Spinning Bodies: A Tutorial

Tadashi Tokieda

Trinity Hall, Cambridge CB2 1TJ, UK
tokieda@dpmms.cam.ac.uk

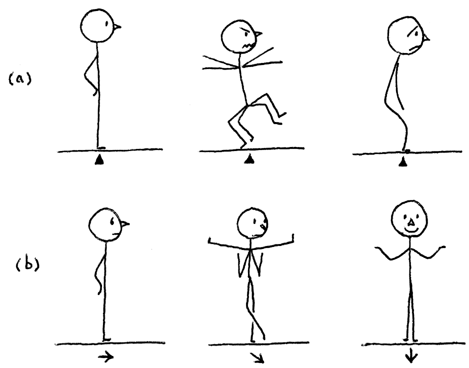
Abstract. The article sets up some of the mathematics underpinning this LNP and is addressed to those who have learnt rigid-body dynamics officially but still feel suspicious toward it. I try to relieve the monotony by discussing unusual examples, and by delving deeper into the usual material than many books.

Contents: 1. Strange rotational phenomena, 2. Inertia matrix, 3. Conservation of angular momentum, 4. Miscellaneous examples, 5. Euler's equations, 6. Euler's top, 7. Lagrange's top, 8. Kovalevskaya's top, 9. Rotational proof of Pythagoras, 10. Further reading.

A Alain Chenciner, maître mécanicien.

1 Introduction

1.1. You are standing on slippery ice. Can you wriggle your body so as to slide and end up standing somewhere else (picture a)?



T. Tokieda: *Spinning Bodies: A Tutorial*, Lect. Notes Phys. **682**, 1–22 (2006)
www.springerlink.com

Well, you can't—no matter how you wriggle yourself, your centre of mass stays on the same spot. Now suppose you try *swiveling* instead of sliding (picture b). Can you end up facing some new orientation? This time you *can*. Stretch out your arms and twist your upper body anticlockwise; your lower body then twists clockwise. Next pull your arms in and untwist your upper body clockwise; your lower body then untwists anticlockwise, *less than it twisted clockwise earlier*. The net effect is, you swivel clockwise by an angle. Denizens of warmer climes may experiment on a swivel chair.

Cats accomplish this feat with instinctive grace: a cat falling upside down twists itself in mid-air and lands upside up, on its paws. I must own that I am too respectful of the feline species to have dared an experiment myself. Instead, here is a design of a cat made of stiff paper. When dropped upside down, this toy cat flips and lands on its paws. (Alas, the physics is unrelated to that of real cats.)

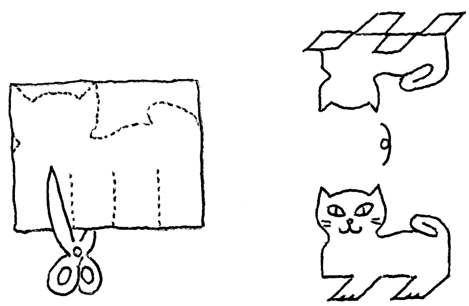


Figure skaters accelerate or decelerate their spin by pulling in or stretching out their limbs.

1.2. Ordinary life offers few opportunities to experience rotational motion. (Never mind for the nonce that we live on a rotating object.) In contrast, translational motion is with us all the time, e.g. when riding a car. But in the days of Galileo & Co., finely controlled translational motion was rare in people's experience; this may explain why dynamics and in particular the law of inertia took long to discover. Controlled rotational motion is not so common to this day, and accordingly dynamics of rotation seems baffling. This article's business is to unbaffle us about dynamics of rotation and to make it as intuitive as dynamics of translation.

1.3. English is rife with pseudo-synonyms of "rotate": "revolve", "spin", "swivel", "turn", "twist", "whirl"... They carry helpful differences of nuance, which we shall turn to our advantage.

1.4. One terminological oddity. Traditionally, rigid bodies are called "tops" (French "toupie", German "Kreisel", Japanese "koma", Latin "turbo", Russian "volchok"). So, from now on,

"Top" and "rigid body" are synonymous,

and a top will be denoted by T. The letter T even looks like a top. Warning: despite the connotation of the word "top", our tops are not *a priori* assumed to be symmetrically shaped.

2 Inertia Matrix

2.1. *Inertia* is a body's resistance to acceleration. In translational motion, it is encoded in mass, a scalar: a napping rhinoceros is hard to budge, a charging rhinoceros is equally hard to halt. In rotational motion, resistance is encoded in a quantity more sophisticated than a scalar because, when it is spun about different axes, a body may differently resist rotational acceleration. Rotational inertia turns out to be a *matrix*.

In particle dynamics, mass m appears as coefficient in two quantities: in momentum $p = mv$ and in kinetic energy $E_{\text{par}} = \frac{1}{2}mv^2$ for a particle moving at velocity v. In rigid-body dynamics, the inertia matrix appears also as coefficient in two quantities (2.2, 2.3).

2.2. Given a top T (1.4), imagine rectangular coordinate axes attached to T whose origin is at a point O which may be inside or outside T. The axes as well as O move together with T.

We always take as O the centre of mass C of the top or some stationary point (pivot).

In these coordinates, each point of T is parametrised by a radius vector $x = (x_1, x_2, x_3)$. Let $\rho(x)$ be the density of T at x, $dx = dx_1 dx_2 dx_3$ the volume element.

A top T of mass M is moving at U = velocity of O, Ω = angular velocity around O, so that a point x of T has velocity $U + \Omega \wedge x$ to an observer at rest. The total angular momentum L of T around O is

$$L = \int_T x \wedge (U + \Omega \wedge x)\rho(x)dx = M(C - O) \wedge U + \int_T x \wedge (\Omega \wedge x)\rho(x)dx\,.$$

The term $M(C - O) \wedge U$ vanishes by our hypothesis that $O = C$ or $U = 0$. The integral term defines an operator, *linear* in Ω hence representable by a matrix, the **inertia matrix** (*alias* inertia tensor) I of T around O :

$$L = I\Omega = \int_T x \wedge (\Omega \wedge x)\rho(x)dx\,.$$

Thus the first quantity in which the inertia matrix I appears as coefficient: the angular momentum $L = I\Omega$. The dimension of L is mass $\times$ length^2 $\times$ time^{-1}; that of I is mass $\times$ length^2.

Note the analogy with $p = mv$ (2.1). Beware however that, because I is a matrix rather than a scalar, in general L is not parallel to Ω. One knack of unbaffling ourselves about dynamics of rotation consists in distinguishing clearly between angular momentum L and angular velocity Ω (e.g. 6.2).

2.3. In the scenario of (2.2) and $\langle\ ,\ \rangle$ denoting the scalar product, the total kinetic energy E of T is

$$\begin{aligned} E &= \frac{1}{2}\int_T (U+\Omega\wedge x)^2\rho(x)dx \\ &= \frac{1}{2}U^2\int_T \rho(x)dx + \langle U, \Omega\wedge\int_T x\rho(x)dx\rangle + \frac{1}{2}\langle\int_T x\wedge(\Omega\wedge x)\rho(x)dx, \Omega\rangle \\ &= \frac{1}{2}MU^2 + \langle U, \Omega\wedge M(C-O)\rangle + \frac{1}{2}\langle I\Omega,\Omega\rangle\ . \end{aligned}$$

E splits into two terms, a translational term that has the form as if the mass of T were concentrated at O, plus a rotational term; the cross term $\langle U, \Omega\wedge M(C-O)\rangle$ vanishes by our hypothesis (2.2). Thus the second quantity in which the inertia matrix I appears as coefficient: the rotational kinetic energy $E_{\mathrm{rot}} = \frac{1}{2}\langle I\Omega,\Omega\rangle$.

Note the analogy with $E_{\mathrm{par}} = \frac{1}{2}\langle mv, v\rangle$ (2.1). Beware however that, because I is a matrix, in general E_{rot} depends not only on the magnitude but also on the direction of Ω.

2.4. The inertia matrix I is *symmetric*. Indeed, for any vectors Ω, $\widetilde{\Omega}$,

$$\begin{aligned} \langle I\Omega,\widetilde{\Omega}\rangle &= \int_T \langle x\wedge(\Omega\wedge x),\widetilde{\Omega}\rangle\rho(x)dx \\ &= \int_T \langle \Omega\wedge x,\widetilde{\Omega}\wedge x\rangle\rho(x)dx = \int_T \langle \Omega, x\wedge(\widetilde{\Omega}\wedge x)\rangle\rho(x)dx = \langle\Omega, I\widetilde{\Omega}\rangle\ , \end{aligned}$$

which expresses that I equals its own transpose. By a theorem of linear algebra, suitable rectangular axes x_1, x_2, x_3 can be chosen so as to diagonalise I; they are called **principal axes** (*alias* principal directions) of the top. With respect to principal axes,

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}.$$

The eigenvalues I_1, I_2, I_3 are the **principal moments of inertia**. The **moment of inertia** about an arbitrary axis, without the epithet "principal", means $\langle Ie, e\rangle$ for a unit vector e along that axis.

One suggestive interpretation of the diagonalisability of I is,

As far as inertial responses are concerned, any top is an ellipsoid.

2.5. Owing to curricula which introduce students to moment of inertia in the context of exercises on multiple integrals, many live under the impression that moment of inertia somehow characterises the mass distribution about an axis. To be sure, it *happens* to be computable from the distribution, but plenty of different distributions result in the same moment of inertia, and anyway mass distribution is not the *raison d'être* of moment of inertia. To repeat, what moment of inertia characterises is the body's resistance to rotational acceleration. As for mass distribution, the good news about rigid bodies is that details more complicated than the ellipsoid of inertia are invisible to dynamics (cf. form of equations 5.1).

2.6. The shape of a top often makes its principal axes readily identifiable: mentally fit an ellipsoid to the top (2.4). A rectangular box has principal axes parallel to the edges. For a circular cylinder, one principal axis is the axis of the cylinder; the remaining two are *any* two axes perpendicular to the first.

An equilateral triangular lamina is instructive. One principal axis is normal to the lamina. About this axis, the lamina has rotational symmetry of order 3, whereas an ellipsoid with distinct semiaxes admits rotational symmetry of order at most 2. Hence the ellipsoid of the lamina must be *of revolution*, and the remaining principal axes are any two axes perpendicular to the first. In general, as soon as a top has rotational symmetry of order > 2 about some axis, its ellipsoid is of revolution about that axis. If this happens about two axes, then the ellipsoid degenerates to a ball, and any three perpendicular axes are principal.

A quiz. About which axis is the moment of inertia of a cube largest? The axis connecting 1) diametrically opposite vertices, 2) midpoints of diametrically opposite edges, 3) midpoints of opposite faces?

2.7. In desperation I could be computed: unpacking the definition (2.2),

$$I = \int_T \begin{pmatrix} x_2^2 + x_3^2 & -x_1x_2 & -x_1x_3 \\ -x_2x_1 & x_3^2 + x_1^2 & -x_2x_3 \\ -x_3x_1 & -x_3x_2 & x_1^2 + x_2^2 \end{pmatrix} \rho(x_1, x_2, x_3) dx_1 dx_2 dx_3 \,,$$

which reveals again the symmetry of I (2.4). Computing moments of inertia is salutary perhaps for the soul but not for much else; please look them up in your favourite reference. We mention just two tips. First, "Routh's rule": the moment of inertia of a homogeneous body about an axis of symmetry is

$$\text{mass} \times \frac{\text{sum of squares of perpendicular semiaxes}}{3, 4, 5} \,,$$

the denominator being 3, 4 or 5 according as the body is rectangular (2D or 3D), elliptical (2D) or ellipsoidal (3D) [14]. Second, if the mass is M and the radius R, the moment of inertia of a homogeneous *solid ball* about its diameter is $\frac{2}{5}MR^2$ (a special instance of Routh), while that of a homogeneous *spherical shell* is $\frac{2}{3}MR^2$ (not an instance of Routh, which does not apply to hollow bodies).

2.8. Faced with a top, our Pavlovian reaction is to think of its moment of inertia around the centre of mass C. Yet it can prove useful to think around other points (e.g. 4.4, 4.5, Sects. 7, 8). The "parallel axes theorem" saves us the trouble of recomputing moments of inertia afresh:

Let I_C [resp. I_O] be the inertia matrix of a top of mass M around C [resp. another point O].

Then $I_O = I_C +$ *inertia matrix around* O *of a particle of mass* M *at* C*.*

The last matrix may be written $M[\,{}^t(C-O)(C-O)\delta - \delta(C-O)\,{}^t(C-O)\,]$, where $C-O$ is a column vector and its transpose ${}^t(C-O)$ a row vector, and δ is the identity matrix (cf. formula of 2.7). The theorem is not used in this article, but it is comforting to know.

3 Conservation Laws

3.1. Dynamics is a drama of conserved quantities: *momentum*, *angular momentum*, *energy*. In dynamics of rotation, the star billing goes to angular momentum and rotational energy. All the mathematics we manipulate in this article are auxiliary to them, all the laws we formulate are ultimately about how they do or do not change in time. In every physical problem, we should zoom in on conservation laws: tyros rush to differential equations, whereas pros stick to conservation laws as far as they can.

3.2. A top T of mass M and inertia matrix I around a point O is moving at $V =$ velocity of its centre of mass C and $\Omega =$ angular velocity around O; our hypothesis (2.2) is that $O = C$ or O is stationary. The momentum and the angular momentum around O of T are $P = MV$, $L = I\Omega$.

Momentum and angular momentum are conserved,
except for external disturbing influences:

$$\frac{d}{dt}P = F, \qquad \frac{d}{dt}L = N.$$

Here F is the force and N the torque acting on T. If each point x of T is subjected to a field of force $f(x)$, then the total force is

$$F = \int_T f(x)dx$$

while the total **torque** (*alias* moment of force) around O is

$$N = \int_T x \wedge f(x)dx\,,$$

the radius vector x being measured from O. The dimension of N is mass $\times$ length^2 $\times$ time^{-2}, the same as that of energy.

3.3. As everywhere in physics,

Energy is conserved.

Of course our accounting must include all forms of energy: kinetic, potential, heat...

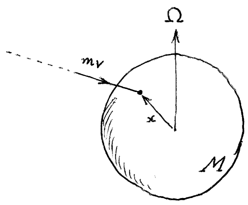
3.4 In many places in the literature, the conservation laws (3.2, 3.3) are "derived" from laws of particle dynamics by regarding a rigid body as an assemblage of particles, etc. Actually it is simpler to adopt the laws (3.2, 3.3) as fundamental in their own right.

3.5. Rigid body is an idealisation, even in macroscopic physics. Relativity teaches that nature knows no such thing as a rigid body. Non-relativistically too, natural matter is more or less *deformable*. Cats actively reposture their bodies and vary their inertia matrices (1.1). A classic example from astronomy is a rotating mass of fluid, e.g. a star; unlike cats, a star passively responds to various forces acting on it and settles into an equilibrium figure. A collection of grains, or rush-hour commuters on the Tokyo underground, can behave like a rigid body or not, depending on how tightly they are packed. This article ignores all these.

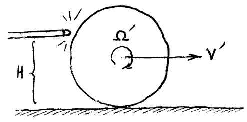
3.6. There is almost nothing on rigid bodies in *Principia*.

4 Miscellaneous Examples

4.1. It is remarkable that simple conservation laws (3.2, 3.3) are already amply powerful to solve many nontrivial problems, without further development of formal machinery. In this section we sample several illustrations.



(c)



(d)

4.2. A meteorite impacts and adheres to a planet. How is the planet's axis of rotation affected (picture c)?

The planet of mass M and moment of inertia I around its centre C is moving at V = velocity of C and Ω = angular velocity around C, when a meteorite of mass m flies in at velocity v and impacts a point x on the planet. Denoting the values after the impact by $'$, we have from conservation of momentum and angular momentum (3.2)

$$MV + mv = MV' + m(V' + \Omega' \wedge x)$$
$$I\Omega + x \wedge mv = I\Omega' + x \wedge m(V' + \Omega' \wedge x)\,.$$

Suppose, reasonably enough, that $m \ll M$, $|v| \gg |V|, |V'|, |\Omega' \wedge x|$. Then the planet's new angular velocity is

$$\Omega' \sim \Omega + \frac{x \wedge mv}{I}\ .$$

The impact could tilt the axis of rotation appreciably. Perhaps this is the fate that befell Uranus, whose axis of rotation is abnormally tilted from the normal to the ecliptic.

4.3. The next one is a chestnut. If you shoot a billiard ball too high [resp. low], the ball skids with forward [resp. backward] spin. At what height must you shoot so as to induce pure rolling (picture d)?

Assume the motion is restricted to a vertical plane; the problem is then planar. The cue horizontally imparts a force F at height H to a ball of mass M, radius R, moment of inertia $I = \frac{2}{5}MR^2$ around its centre (2.7). Before, the ball had velocity $V = 0$ and angular velocity $\Omega = 0$; after, these will change to V', Ω', *both of which we can leave unknown and yet solve the problem.* If the shot occurs during a brief interval Δt, then

$$MV' = F\Delta t, \qquad I\Omega' = (H - R)F\Delta t\,;$$

eliminating $F\Delta t$,

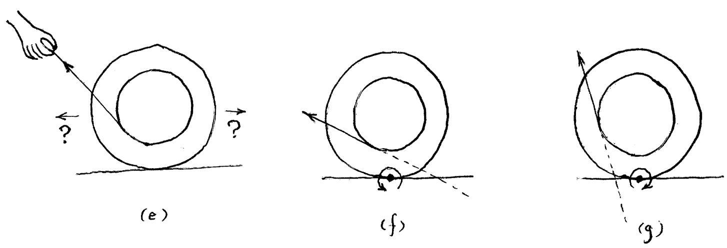
$$\Omega' R = \frac{5}{2}\frac{H-R}{R}V'$$

whence the velocity of the point of contact with the table is

$$V' - \Omega' R = \frac{7R - 5H}{2R}V'.$$

$H < \frac{7}{5}R$ induces backward spin, $H > \frac{7}{5}R$ forward spin, $H = \frac{7}{5}R$ pure rolling.

4.4. Gently tug on the string of a spool (picture e). Which way will the spool roll?



Two theories: 1) you input momentum in the direction of tugging, so the spool rolls left; 2) tugging induces clockwise spinning, so the spool rolls right.

Which way the spool rolls depends on the inclination of the tug. In picture (f), the line of force passes *above* the point of contact with the ground, so the tug creates *anticlockwise* angular momentum around the point of contact; the spool rolls left, reeling the string *in*. Likewise in picture (g), the spool rolls right, reeling the string *out*.

4.5. Place a ball on a sheet of paper, and withdraw the sheet from under the ball. Which way will the ball end up rolling? Two competing theories again. The answer is that the ball stops dead.

Moral of (4.4, 4.5): it can prove useful to consider angular momentum around points other than the centre of mass (2.8).

4.6. A *superball* is a perfectly elastic ball whose surface is non-slipping; *elastic* means no loss of energy upon bouncing, so a superball bounces excitingly high. We analyse the bouncing of a superball of mass M, radius R, moment of inertia $I = \frac{2}{5}MR^2$ around its centre (2.7).

Assume the problem is planar. The superball comes in at velocity whose *horizontal component* is V and angular velocity Ω around its centre, and bounces off a horizontal floor or ceiling; the vertical component of the velocity merely reverses upon bouncing. During the brief interval Δt of a bounce, the floor or ceiling exerts on the ball not only a normal reaction but also a friction F. Denoting the values after a bounce by $'$, we have from conservation of momentum and angular momentum (3.2)

$$M(V' - V) = F\Delta t, \qquad I(\Omega' - \Omega) = -RF\Delta t$$

and from conservation of energy (3.3)

$$\frac{1}{2}MV'^2 + \frac{1}{2}I\Omega'^2 = \frac{1}{2}MV^2 + \frac{1}{2}I\Omega^2\,;$$

eliminating and factoring, we get two equations

$$M(V' - V) = -\frac{I}{R}(\Omega' - \Omega), \qquad M(V' - V)(V' + V) = -I(\Omega' - \Omega)(\Omega' + \Omega).$$

The dull solution is $V' = V$, $\Omega' = \Omega$, $F\Delta t = 0$. The other solution, worthy of a superball, is

$$V' - \Omega' R = -(V - \Omega R), \qquad F\Delta t = -\frac{2MI}{I + MR^2}(V - \Omega R),$$

i.e. upon bouncing the velocity of the point of contact instantaneously reverses: *a superball bounces not only normally but also tangentially*. The law of bouncing is then

$$\begin{aligned} V' &= \frac{3}{7}V + \frac{4R}{7}\Omega \\ \Omega' &= \frac{10}{7R}V - \frac{3}{7}\Omega \end{aligned}$$

for bounce off the floor, and

$$\begin{aligned} V' &= \frac{3}{7}V - \frac{4R}{7}\Omega \\ \Omega' &= -\frac{10}{7R}V - \frac{3}{7}\Omega \end{aligned}$$

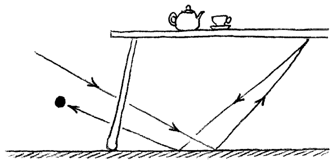
for bounce off the ceiling. Both these linear operators have determinant -1.

Throw a superball under a table (the underside of the table serving as ceiling). It bounces successively off: floor, ceiling, floor, ceiling. . .

$$V \qquad V' = \tfrac{3}{7}V \qquad V'' = -\tfrac{31}{49}V \qquad V''' = -\tfrac{333}{343}V \ldots$$

$$\Omega = 0 \qquad \Omega' = \tfrac{10}{7R}V \qquad \Omega'' = -\tfrac{60}{49R}V \qquad \Omega''' = -\tfrac{130}{343R}V \ldots$$

The superball comes back out from under the table.



4.7. Lay a boiled egg, and give it a vigorous spin. It rises and spins upright (picture h). In fact, just about any convex object spun on a frictional surface tends to raise its centre of mass.

The simplest model of this phenomenon is as follows. To a hoop affix a wad of clay, and set it spinning about its diameter with the clay at the bottom. As the hoop spins, the clay rises to the top. In picture (i), the clay shifted the centre of mass C off the centre of curvature K of the hoop of radius R. The hoop plus the clay have mass M and a roughly spherical inertia matrix I around C. Gravity Mg presses the hoop down, provoking friction μMg at the point $\otimes$ directly beneath K. The angular momentum L around C is roughly vertical. In the configuration of picture (i), the spin plunges $\otimes$ *into* the page, so the friction protrudes *out* at $\otimes$. Its torque N around C is roughly horizontal. N makes L tremble, but because N whirls rapidly about L during the spin, L varies little on a long time scale—as observed in experiments.

We analyse the change in time of θ, the angle between L and the axis CK. For the component of L along CK (3.2),

$$\frac{d}{dt}|L|\cos\theta = -|N|\sin\theta\,.$$

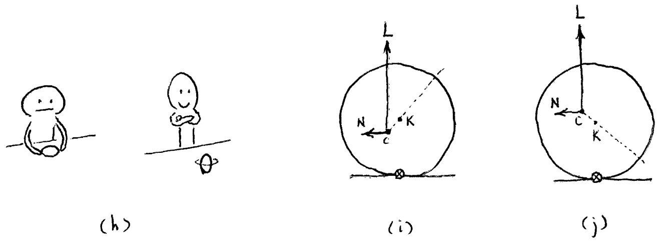
In the approximation of constant L

$$\frac{d\theta}{dt} \sim \frac{|N|}{|L|} \sim \frac{R\mu Mg}{I|\Omega|} > 0\,,$$

where in the same approximation Ω is the initial angular velocity given to the hoop. θ *increases*, which means CK *rises*. Lest readers worry what ensues once CK is horizontal, in picture (j) too θ goes on increasing; this shows incidentally that centrifugal force alone does not explain the phenomenon. The hoop tips over in time

$$\pi \Big/ \frac{d\theta}{dt} \sim \frac{\pi I|\Omega|}{R\mu Mg}\,.$$

For a commercially available **tippy top**, a wooden off-centered ball-like top, our theoretical value for the tip-over time is of the order of $\pi \frac{2}{5} MR^2 |\Omega| / R\mu Mg \sim \pi \frac{2}{5} \cdot 2\,\mathrm{cm} \cdot 2\pi\, 50\,\mathrm{Hz} / \frac{1}{3} \cdot 1000\,\mathrm{cm\,sec^{-2}} \sim 2\,\mathrm{sec}$.



The seemingly reckless approximations above are justifiable by a more precise analysis. For a physically important example, if you spin an egg too sluggishly, it rises only part of the way; the reason is that sliding at $\otimes$ transits to rolling and the friction coefficient μ drops. A precise analysis handles the sliding/rolling transition, among other things.

I also announce, for the first time in the literature, the existence of *chiral tippy tops*, which tip over when spun one way but not when spun the opposite way. They indicate that some crucial physical insight is missing from all previous theories of tippy top, none of which accommodates, let alone predicts, any chirality. I plan to publish a full discussion soon.

4.8. Too many books already treat gyroscopes.

4.9. How does a yo-yo work?

4.10. When leaves stop falling, fall starts leaving. Most falling leaves dance to and fro, zigzagging randomly earthbound. But there are some elongated leaves that spin busily about the long axis and fall along a fairly straight trajectory; the angular velocity is very large and roughly horizontal, the direction of the fall is roughly perpendicular to the angular velocity. Ditto for rectangular strips of paper: beyond a certain aspect ratio of the rectangle, they "tumble rather than flutter". Why?

5 Euler's Equations

5.1. A top T of inertia matrix I around a point O is spinning at angular velocity Ω around O. Let e_1, e_2, e_3 be the orthonormal basis vectors that define coordinates x_1, x_2, x_3 attached to T whose origin is at O. For any vector-valued function $A = A(t) = A_1 e_1 + A_2 e_2 + A_3 e_3$,

$$\frac{d}{dt} A = \Big(\frac{dA_1}{dt} e_1 + \frac{dA_2}{dt} e_2 + \frac{dA_3}{dt} e_3 \Big) + \Big(A_1 \frac{de_1}{dt} + A_2 \frac{de_2}{dt} + A_3 \frac{de_3}{dt} \Big) .$$

We shall denote the first $(\cdots)$ by $\frac{\partial}{\partial t}A$; on account of $\frac{d}{dt}e_i = \Omega \wedge e_i$, the second $(\cdots)$ is $\Omega \wedge A$. Symbolically,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \Omega \wedge \ .$$

Writing out $\frac{d}{dt}L = \frac{\partial}{\partial t}L + \Omega \wedge L = I\frac{\partial}{\partial t}\Omega + \Omega \wedge I\Omega = N$ (3.2) with respect to principal axes, we obtain **Euler's equations** [5]

$$\begin{aligned} I_1 \frac{\partial}{\partial t}\Omega_1 &= (I_2 - I_3)\Omega_2\Omega_3 + N_1 \\ I_2 \frac{\partial}{\partial t}\Omega_2 &= (I_3 - I_1)\Omega_3\Omega_1 + N_2 \\ I_3 \frac{\partial}{\partial t}\Omega_3 &= (I_1 - I_2)\Omega_1\Omega_2 + N_3 \,, \end{aligned}$$

the torque N being around O. Though something of an elephant in a china shop when applied to concrete problems, Euler's equations are effective in theoretical investigations: cf. Sects. 6, 7, 8.

5.2. Euler's equations in hydrodynamics for an ideal fluid are interpretable as Euler's equations for an infinite-dimensional rigid body [2].

5.3. Essentially three kinds of tops have been studied in the literature:

- Euler's top
- Lagrange's top
- Kovalevskaya's top.

Moreover, it is a theorem that these tops and these alone are *algebraically integrable*. We shall study them in turn: Euler in Sect. 6, Lagrange in Sect. 7, Kovalevskaya in Sect. 8.

6 Spinning under No Torque: Euler's Top

6.1. Throughout this section, the force and the torque are absent

$$F = 0, \qquad N = 0,$$

which implies constant momentum, angular momentum, energy; modulo a Galilean transformation we may even assume that P is zero:

$$P = 0, \qquad L = I\Omega = \text{const.}, \qquad E = E_{\text{rot}} = \text{const.}$$

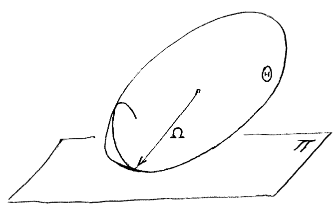
Such a rigid body, in "free rotation" around its immobile centre of mass, is called **Euler's top** [6]. Isolated celestial bodies are examples, as are gyroscopes supported at their centres of mass. We describe the motion of Euler's top in two ways: pictorial (6.2, 6.3) and analytical (6.4, 6.5, 6.6, 6.7).

6.2. Poinsot [13] devised a pictorial description of Euler's top. The ingredients of the picture are built from the constants of the top: matrix I, scalar E, vector L. The description revolves around the distinction between L and Ω (2.2): L is constant but in general Ω moves.

Imagine an ellipsoid *attached to the top*

$$\Theta \, : \, \langle Ix, x \rangle = 2E$$

and a plane *fixed in space*



$$\Pi \, : \, \langle L, x \rangle = 2E \, .$$

The trick now is to consider the point $x = \Omega$ of Θ. On one hand, the tangent plane to Θ at $x = \Omega$ is Π (its equation being $2\langle I\Omega, x\rangle = 2\langle I\Omega, \Omega\rangle$). On the other hand, since the top is instantaneously spinning about Ω, $x = \Omega$ is instantaneously at rest. These together mean that

Euler's top moves as if the ellipsoid Θ *were rolling on the plane* Π.

The curve traced on Θ [resp. Π] by the point of rolling contact $x = \Omega$ is the **polhode** [resp. **herpolhode**]. In principle the motion of the top can be reconstructed from the polhode.

6.3. With respect to principal axes

$$E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2) \, , \qquad L^2 = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2 \, ,$$

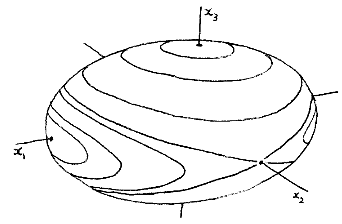
so

$$\text{polhode} = \{E = \text{const.}\} \cap \{L^2 = \text{const.}\}.$$

Switching to the variables L_1, L_2, L_3 facilitates visualisation:

$$\text{polhode} = \left\{ \frac{L_1^2}{2EI_1} + \frac{L_2^2}{2EI_2} + \frac{L_3^2}{2EI_3} = 1 \right\} \cap \left\{ L_1^2 + L_2^2 + L_3^2 = L^2 \ \text{(const.)} \right\} ,$$

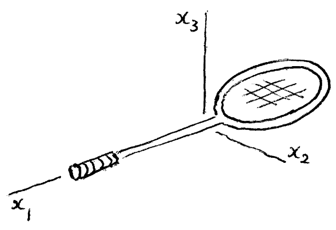
i.e. a polhode is a curve along which an ellipsoid and a sphere intersect. As various values of L and E are picked, a family of such curves are cut out. The choice of an initial condition puts Ω on one of these curves, and from then on Ω follows that curve.



The picture has been drawn assuming $I_1 < I_2 < I_3$. It shows that a polhode starting near the x_3- or x_1-axis dawdles near that axis, whereas a polhode starting near the x_2-axis wanders far from that axis and swings over to the other side of the ellipsoid.

Suppose $I_1 < I_2 < I_3$. Then the rotation of the top is stable about x_3 and x_1, unstable about x_2.

This stability result is nicknamed "tennis racket theorem": a racket tossed spinning is easy to catch if spun about x_3 or x_1, but it wobbles out of control if spun about x_2.



Poinsot's picture tells us the *trajectory* of Euler's top. What it leaves untold is at what pace the top follows the trajectory *in the course of time.* The time-evolution is rendered explicit by the analytical description. We analyse cases of increasing generality.

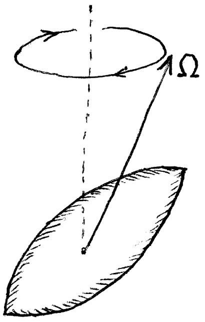
6.4. Case of a *spherical* top, $I_1 = I_2 = I_3$. Euler's equations (5.1) reduce to $\frac{\partial}{\partial t}\Omega = 0$, $\Omega = \text{const.}$: the top continues to spin about the same axis at the same rate—quite uneventful.

6.5. Case of a *symmetric* top, $I_1 = I_2 \neq I_3$—slightly more eventful. Euler's equations (5.1) may be recast as

$$\frac{\partial}{\partial t}(\Omega_1 + i\Omega_2) = i\Omega_3\Big(\frac{I_3}{I_1} - 1\Big)(\Omega_1 + i\Omega_2), \qquad \frac{\partial}{\partial t}\Omega_3 = 0\ ,$$

which integrate to

$$\Omega_1 + i\Omega_2 = (\Omega_1(0) + i\Omega_2(0))\exp\Big[i\Omega_3(0)\Big(\frac{I_3}{I_1} - 1\Big)t\Big]\,, \qquad \Omega_3 = \Omega_3(0) \quad (\text{const.}).$$



The top **precesses** with period $2\pi/|\Omega_3(0)(I_3/I_1 - 1)|$; the period does not depend on Ω_1, Ω_2, i.e. not on how widely Ω is tilted away from $(0, 0, \Omega_3)$.

Since the oblateness of the Earth (extra bulge at the equator) is $I_3/I_1 \sim 301/300$, and $\Omega_3 = 2\pi/1$ day, our theoretical value for the precession period of the Earth is $\sim$ 300 days. The observed value, the "Chandler period", is $\sim$440 days.

The limit $I_3 \to I_1$ yields $I_3/I_1 - 1 \to 0$, trigonometric functions degenerate to costants, recovering the spherical case (6.4).

6.6. Generic case of Euler's top. It turns out the problem is integrable in terms of Jacobian elliptic functions [8] (reference on elliptic functions: [9]).

Recall the conservation laws

$$E = \frac{1}{2}(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2)\,, \qquad L^2 = I_1^2\Omega_1^2 + I_2^2\Omega_2^2 + I_3^2\Omega_3^2\,.$$

The principal moments of inertia are all distinct, say $I_1 < I_2 < I_3$. Then $I_1 < L^2/2E < I_3$. In the picture (6.3), the separatrices slice the ellipsoid into 4 eye-shaped sectors $\Omega_3 > 0$, $\Omega_3 < 0$ and $\Omega_1 > 0$, $\Omega_1 < 0$, the former two satisfying $L^2/2E > I_2$ and the latter two $L^2/2E < I_2$. Let us analyse a motion during which Ω_3 keeps a constant sign (for Ω_1 constant sign swap the indices 3 and 1). Extracting Ω_3^2, Ω_1^2 between the conservation laws,

$$\Omega_3^2 = \frac{L^2 - 2EI_1 - (I_2 - I_1)I_2\Omega_2^2}{(I_3 - I_1)I_3}\,, \qquad \Omega_1^2 = \frac{L^2 - 2EI_3 - (I_2 - I_3)I_2\Omega_2^2}{(I_1 - I_3)I_1}\,,$$

which separate the second of Euler's equations (5.1)

$$\frac{\partial}{\partial t}\Omega_2 = \frac{I_3 - I_1}{I_2}\Omega_3\Omega_1 = \sqrt{\text{polynomial of degree 4 in } \Omega_2}\,.$$

In rescaled variables

$$\tau = t\sqrt{\frac{(I_3 - I_2)(L^2 - 2EI_1)}{I_1 I_2 I_3}}, \qquad \omega = \Omega_2\sqrt{\frac{(I_2 - I_3)I_2}{L^2 - 2EI_3}}$$

and a new constant (*modulus*)

$$k^2 = \frac{(I_1 - I_2)(L^2 - 2EI_3)}{(I_3 - I_2)(L^2 - 2EI_1)} \qquad (0 < k^2 < 1)\,,$$

the equation $\frac{\partial}{\partial t}\Omega_2 = \cdots$ integrates to

$$\tau = \int_0^\omega \frac{d\omega}{\sqrt{(1-\omega^2)(1-k^2\omega^2)}}\,.$$

Inverting, we find $\omega = \operatorname{sn}\tau$, which as a function of t determines Ω_2 and thereby Ω_3, Ω_1 :

$$\Omega_1 = \sqrt{\frac{L^2 - 2EI_3}{(I_1 - I_3)I_1}}\operatorname{cn}\tau\,, \quad \Omega_2 = \sqrt{\frac{L^2 - 2EI_3}{(I_2 - I_3)I_2}}\operatorname{sn}\tau\,, \quad \Omega_3 = \sqrt{\frac{L^2 - 2EI_1}{(I_3 - I_1)I_3}}\operatorname{dn}\tau\,.$$

The period in t is

$$4K(k)\sqrt{\frac{I_1 I_2 I_3}{(I_3 - I_2)(L^2 - 2EI_1)}}\,.$$

The limit $I_2 \to I_1$ yields $k^2 \to 0$, elliptic functions degenerate to trigonometric ones, recovering the symmetric case (6.5).

6.7. Tennis racket revisited. Earlier the stability result (6.3) was deduced pictorially. Analytically it could be excavated from the exact solution (6.6). More cheaply, perturb $\Omega = (0, \Omega_2(0), 0)$, a steady rotation about x_2, to $(\Delta\Omega_1, \Omega_2(0) + \Delta\Omega_2, \Delta\Omega_3)$. Neglecting terms of order Δ^2 or higher in Euler's equations (5.1),

$$\frac{\partial}{\partial t}\Omega_2 = 0, \quad \frac{\partial^2}{\partial t^2}\Delta\Omega_i = \lambda\Delta\Omega_i (i = 3, 1) \text{ with } \lambda = (I_1 - I_2)(I_2 - I_3)/I_3 I_1 > 0\,.$$

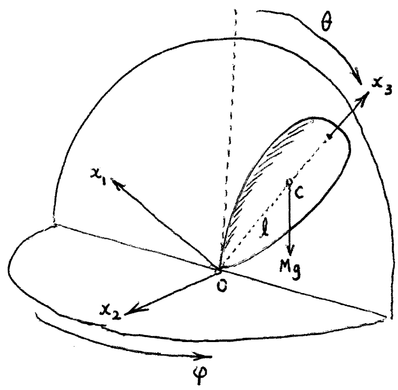
Unless the perturbation puts Ω on an incoming separatrix in Poinsot's picture (6.3), $\Delta\Omega_i$ contains an exponential term with exponent $+\sqrt{\lambda} > 0$, so rotation about x_2 is *unstable*. Similarly rotation about x_3 or x_1 is *stable*.

6.8. It is no accident that integrable problems involve elliptic—or rather theta—functions, for geometrically integrability means foliation of the phase space into invariant tori, and theta functions are the very creatures, via Abel-Jacobi embeddings, that give us holomorphic functions on a torus. But I digress.

7 Some Cases of Spinning under Torques: Lagrange's Top

7.1. This section studies a top friendlier than Euler's but in a more hostile environment: **Lagrange's top** [12] is symmetric, $I_1 = I_2$, pivoted at a point *on the axis of symmetry* but *not at the centre of mass* and spinning under gravity. Gravity acts at the centre of mass and exerts a torque around the pivot. This comes closer to a realistic model of a top in the colloquial sense of a conical toy we play with. As with Euler's top (6.6), Lagrange's top is integrable in terms of elliptic functions.

7.2. Lagrange's top T of mass M is spinning, tilted at an angle θ (colatitude) from the vertical. T swings about the vertical by an angle φ (longitude). Let ℓ be the distance from the pivot O to the centre of mass C of T. At the instant under consideration, take x_3 along the top's axis of symmetry, x_2 horizontal and perpendicular to x_3, x_1 perpendicular to the x_2x_3-plane, the axes having their origin at O. The inertia matrix I is around O, not around C.



Since gravity exerts zero torque about x_3 and about the vertical, L_3 and the vertical component L_{vert} of L are conserved (cf. Euler's equations (5.1) with $I_1 = I_2$):

$$\begin{aligned} L_3 &= I_3\Omega_3 = \text{const.}, \\ L_{\text{vert}} &= I_1\Omega_1 \sin\theta + I_3\Omega_3\cos\theta = I_1\frac{d\varphi}{dt}\sin^2\theta + L_3\cos\theta = \text{const.} \end{aligned}$$

The conservation of energy (3.3) now includes the potential energy due to gravity:

$$\begin{aligned} E &= \frac{1}{2}I_1(\Omega_1^2 + \Omega_2^2) + \frac{1}{2}I_3\Omega_3^2 + \text{potential} \\ &= \frac{1}{2}I_1\left[\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{d\varphi}{dt}\right)^2\sin^2\theta\right] + \frac{L_3^2}{2I_3} + Mg\ell\cos\theta = \text{const.} \end{aligned}$$

Eliminate $d\varphi/dt$ between the conservation laws; in a new variable

$$h = \cos\theta$$

we get

$$I_1^2\Big(\frac{dh}{dt}\Big)^2 = 2I_1\Big(E - \frac{L_3^2}{2I_3} - Mg\ell h\Big)(1-h^2) - (L_{\text{vert}} - L_3 h)^2\,.$$

The right-hand side $f(h)$ is a cubic polynomial in h, with roots say h_1, h_2, h_3. The equation integrates to

$$h = h_1 + (h_2 - h_1)\,\mathrm{sn}^2\left(t\sqrt{\frac{Mg\ell(h_3-h_1)}{2I_1}}\right)$$

with modulus

$$k^2 = \frac{h_2 - h_1}{h_3 - h_1}\,.$$

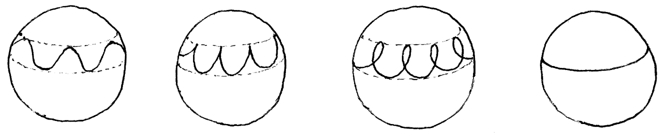
This determines h, thereby θ, as a periodic function of t, **nutation**; its period is

$$2K(k)\sqrt{\frac{2I_1}{Mg\ell(h_3-h_1)}}\,.$$

In its turn, φ is determined as an elliptic integral

$$\varphi = \int_0^h \frac{L_{\text{vert}} - L_3 h}{(1-h^2)\sqrt{f(h)}}dh\,.$$

Generically the axis of symmetry of T traces waves (picture k) or swirls (picture m).



(k) (l) (m) (n)

7.3. In *pure precessions*, i.e. precessions with zero nutation (picture n), θ, or h, is constant, so $h_2 - h_1 = 0$. Therefore pure precessions are sustained at a tilt angle $\theta_{\text{pr}} = \arccos h_{\text{pr}}$ that satisfies the double-root condition

$$f(h_{\text{pr}}) = f'(h_{\text{pr}}) = 0\,.$$

Combining this with the conservation laws leads to

$$I_1 \cos\theta_{\text{pr}}\Big(\frac{d\varphi_{\text{pr}}}{dt}\Big)^2 - L_3\frac{d\varphi_{\text{pr}}}{dt} + Mg\ell = 0\,,$$

an equation quadratic in the rate of pure precession $d\varphi_{\text{pr}}/dt$. Suppose that "spin overwhelms gravity": $L_3^2 \gg I_1 \cos\theta_{\text{pr}} Mg\ell$. The binomial expansion of the roots then yields

$$\text{slow precession} \quad \frac{d\varphi_{\text{pr}}}{dt} \sim \frac{Mg\ell}{L_3}, \qquad \text{fast precession} \quad \frac{d\varphi_{\text{pr}}}{dt} \sim \frac{L_3}{I_1 \cos\theta_{\text{pr}}}.$$

The fast precession tends to be damped away quickly.

7.4. If a spinning top is released from a tilted position, it *dips* at first, then goes into precession and nutation (picture l). The graph of θ against φ is approximately a cycloid. In a real top, as friction at the pivot damps the nutation, the motion asymptotes to a pure precession.

7.5. If a spinning top is released upright, $\theta = 0$, $h = 1$, it may be able to stay upright; this is the **sleeping top**. A sleeping top is stable provided $f(h) < 0$ near $h = 1$, i.e.

$$\Omega^2 = \Omega_3^2 > \frac{4I_1 Mg\ell}{I_3^2}.$$

So a top needs to be spun sufficiently fast to go to sleep. In a real top, friction decelerates Ω ; when eventually Ω violates the above inequality, the top *wakes up* and goes into precession and nutation. Conversely, if a top is spun sufficiently fast, even from a tilted position it snaps upright and goes to sleep, by the tippy-top mechanism (4.7).

7.6. In the limit $I_3 \to 0$, Lagrange's top degenerates to a spherical pendulum. As a corollary a spherical pendulum is integrable in terms of elliptic functions.

8 Kovalevskaya's Top

8.1. Our final top T also spins under gravity. As with Lagrange's top (7.1), pivot T at a point O not its centre of mass C and take x_1, x_2, x_3 principal axes attached to T with their origin at O. The inertia matrix I is around O. Let (C_1, C_2, C_3) be the (constant) coordinates of the centre of mass, (z_1, z_2, z_3) the (variable) components of the upward unit vector z. Euler's equations (5.1) are

$$\begin{aligned}
I_1 \frac{\partial}{\partial t}\Omega_1 &= (I_2 - I_3)\Omega_2\Omega_3 - Mg(C_2 z_3 - C_3 z_2) \\
I_2 \frac{\partial}{\partial t}\Omega_2 &= (I_3 - I_1)\Omega_3\Omega_1 - Mg(C_3 z_1 - C_1 z_3) \\
I_3 \frac{\partial}{\partial t}\Omega_3 &= (I_1 - I_2)\Omega_1\Omega_2 - Mg(C_1 z_2 - C_2 z_1).
\end{aligned}$$

T has 3 degrees of freedom and 2 conserved quantities E, L_{vert} (7.2). In comparison with Lagrange's top, we *lose the conserved quantity* L_3 because we are no longer assuming that OC is an axis of symmetry of T. In order to integrate the problem, we need 1 more conserved quantity. **Kovalevskaya's top** [11] is exactly rigged so as to allow the existence of a third conserved quantity.

Kovalevskaya's top: $I_1 = I_2 = 2I_3$
and the centre of mass C *is on the* x_1x_2*-plane.*

E.g. a homogeneous ellipsoid with semiaxes $1, \sqrt{3}, 3$ pivoted on the x_1-axis at a distance $\sqrt{2/5}$ from the centre.

8.2. Without loss of generality set $I_1 = I_2 = 2, I_3 = 1$, $C_2 = C_3 = 0$. Euler's equations (8.1) become

$$\begin{aligned} 2\frac{\partial}{\partial t}\Omega_1 &= \Omega_2\Omega_3 \\ 2\frac{\partial}{\partial t}\Omega_2 &= -\Omega_3\Omega_1 \quad +MgC_1z_3 \\ \frac{\partial}{\partial t}\Omega_3 &= \qquad\qquad\; -MgC_1z_2\ . \end{aligned}$$

Writing out $0 = \frac{d}{dt}z = \frac{\partial}{\partial t}z + \Omega \wedge z$ (5.1) in coordinates,

$$\frac{\partial}{\partial t}z_1 = z_2\Omega_3 - z_3\Omega_2, \qquad \frac{\partial}{\partial t}z_2 = z_3\Omega_1 - z_1\Omega_3, \qquad \frac{\partial}{\partial t}z_3 = z_1\Omega_2 - z_2\Omega_1\ .$$

Claim:
Kovalevskaya's top has the conserved quantity $|(\Omega_1 + i\Omega_2)^2 - MgC_1(z_1 + iz_2)|$.
Indeed,

$$\begin{aligned} 2\frac{\partial}{\partial t}(\Omega_1 + i\Omega_2) &= -i[(\Omega_1 + i\Omega_2)\Omega_3 - MgC_1z_3]\ , \\ \frac{\partial}{\partial t}(z_1 + iz_2) &= -i[(z_1 + iz_2)\Omega_3 - z_3(\Omega_1 + i\Omega_2)]\ , \end{aligned}$$

therefore

$$\frac{\partial}{\partial t}\{(\Omega_1 + i\Omega_2)^2 - MgC_1(z_1 + iz_2)\} = -i\Omega_3\{(\Omega_1 + i\Omega_2)^2 - MgC_1(z_1 + iz_2)\}\ .$$

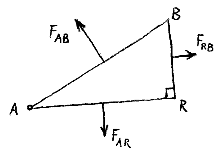
Since the velocity of $\{\cdots\}$ is perpendicular to $\{\cdots\}$, the absolute value $|\ \{\cdots\}\ | = \text{const.}$

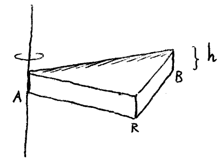
The integration is completed in terms of hyperelliptic functions [7]. In the limit $C_1 \to 0$, Kovalevskaya's top degenerates to a special case of Lagrange's top.

8.3. Kovalevskaya's top was the last integrable system of the 19th century. The discovery of the next integrable system had to wait 78 years, until Toda lattices arrived on the scene [16].

9 Appendix

9.1. Let ARB be a right triangle. We wish to prove that $AR^2 + RB^2 = AB^2$. Upon ARB as base build a box of height h and hinge it at A to a vertical axis, around which it can revolve smoothly.





Now fill the box with gas of pressure p. The gas exerts forces that may be regarded as acting at the centre of, and normal to, each face of the box. The forces on the lid and the bottom don't interest us. The forces F_{AR}, F_{RB} on the sides AR, RB try to revolve the box clockwise, whereas the force F_{AB} on the side AB tries to revolve it anticlockwise. But filling with gas can't coax a box into moving: the torques about the axis must balance. The torques due to F_{AR}, F_{AB} are $AR/2 \times F_{AR}, AB/2 \times F_{AB}$, and *because R is a right angle* the torque due to F_{RB} is $RB/2 \times F_{RB}$:

$$\frac{AR}{2} \times F_{AR} + \frac{RB}{2} \times F_{RB} = \frac{AB}{2} \times F_{AB}.$$

Force is pressure times area, $F_{AR} = phAR$, etc. Dividing through by $ph/2$, we are home.

9.2. Recycling the argument on a not necessarily right triangle proves the "cosine law".

10 Further Reading and Acknowledgement

Dynamics of rigid bodies in rotation is a staple diet of textbooks on mechanics [1]. Among specialised monographs, the richest cache of examples is [14, 15]. *μέγα βιβλίον μέγα κακόν* to [10], though admittedly it makes available material not collected elsewhere. [4] is elementary and charming; inevitably for elementary charming books, it is out of print. [7] exposes the relationship between spinning tops and elliptic/theta functions. To acquaint yourself with the current mathematical take on the subject, [3].

This article reproduces lectures, minus toy demonstrations, from the CNRS *école d'hiver* at Lanslevillard, March 2003. I thank its organiser J. Souchay for his kind invitation and J. Laskar for first suggesting that these lectures be given. I am also obliged to R. and D. Gonczi for their hospitality in Nice.

References

1. P. Appell, *Traité de Mécanique Rationnelle*, tome II, Gauthier-Villars, 1896; repr. J. Gabay, 1991.
2. V. Arnold, Ann. Inst. Fourier, **16** (1966) 319–361.

3. M. Audin, *Spinning Tops*, Cambridge UP, 1996.
4. H. Crabtree, *Elementary Treatment of the Theory of Spinning Tops and Gyroscopic Motion*, Longmans, 1909; repr. Chelsea, 1967.
5. L. Euler, Mém. de l'Acad. Sci. Berlin **14** (1758) 154–193 ≃ *Opera* II **7**, 200–235.
6. L. Euler, *Theoria Motus Corporum Solidorum seu Rigidorum…*, A. F. Roese, 1765 ≃ *Opera* II **3**, **4**.
7. V. V. Golubev, *Lektsii po Integrirovaniyu Uravneniĭ Dvizheniya Tyazhelogo Tverdogo Tela okolo Nepodvizhnoĭ Tochki*, Gostekhizdat, 1953.
8. C. G. J. Jacobi, Crelle **39** (1849) 293–350 ≃ *Werke* **2**, 289–352.
9. C. Jordan, *Cours d'Analyse*, tome II, Gauthier-Villars, 2e ed. 1894; repr. J. Gabay, 1991.
10. F. Klein & A. Sommerfeld, *Über die Theorie des Kreisels*, Band I–IV, Teubner, 1897–1910.
11. S. Kowalevski, Acta Math. **12** (1889) 177–232.
12. J.-L. Lagrange, *Méchanique Analitique*, Veuve Desaint, 1788 ≃ *Œuvres* **11**, **12**; repr. J. Gabay, 1989.
13. L. Poinsot, *Théorie Nouvelle de la Rotation des Corps*, Paris, 1834; repr. J. Math. Pures Appl., série I, **16** (1851) 9–129, 289–336.
14. E. J. Routh, *Dynamics of a System of Rigid Bodies: Elementary Part*, Macmillan, 1860; repr. Dover, 1960.
15. E. J. Routh, *Dynamics of a System of Rigid Bodies: Advanced Part*, Macmillan, 1860; repr. Dover, 1955.
16. M. Toda, J. Phys. Soc. Japan **22** (1967) 431–436.