hyperbolas, suppose that $\det(C_2 - C_1) > 0$. Then $\Delta C := C_2 - C_1$ is a definite quadratic form and, by interchanging the conics if necessary, it is positive. Hence $\Delta C, C_1$ can be transformed to $ax^2 + by^2 = 1, x^2 - y^2 = 1$ (respectively), with $a, b > 0$. Then C_2 is $(a+1)x^2 - (1-b)y^2 = 1$, and since it is a hyperbola, we have $0 < b < 1$. Renaming the constants, C_2 is $(x/a)^2 - (y/b)^2 = 1$, $0 < a < 1 < b$. It is now easy to see that C_1 is nested in C_2. See Figure 4.

The case $\det(\Delta C) = 0$ is a limit of $\det(\Delta C) > 0$, since being "nested" is a closed condition. $\square$

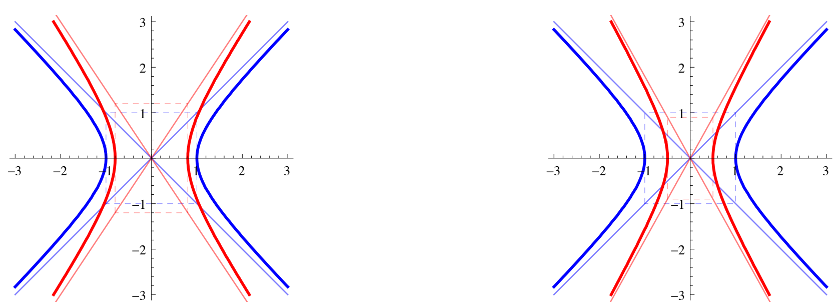

Figure 4: The proof of Lemma 2.2. A pair of nested hyperbolas is shown in each figure, $x^2 - y^2 = 1$ (blue) and $(x/a)^2 - (y/b)^2 = 1$ (red). Left: if the pair has timelike difference, $\det(C_2 - C_1) > 0$, then either $0 < a < 1 < b$ or $0 < b < 1 < a$ and the pair is nested. Right: timelike difference is not a necessary nesting condition; if $a < b < 1$ or $1 < b < a$ then the difference is spacelike but the hyperbolas are still nested.

Remark 2.3. Although we do not use it, we note that the converse of Lemma 2.2 holds for ellipses (as follows easily from the above proof), but not for hyperbolas. For example, the hyperbolas $x^2 - y^2 = 1$ and $(x/3)^2 - (y/2)^2 = 1$ are nested, but the respective determinant is negative. We do not dwell on the precise algebraic condition for a pair of quadratic forms to determine a pair of nested hyperbolas.

Lemma 1.3 also has an analog (with the same hand-waving explanation as before). Again we denote by Γ the curve in the space of Hooke conics that osculate a centroaffine curve γ.

Lemma 2.4. *The curve Γ is a null curve.*

Proof. Let $\gamma(t) = (x(t), y(t))$ where $xy' - yx' = 1$, and let $p(t)$ be the centroaffine curvature. A direct calculation shows that the osculating central conic is given by the equation $aX^2 + 2bXY + cY^2 = 1$, with

$$a = py^2 + (y')^2, \ b = -(pxy + x'y'), \ c = px^2 + (x')^2$$

(one needs to check that $ax^2 + 2bxy + cy^2 = 1, \ (ax + by)x' + (bx + cy)y' = 0,$ and $p = ac - b^2$).

Then another calculation shows that

$$a' = p'y^2, \ b' = -p'xy, \ c' = p'x^2,$$

and hence $a'c' - (b')^2 = 0$, as claimed. □

Now Lemma 1.4 applies, both for ellipses and hyperbolas, thereby completing the proof of Theorem 2.1.

Let us add that the Tait-Kneser theorem is closely related to another classical result, the 4-vertex theorem that, in its simplest form, states that a plane oval has at least four vertices. As the example in Figure 3 shows, its analog does not hold in centroaffine geometry: the circle of radius 1 centered at $(0.5, 0)$ has only 2 "vertices", that is, hyper-osculating central conics.

Remark 2.5. The Lorentzian geometry of the space of central conics is studied in [16].

3 Kepler orbits

A Kepler orbit is a plane conic (ellipse, parabola or hyperbola) with a focus at the origin. These are the trajectories of mass points subject to Newton's force law (either attractive or repulsive): the radial force is proportional to the inverse square of the distance to the origin. For attractive force the orbits are ellipses, parabolas or hyperbola branches bending around the origin. For repulsive force only hyperbolas appear, the 'other' branches left out by the attractive force hyperbolas. See Figure 5.

A Kepler conic is the orthogonal projection on the horizontal plane of the intersection of the cone $x^2 + y^2 = z^2$ in $\mathbb{R}^3$ with a plane $ax + by + cz = 1, \ c > 0$ [8]. Thus the space of Kepler conics is parametrized by the space $\mathbb{R}^{1,2}_+$ with coordinates (a, b, c) and the quadratic form $|(a, b, c)|^2 = -a^2 - b^2 + c^2$. The null

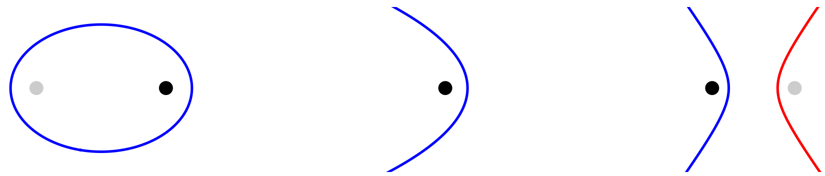

Figure 5: Kepler orbits share a focus at the origin (the black dot; the gray dot is the other focus, not shared). The blue curves (ellipse, parabola, hyperbola) bend around the origin, due to an attractive force. The red curve (hyperbola) bends away from the origin, due to a repulsive force.

cone $|(a,b,c)|^2 = 0$ parametrizes Kepler parabolas, its interior $|(a,b,c)|^2 > 0$ parametrizes ellipses, and its exterior $|(a,b,c)|^2 < 0$ parametrizes hyperbolas.

As before, a smooth star-shaped curve with non-vanishing centroaffine curvature can be 2nd order approximated at every point by its osculating Kepler conic; this osculating conic may hyper-osculate, that is, approximate the curve to higher order.

An analog of the Tait-Kneser theorem holds:

Theorem 3.1. *Consider a star shaped curve, free from hyper-osculating Kepler conics. Then its osculating Kepler conics are pairwise nested. See Figure 6.*

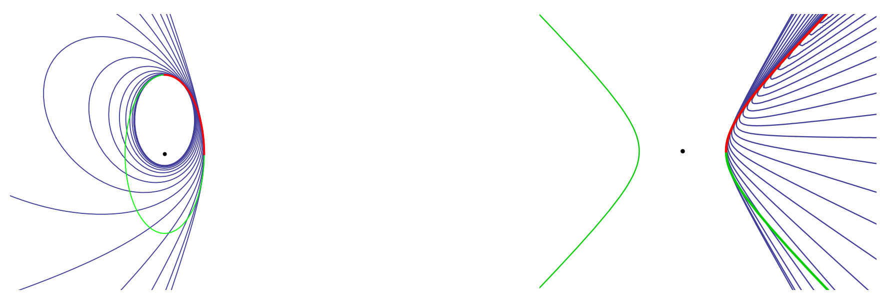

Figure 6: Osculating Kepler conics (blue) along a vertex free arc (red) of a Hooke conic (green). Left: along a Hooke ellipse. Right: along a Hooke hyperbola.

One can prove Theorem 3.1 along the same lines as before. We only present an analog of Lemmas 1.2 and 2.2.

Lemma 3.2. *Two Kepler conics, corresponding to points* $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^{1,2}_+$, *are disjoint if and only if* $|\mathbf{v}_1 - \mathbf{v}_2|^2 > 0$.

(For Kepler conics, being nested and disjoint are equivalent).

Proof. The system of equations

$$a_1x + b_1y + c_1z = 1,\ a_2x + b_2y + c_2z = 1,\ x^2 + y^2 = z^2$$

has no solutions if and only if the system

$$(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z = 0,\ x^2 + y^2 = z^2$$

has no non-zero solutions, if and only if the vector $(a_1 - a_2, b_1 - b_2, c_1 - c_2)$ lies in the interior of the cone $x^2 + y^2 = z^2$, as needed. See Figure 7. $\square$

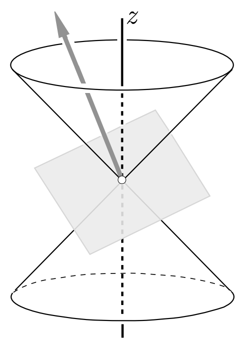

Figure 7: Proof of Lemma 3.2. The plane $ax + by + cz = 0$ intersects the cone $x^2 + y^2 = z^2$ only at its vertex whenever the normal vector (a, b, c) lies inside the cone.

Unlike Hooke conics, one has a version of the 4-vertex theorem for Kepler conics. Let γ be a simple closed star-shaped curve, and let us call a vertex a point at which it is approximated by a Kepler conic to 3rd order (higher than usual).

Theorem 3.3. *The curve γ has at least four distinct vertices.*

Proof. A Kepler conic, in polar coordinates (α, r), is given by the formula

$$r = \frac{c}{1 + e\cos(\alpha + \varphi)},$$

where c, e, and φ are constants. Let $\rho = 1/r$. Then $\rho(\alpha)$ is a first harmonic, hence $\rho''' + \rho' = 0$ for all α. This equation characterizes Kepler conics.

The curve γ is given by its function $\rho(\alpha)$, and its vertices are precisely the points where $\rho''' + \rho' = 0$. It remains to use the fact that such an equation has at least four distinct roots for every periodic function ρ – a particular case of the Sturm-Hurwitz theorem that asserts that a smooth 2π-periodic function has at least $2n$ roots where n the number of the first harmonic in its Fourier expansion (see, e.g., [15], Appendix 1). The case at hand is $n = 2$: the function $\rho''' + \rho'$ is free from the constant term and the first harmonics. $\square$

4 Odds and ends

Identifying the plane with $\mathbb{C}$, consider the square map $z \mapsto z^2$. This map takes Hooke conics to Kepler conics [1,3]. It thus provides a direct connection between the results of Sections 2 and 3.

A more general statement relates the trajectories of mass points subject to radial forces proportional to powers of distances to the origin; see [1] for a modern treatment.

Theorem 4.1 (Bohlin-Kasner [3, 11])**.** *Consider two central force laws in the plane, with the force proportional to r^a and to r^b, where r is the distance to the origin. Let $(a+3)(b+3) = 4$. Then the map $z \mapsto z^{(a+3)/2}$ takes the trajectories of motion in the first field to those in the second field.*

The Hooke and Newton attraction laws are $a = 1$ and $b = -2$ (respectively). These cases are distinguished among central force laws.

Theorem 4.2 (Bertrand [2])**.** *Assume that all the trajectories of a mass point, subject to a central force that depend on the distance to the origin and whose energy does not exceed a certain limit, are closed. Then the law of attraction is either Hooke's or Newton's.*

One notes that the family of circles, of Hooke conics, and of Kepler conics each depends on three parameters – this is why they can approximate smooth curves to 2nd order.

More generally, given a field of forces in the plane, the trajectory of a mass point depends on its initial position and velocity, and hence the trajectories form a 3-parameter family. Which 3-parameter families of plane curves are obtained this way? This problem was thoroughly studied by E. Kasner who obtained a complete answer to this question [10, 11].

Other examples of 3-parameter families of curves for which a version of the Tait-Kneser theorem holds are parabolas $y = ax^2+bx+c$, the graphs of quadratic polynomials, and hyperbolas $(x-a)(y-b) = c^2$, the graphs of fractional-linear functions (see [9]). Our proof works in these cases as well, with the metrics given by $ds^2 = db^2 - 4(da)(dc)$ and $ds^2 = dc^2 - (da)(db)$, respectively.

One can describe the graphs of 3-parameter families of functions $y(x)$ by 3rd order differential equations $y''' = F(x, y, y', y'')$. For example, the equation $y''' = 0$ describes the vertical parabolas, and the graphs of fractional-linear functions are described by the vanishing of the Schwarzian derivative:

$$\frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2 = 0.$$

Given such a 3-parameter family of curves $\mathcal{F}$, one defines null cones whose rulings consist of the curves that are tangent to a fixed line at a fixed point. A smooth plane curve defines an associated curve in $\mathcal{F}$, and it is still true that this associated curve is null. However, for a general family $\mathcal{F}$, these cones may fail to be quadratic.

The families $\mathcal{F}$ for which the nullity condition is quadratic, and hence defines a conformal Lorentzian metric on $\mathcal{F}$, are characterized by a complicated non-linear PDE on the function $F(x, y, y', y'')$ that defines this family. This condition was studied by K. Wünschmann [18] and later by É. Cartan [5] and S.S. Chern [6, 7], using the method of equivalence. For recent presentations of this deep result see [12, 14].

* * *

We end with the observation that the Tait-Knesser and 4 vertex theorems for Kepler conics (Theorems 3.1 and 3.3) can be derived from their Euclidean analogues via projective duality. Here is a sketch (more details will appear in [4]). The equation $ax + by = 1$ associates to a point in the ab plane a line in the xy plane and vice versa. To a curve C in one plane corresponds its dual curve C^* in the other plane, whose points parametrize the lines tangent to C. One then shows that the dual of a Kepler conic is a circle and that duality preserves nesting and order of contacts of curves. It follows that the dual of the osculating Kepler conic to C is the osculating circle to C^* and the same holds for hyperosculating conics. Thus duality interchanges Euclidean and Keplerian vertices, reducing the Kepler version of the Tait-Knesser and 4 vertex theorems to their Euclidean analogues.

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