In addition to topics related to celestial mechanics, this conference brought together researchers from a wide spectrum of areas of contemporary research in Hamiltonian dynamics. Just a small sample includes the existence of periodic orbits with variation methods, twist and annulus maps, stable manifold theory, almost periodic motion, heteroclinic and homoclinic orbits, etc. It is our hope that by bringing together papers from such a diverse range of topics will serve as a stimulant for further development in Hamiltonian dyamics.

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March, 1988

Contemporary Mathematics
Volume 81, 1988

Some Qualitative Features of the Three-Body Problem

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1.Introduction. This paper is a survey of certain features of the three-body problem that I find particularly appealing. The emphasis will be on presenting the "big picture" of what is going on in the three-body problem. It is shown in figure 1.

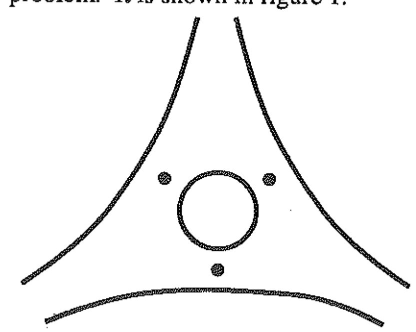

FIGURE 1: The Three-Body Problem

This picture was first drawn for me by Charles Conley (probably on a napkin at the Gourmandaise restaurant in Madison, Wisconsin) when I was a graduate student and I have spent a good deal of time since then trying to figure it out. It turns out that it is somewhat oversimplified but it captures the main features !

Looking at figure 1, the expert will wonder which three-body problem it depicts. We will consider only the planar three-body problem with unrestricted masses. Many interesting results about the restricted problem are omitted. One of the main purposes of this paper and of the lecture from which it derives is to show how little is really known about this problem. Thus as a counterpoint to the theorems we will list many open problems (some of which may actually be solvable).

2.The Equations. The planar three-body problem concerns the motion of three point masses in a plane under the influence of their mutual gravitational attraction. We let $q_j \in \mathbf{R}^2$

* Research supported by the National Science Foundation

stand for the position of the j^{th} point, $p_j \in R^2$ for its momentum and $m_j \in R^+$ for its mass. The system is governed by the Hamiltonian function:

$$H(q,p) = \frac{1}{2} p^T M^{-1} p - U(q)$$

where $q = (q_1, q_2, q_3) \in R^6$, $p = (p_1, p_2, p_3) \in R^6$, $M = \mathrm{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$ and:

$$U(q) = \frac{m_1 m_2}{|q_1 - q_2|} + \frac{m_1 m_3}{|q_1 - q_3|} + \frac{m_2 m_3}{|q_2 - q_3|} .$$

U(q) is minus the Newtonian gravitational potential energy. Hamilton's differential equations are:

$$\dot{q} = M^{-1} p \qquad (2.1)$$

$$\dot{p} = \nabla U(q) .$$

These equations define a dynamical system in R^{12}. However, it is possible to reduce the problem to a five-dimensional system by making use of the well-known integrals of motion.

The first integral is the total momentum. We assume without loss of generality that:

$$p_1 + p_2 + p_3 = 0 .$$

This assumtion implies that the center of mass will be constant and we can take it to be the origin:

$$m_1 q_1 + m_2 q_2 + m_3 q_3 = 0 .$$

These equations restrict the momentum vector,p, and position vector,q, to four-dimensional subspaces of R^6 so together they reduce the dimension of the system by 4.

Next we consider angular momentum. The equations 2.1 are invariant under simultaneous rotation of all positions and momenta in R^2. As a result, total angular momentum is constant:

$$p_1 \times q_1 + p_2 \times q_2 + p_3 \times q_3 = \omega .$$

Here we view the cross products as scalars. This reduces the dimension of the system by 1. Since the system is symmetric under rotations, we can pass to a quotient space in which all vectors (q,p) which differ only by a simultaneous rotation of all q_j and p_j are identified. This eliminates 1 more dimension.

Finally, the Hamiltonian itself is the total energy of the system and is conserved:

$$H(q,p) = \frac{1}{2} p^T M^{-1} p - U(q) = h .$$

This eliminates 1 more dimension. All of these equations together define a five-dimensional manifold, $M(h,\omega)$, the quotiented energy and angular momentum manifold.

The topology of these manifolds depends on the energy, h, the angular momentum, ω, and the masses, m_j [Eas,Sm].

To facilitate the geometrical discussion below, it is convenient to introduce a coordinate system discovered by McGehee [Mc1]. Since the center of mass is at the origin, the moment of inertia about the origin plays a central role:

$$\mathfrak{I} = q^T M q = m_1 |q_1|^2 + m_2 |q_2|^2 + m_3 |q_3|^2 .$$

The variable $r = \sqrt{\mathfrak{I}}$ will be the radial variable of a kind of polar coordinate system in R^6. It is a measure of the size of the triangle formed by the three point masses; in particular, r = 0 represents a triple collision at the origin. The normalized position vector, $s = \frac{q}{r}$ measures the shape and angular position of the triangle. Note that by definition, s satisfies: $s^T M s = 1$. In the quotient manifold we lose information about the angle and we think of s as representing only shape. It turns out to be advantageous to normalize momentum differently. Define $z = \sqrt{r}\, p$. The variables (r,s,z) are superior to (q,p) because of their behavior near the triple collision singularity. The energy and angular momentum equations in these coordinates are:

$$H(s,z) = \frac{1}{2} z^T M^{-1} z - U(s) = h\, r \qquad (2.2)$$

$$z_1 \times s_1 + z_2 \times s_2 + z_3 \times s_3 = \omega \sqrt{r} .$$

We are able to factor the r dependence out of the Hamiltonian because of the homogeneity of the Newtonian potential function. The possibility of writing the energy equation in this way motivates the choice of scaling for the momentum. When the differential equations are expressed in these coordinates it is found that they contain a singular common factor of $r^{-\frac{3}{2}}$. Multiplying through by a factor of $r^{\frac{3}{2}}$ changes only the parametrization of solution curves; the result is:

$$r' = v\, r$$

$$s' = z - \frac{1}{2} v\, s \qquad (2.3)$$

$$z' = \nabla U(s) + \frac{1}{2} v\, z$$

where $v = s \cdot z$ and ' denotes differenttiation with respect to the new parameter. Note that the last two equations, describing the rate of change of the shape and normalized momentum, are independent of r.

3. Hill's Regions. We have reduced the dimension of the dynamical system describing the planar three-body problem from twelve dimensions to five. Unfortunately, five is still too many to sketch. Since the behavior of the size and shape of the triangle formed by the three bodies has a more direct intuitive meaning than the behavior of the momenta we will focus attention on the configuration space. Define:

$$C = \{ (r,s) : r \geq 0,\ s^T M s = 1,\ m_1 s_1 + m_2 s_2 + m_3 s_3 = 0 \} / S^1$$

the space of all admissable configurations with the rotation symmetry quotiented out. It is not difficult to see that this space is homeomorphic to $R^+ \times S^2$; the two equations in s define a three-dimensional ellipsoid in R^6 and the quotient space of this ellipsoid under the circle action is homeomorphic to a two-sphere. We can visualize this as in figure 2.

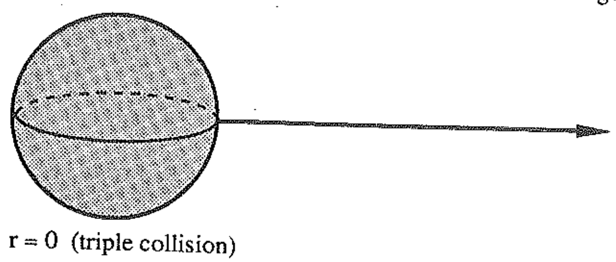

FIGURE 2: Configuration Space

Once again we note that r represents the size of the triangle formed by the three bodies while s represents its shape. A ray represents a family of similar triangle of varying size. We will draw the shape two-sphere in more detail (figure 3). There are several interesting features. First, the collinear "triangles" form a circle (depicted here as the equator) in the two-sphere. The isosceles triangles form three circles distinguished by which mass lies on the axis of symmetry. These three circles meet at the equilateral triangle configurations (shown here as the poles); note that there are two rotationally inequivalent equilateral triangles with the masses 1,2,3 appearing in either clockwise or counterclockwise fashion. Each circle of isosceles triangles intersects the collinear circle in two points; one represents a collinear configuration with one mass at the midpoint of the other two and the other represents a double collision configuration.

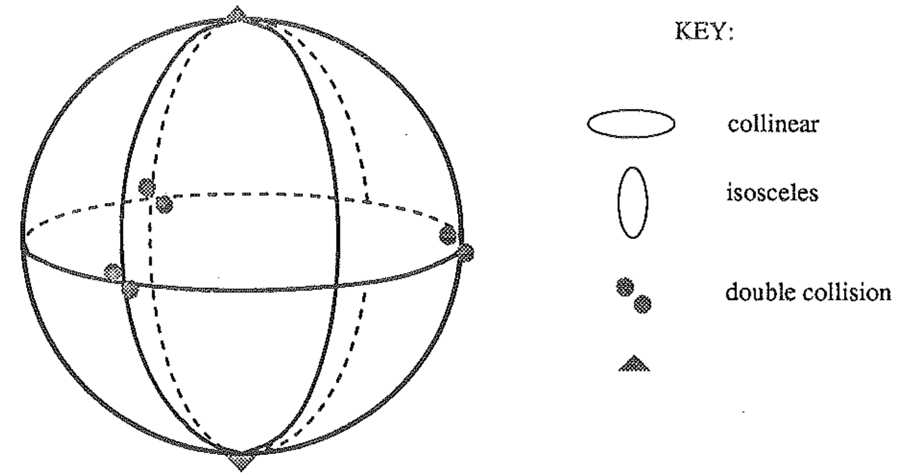

FIGURE 3 : The Shape Sphere

It was G.W. Hill who first realized that the energy and angular momentum integrals impose constraints on the configuration [H]. Although he worked in the restricted three-body problem, the idea is fruitful in the planar problem as well. Define the Hill's regions:

$$C(h,\omega) = \{ (r,s) \in C : \text{for some } z \in \mathbf{R}^6,\ (r,s,z) \in M(h,\omega) \}.$$

Thus $C(h,\omega)$ is the projection onto the configuration space of the integral manifold $M(h,\omega)$.

The Hill's regions, $C(h,\omega)$, will provide an organizing center for this paper. We will study how they vary as the energy and angular momentum are changed. For each choice of h and ω we will describe some features of the dynamical system on $M(h,\omega)$ and how they look in $C(h,\omega)$. The shapes of the Hill's regions will suggest several open problems.

We do not need to survey the entire two-parameter family of Hill's regions. First, we can restrict attention to negative energies: $h < 0$. If $h \geq 0$ all orbits scatter to infinity in both time directions, so no recurrence is possible and the problem holds little interest. Second, it is easy to show that the dynamics depends only on the quantity $\lambda = -h\,\omega^2$ (of course the dynamics also depends on the choice of masses). Thus to see the whole story it suffices to fix some $h < 0$ and let ω vary over $[0,\infty)$.

Before turning to the case by case description we derive the inequalitites characterizing Hill's regions. The energy and angular momentum equations 2.2 are the key to deriving these. The energy equation alone imposes some restrictions on the configuration; namely, since the kinetic energy term, $\frac{1}{2}z^TM^{-1}z$, is non-negative we have:

$$U(s) \geq |h|\, r \tag{3.1}$$

and so for any fixed shape s_0, the size is restricted to the range $0 \leq r \leq \frac{U(s_0)}{|h|}$. Thus all configurations arising from a state with energy h lie in the region shown in figure 4.

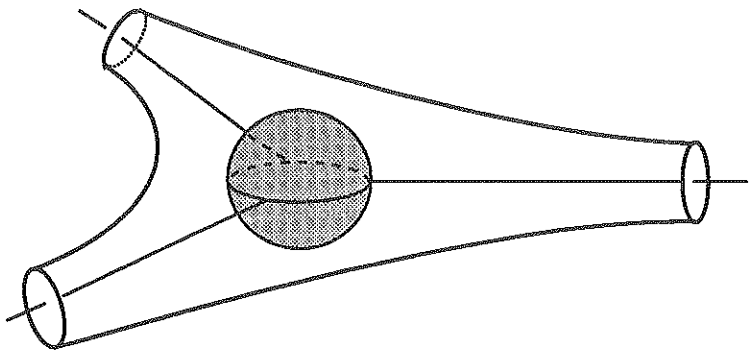

FIGURE 4 : Constraints due to energy

The potential function U(s) on the shape sphere has maximum value ∞ at the double collision configurations and attains minima at the equilateral configurations. Thus the energy imposes no restriction on the size of a double collision configuration but rules out all sufficiently large triangles with any other shape, the greatest restrictions being on the equilateral triangles.

The inequality 3.1 was derived from the observation that the kinetic energy is non-negative. One can show without difficulty that when the angular momentum is fixed the following sharper estimate holds:

$$\frac{1}{2}z^TM^{-1}z \geq \frac{1}{2}\frac{\omega^2}{r}.$$

When this is plugged into the energy equation we find:

$$U(s) \geq |h|\, r + \frac{\omega^2}{2r} \tag{3.2}$$

which characterizes C(h,ω).

C(h,ω) is a solid region in $R^+ \times S^2$. Its boundary is given by the equality in 3.2. Since this is a quadratic equation for r given s we see that ∂C(h,ω) lies in two sheets over

some subset of the shape two-sphere. The projection of ∂C(h,ω) to the two-sphere is the set of all s such that 3.2 holds for some $r \geq 0$. Minimizing the right side of 3.2 gives:

$$U(s) \geq \sqrt{2\,|h|\,\omega^2} = \sqrt{2\,\lambda} \tag{3.3}$$

which defines the projection. Thus the Hill's region lies over a region of the shape two-sphere bounded by an equipotential curve. Its boundary lies in two sheets over the projection and these two sheets come together over the equipotential curve. These observations will underlie the pictures which follow.

4. Large Angular Momentum. We will begin our survey with the case of large ω, or equivalently, large λ. Inequality 3.3 forces the shape, s, to lie in one of three disks around the double collision configurations (where U(s) = ∞). This means that two of the bodies are very close relative to their distance from the third body; we call this a tight binary configuration.The Hill's region consists of three lobes over the disks. It is shown in figure 5.

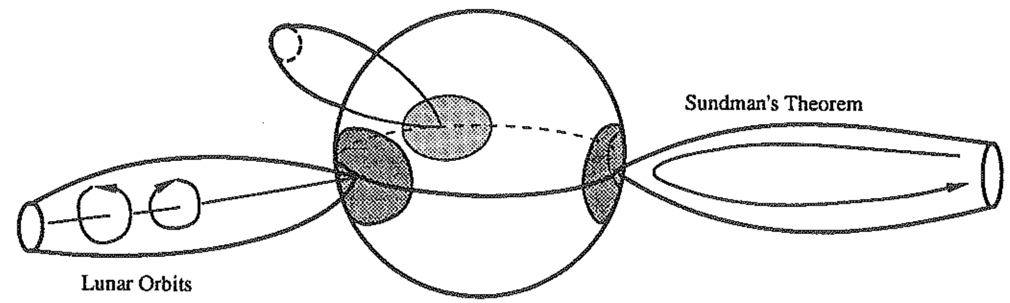

FIGURE 5 : Hill's Region for Large Angular Momentum

The lobes touch triple collision (r = 0) and infinity over the double collision configurations. The behavior of orbits near these two extremes of r is similar for all non-zero angular momenta so we will describe this before turning to the features specific to the large angular momentum case.

One of the nicest results in the whole theory is:

Theorem: Let the angular momentum be non-zero. Then any orbit passing sufficiently close to triple collision is of the following type: the configuration is a tight binary for all time and the short side of the triangle remains bounded while the other two sides tend to infinity in both forward and backward time.

This theorem is due to Sundman [Su] with refinements due to Birkhoff [Bir]. In particular, it implies that triple collision is impossible for $\omega \neq 0$. An orbit obeying Sundman's theorem is shown in the right lobe of figure 5. Although we will not draw them, such orbits occur in all of the pictures referring to non-zero angular momenta.

At the other end of the lobe we have the two-body problem at infinity. As $r \to \infty$, 3.1 shows that the configuration is forced into tight binary. Although it is conceivable a priori that all three sides of the triangle could become infinite, an appeal to the unrescaled energy equation shows that the short side remains bounded while the other two become infinite. It stands to reason that the influence of the third mass on the binary will become negligible and that the binary will behave essentially as a two-body problem. In fact, using rescalings similar to those in section 2 it is possible to paste a copy of the two-body problem onto each lobe of $M(h,\omega)$ at infinity [Mc2,Mc-Eas,Rob]. Actually, there are many two-body problems at infinity distinguished by the asymptotic speed of separation of the binary from the third mass. Intuitively, there are three cases. Either the third mass has just enough energy to escape from the binary and so reaches infinity with zero asymptotic speed (parabolic case) or it has plenty of energy and reaches infinity with positive asymptotic speed (hyperbolic case) or it does not have enough energy and returns for another approach to the binary (elliptic case). Clearly the parabolic case separates the other two. It is shown in the references above that the set of orbits tending parabolically to infinity forms a four-dimensional invariant manifold in $M(h,\omega)$ which we call the stable manifold of infinity (even though a whole open set of orbits tends to infinity hyperbolically). Similarly there is a four-dimensional unstable manifold of parabolic infinity. These can be viewed as the stable and unstable manifolds of an invariant three-sphere pasted onto $M(h,\omega)$, the parabolic two-body problem at infinity. These invariant sets are present for all angular momenta, even $\omega = 0$. Several open problems which can be posed for all ω concern these manifolds. Do there exist orbits homoclinic to parabolic infinity ? Do there exist orbits which tend parabolically to infinity in one time direction but which remain bounded in the other (capture or escape orbits) ? Do there exist orbits which oscillate to infinity, i.e., orbits with $\overline{\lim}\, r(t) = \infty$ but $\underline{\lim}\, r(t) < \infty$? Such orbits have been found in special cases of the three-body problem [Sit,Mos]. It is shown in [Mc-Eas] that if there are favorable homoclinic intersections of the invariant manifolds of parabolic infinity in the planar problem then capture/escape orbits and oscillation orbits also exist.

We turn now to those features which are specific to large angular momenta. First, recall that 3.3 forces the masses into a tight binary configuration from which they cannot escape. Thinking of the binary as the earth and moon and of the third body as the sun we see that the moon will always be a bounded distance from the earth and much closer to the earth than it is to the sun. This is a planar version of Hill's proof of the stability of the earth-moon system [H], one of the first applications of qualitative, geometrical reasoning to mechanics !

Continuing the earth-moon analogy we could look for "lunar" periodic orbits, that is, orbits such that the two bodies in the binary move in nearly circular orbits around their center of mass while the binary and the third mass move in nearly circular orbits around their center of mass. Such an orbit is called prograde if both circular motions have the same orientation and it is called retrograde if the orientations are different.

Theorem: For all sufficiently large angular momenta there is at least one prograde lunar orbit and at least one retrograde lunar orbit in each lobe of $M(h,\omega)$.

This result is due to Hill [H] in the restricted three-body problem and to Moulton [Mou1] in the planar problem. A nice proof can be found in [Mey]. A prograde and a retrograde lunar orbit are depicted in the front lobe of figure 5. The reason for requiring large ω is that if the configuration is a very tight binary then the third body can be viewed as a perturbation on the binary. It is an open question whether these orbits persist to lower angular momenta. In the restricted problem Conley [C] used the lunar orbits as the boundaries for an annular cross-section to the three-dimensional phase space. It is not clear how to generalize his work to the five-dimensional planar setting.

5. The First Critical ω. As we lower ω, the Hill's regions behave continuously until we reach a certain critical value. From the description in section 3 of how the Hill's regions lie over their projections to the shape sphere it is clear that bifurcations of the Hill's regions arise from bifurcations of the equipotential curves in the shape sphere. Thus critical values of ω

correspond to critical values of U(s). Critical points of U(s) are called central configurations or relative equilibria. We have already pointed out that the equilateral configurations are minima of U(s). There are also three collinear configurations which are saddle points of U(s). These are distinguished by the order in which the three bodies appear along the line; the exact spacing depends in a complicated way on the masses [Mou2]. The five central configurations of the three-body problem are shown in figure 6.

Each central configuration determines a restpoint in M(h,ω) for the corresponding critical ω. Actually these represent periodic orbits of the three-body problem for which the triangle formed by the three masses rotate rigidly around the center of mass; they appear as restpoints in M(h,ω) because we have quotiented out the rotational symmetry. These periodic orbits, which were discovered by Lagrange [Lag], are also shown in figure 8.

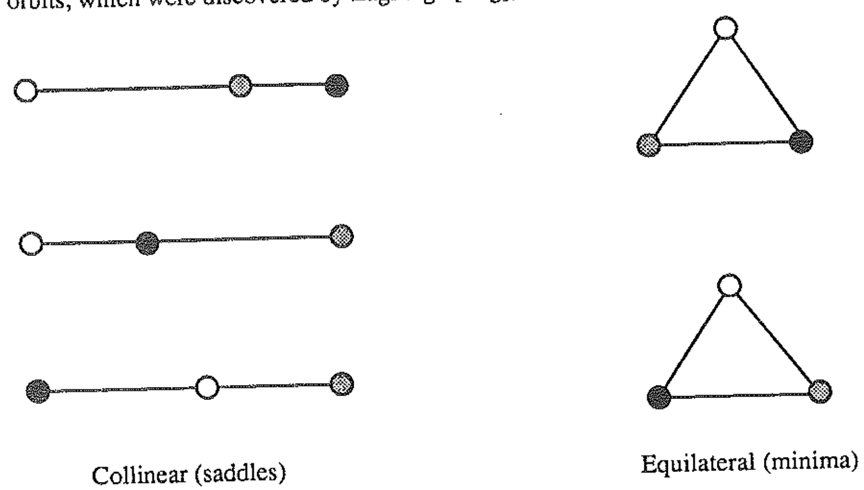

FIGURE 6 : Central Configurations

As we reach the first critical value of ω, two of the three disks around the double collisions in the shape sphere meet at one of the collinear saddle points, s_c. Over this saddle point at radius $\frac{U(s_c)}{2|h|}$, two of the three lobes of the Hill's region meet at the point representing the configuration of the Lagrangian periodic orbit. As we pass through the critical ω a tunnel opens between the two lobes. The critical Hill' region and tunnel are shown in figure 7. Other than the Lagrangian periodic orbit and the orbits arising from Sundman's theorem, not much is known about the dynamics for the critical angular

momentum. However, an interesting invariant set lives in the tunnel. As we pass through the critical ω a hyperbolic invariant three-sphere bifurcates from the restpoint; this invariant set has four-dimensional stable and unstable manifolds. The existence of this invariant set follows from a linear analysis of the restpoint together with standard perturbation results for hyperbolic invariant manifolds. Inside the invariant three-sphere there is at least one periodic orbit, the elliptical orbit of Lagrange. This is the continuation to lower angular momenta of the circular orbit described above. The configuration remains similar to the central configuration but instead of rigidly rotating, the size expands and contracts as the three bodies orbit on similar ellipses around the center of mass obeying Kepler's laws for the two-body problem (figure 8). Since the shape remains constant, such an orbit appears in the Hill's region as a radial line segment over the central configuration; this segment runs completely across the tunnel. The elliptical Lagrange orbit appears in figure 7 along with a crazy orbit from the invariant three-sphere.

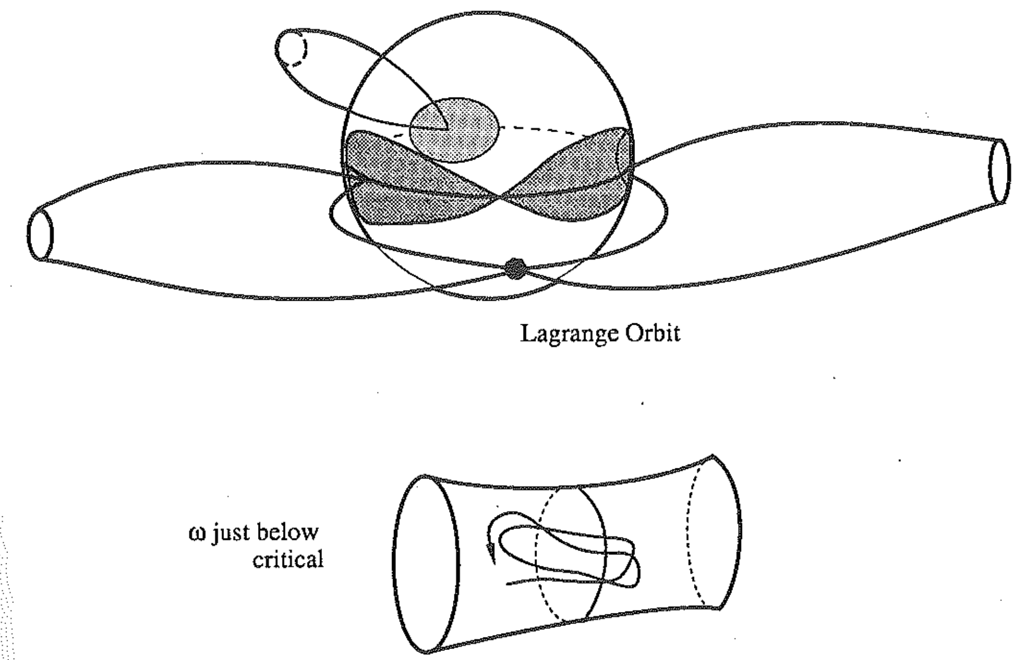

FIGURE 7 : First Critical Hill's Region and the Tunnel

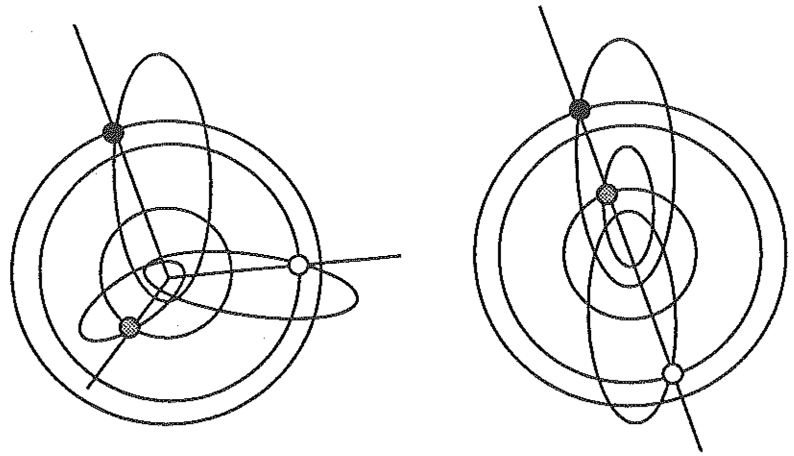

FIGURE 8 : Circular and Elliptical Lagrange Orbits

The shape of the Hill's region for ω below the critical level suggests a problem: do there exist binary exchange orbits, that is, orbits heteroclinic between the two lobes ? Such an orbit would exhibit different tight binary configurations in forward and backward time. More specifically, one could ask for heteroclinic orbits connecting the two parabolic infinities. Perhaps the invariant three-sphere in the tunnel is involved in such a network of homoclinic and heteroclinic orbits. An orbit connecting a three-sphere at infinity to the three-sphere in the tunnel would be an interesting type of capture orbit. A final open problem concerns the persistance of the invariant three-sphere or at least of some large invariant set as the angular momentum is lowered.

6. Below the Third Critical ω. The other two collinear central configurations are associated with bifurcations similar to the one described in section 5. At the critical levels, circular periodic orbits appear and develop into hyperbolic invariant three-spheres as the angular momentum is lowered further. New tunnels develop connecting the third lobe to the two which were already joined. After the third collinear orbit has developed the projection of the Hill's region is an equitorial band on the shape sphere and we have a Hill's region as in figure 9.

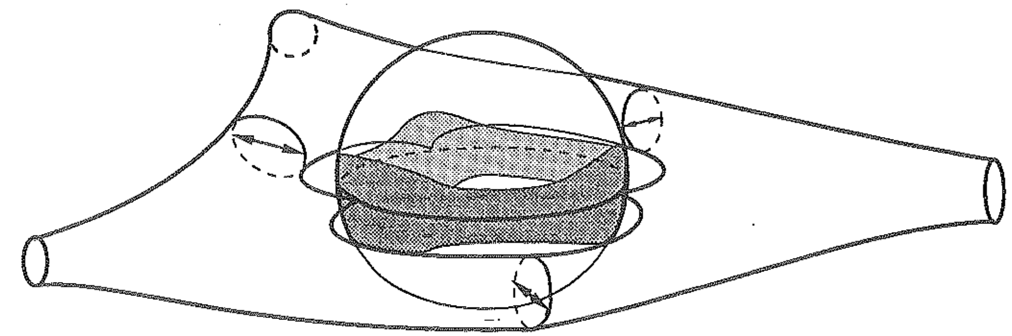

FIGURE 9 : Hill's Region Below the Third Critical ω

Very little is known about the dynamics for these intermediate values of angular momentum. About all that can be said is that there are three collinear, elliptical Lagrangian periodic orbits; these are the radial line segments crossing the tunnels in the figure. For parameters near the critical values there will also be invariant three-spheres but as mentioned in section 5, their persistance is in doubt. If there are interesting invariant sets in the three tunnels which behave in some sense (Conley index?) like hyperbolic invariant three- spheres, then there would be the possibility of heteroclinic connections. A less daunting open problem is suggested by the topology of the Hill's region. Do there exist periodic orbits which are homotopically nontrivial in the sense that they run around the Hill's region passing through all three tunnels to form a noncontractible closed curve ?

7. Below the Last Critical ω. As we lower ω still further the equitorial band on the shape sphere becomes wider until at the last critical value of ω it finally covers the north and south poles (the equilateral central configurations). At the critical level two restpoints develop in M(h,ω) corresponding to the two circular, equilateral Lagrangian periodic orbits. As we lower ω further these become elliptical just as in the collinear case. A detail of the bifurcation of the Hill's region over one of the equilateral points is shown in figure 10. After the bifurcation, the elliptical Lagrange orbits appear as radial line segments connecting the two sheets. Globally, the boundary of the Hill's region splits into two two-spheres (figure11). This topology persists all the way down to $\omega = 0$. As $\omega \to 0$, the inner surface converges to the triple collision sphere $r = 0$ and the Hill's region tends to the region of figure 4.

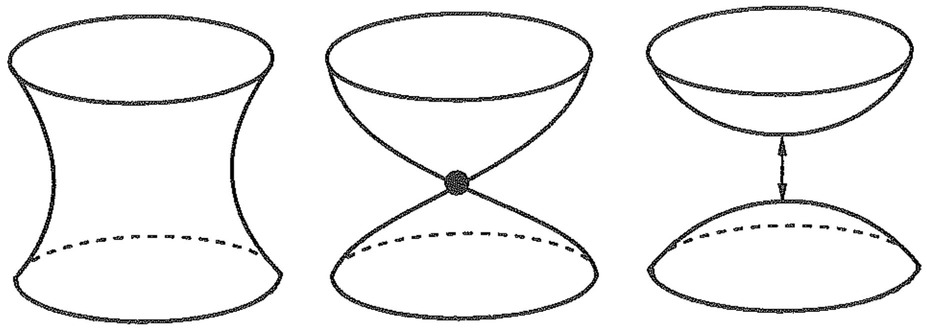

FIGURE 10 : Bifurcation over the Poles

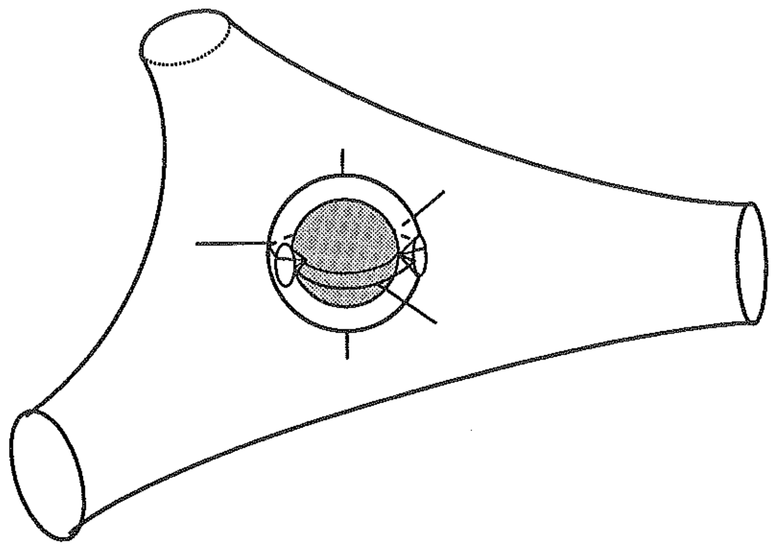

FIGURE 11 : Hill's Region after the Last Bifurcation

For all nonzero ω below the last bifurcation, there will be five elliptical Lagrangian orbits, one for each central configuration. For ω just below the critical level there may also be complicated invariant sets near the equilateral orbits. This depends on the choice of the masses; for some masses (the minority) KAM theory applies near the equilateral restpoints and there will be invariant tori, long-period orbits, etc. For most choices of the masses, however, the equilateral periodic orbits are born hyperbolic with three-dimensional stable and unstable manifolds in $M(h,\omega)$. In this case they are isolated invariant sets.

Since the dimension of $M(h,\omega)$ is so large, the KAM theory does not imply stability and this would be the place to look for Arnold diffusion in the three-body problem. If the equilateral orbits are hyperbolic it is natural to look for transverse homoclinic and heteroclinic points connecting them. We will see in section 8 that such orbits do exist for small non-zero ω, but for the nearly circular case the problem is open.

8. Low Angular Momentum. In this section we will consider the case $\omega = 0$ and the case of small non-zero ω. The interesting dynamics involves orbits which pass near the triple collision singularity. Thanks to the coordinate system of McGehee, which blows up the singularity into the invariant set $\{r = 0\}$ of 2.3, one can effectively study orbits passing near the singularity. As a result, more can be said about this case than about all the others combined. The study of triple collision began with Sundman[Su] and was carried on by Siegel [S-M]. McGehee's study of the collinear three-body [Mc3] problem led to much further work. The isosceles three-body problem, a subsystem present when two of the three masses are equal, has been studied by a number of authors [Dev1,Dev2,Si,L-L,M1,M2]. Finally [M3,M4] treat the planar case.

We will begin with the case $\omega = 0$. The Hill's region is reproduced in figure 12. Sundman's theorem about orbits near $r = 0$ no longer applies and triple collisions are possible. Quite a lot can be said about orbits which begin or end in triple collision. First of all, such orbits exist. In fact the elliptical orbits of Lagrange become more and more eccentric as $\omega \to 0$ and in the ellipses degenerate into line segments (figure 8). The limiting orbits are homothetic expansions and contractions; the shape is always the central configuration while the size increases from zero to some maximum and back to zero again. Thus these orbits are homoclinic to triple collision. In figure 12 they appear as the five line segments from the sphere $r = 0$ to the outer surface. It turns out that every triple collision orbit must approach one of the five central configurations. Some other triple collision orbits are shown in the figure. The orbits which tend to a given central configuration form a smooth submanifold of M(h,0) of dimension 2 in the collinear case and 3 in the equilateral case. The manifold of collinear collision orbits consists entirely of orbits whose configuration is collinear for all time. Such orbits would lie in the equitorial plane in figure 12. However, there are orbits ending at the equilateral collision which look nearly collinear until the last moment when the middle mass slips out from between the other two to form the required equilateral triangle. Such an orbit is shown in the figure.

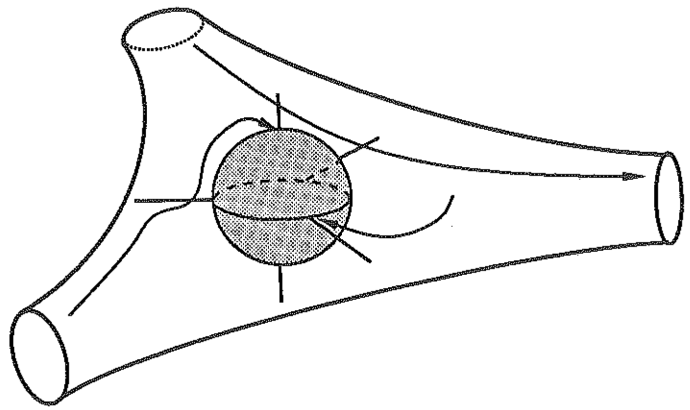

FIGURE 12 : Some Zero Angular Momentum Orbits

Among the orbits tending to triple collision in one time direction there are orbits which tend parabolically to each of the three infinities in the other time direction. By making use of these connections from infinity to triple collision one can show that there are binary exchange orbits; the exchange is carried out during a close approach to triple collision.

We already mentioned that the limiting Lagrange orbits can be viewed as orbits homoclinic to triple collision. There are infinitely many other orbits homoclinic to the two equilateral triple collisions and heteroclinic between them (for technical reasons this is only a theorem for masses in a certain open set in mass space but it probably holds for all choices of masses). Some of these pass very near to the collinear Lagrange orbits switching to the equilateral configurations only very near triple collision.

Finally we mention another kind of oscillation orbit which is known to exist in the isosceles subsystem. There are orbits which approach arbitrarily close to triple collision without actually colliding; in fact they converge to one of the collinear Lagrangian orbits. They feature infinitely many increasingly close approaches to collision between which they

expand nearly homothetically like the Lagrange orbit. Such an orbit satisfies $\underline{\lim}\ r(t) = 0$ but $\overline{\lim}\ r(t) > 0$. Such an orbit is shown in figure 13 along with some of the orbits homoclinic to triple collision.

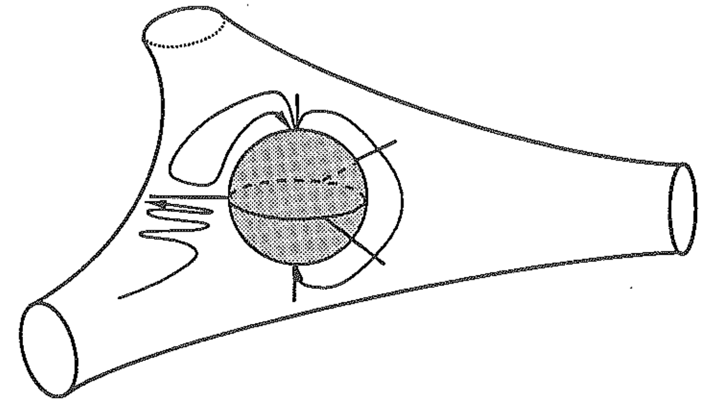

FIGURE 13 : Orbits Homoclinic to Triple Collision

The case of small non-zero angular momentum combines close approaches to triple collision with the recurrence of the highly elliptical Lagrange orbits to produce chaotic results. We will concentrate on the equilateral Lagrange orbits. When ω is sufficiently small, these orbits are hyperbolic with three-dimensional stable and unstable manifolds. These two orbits are connected by heteroclinic orbits to one another and to the three infinities (at least for masses chosen in a suitable open set as mentioned above). These are shown in figure 14.

First there are orbits running from the equilateral Lagrange orbits to parabolic infinity. Depending on the direction in which they run they are either capture orbits or escape orbits. The capture orbits, for example, evolve as follows: the particles are in a tight binary configuration, but the third mass approaches the binary, interacts closely with it and begins a very regular bounded motion which approaches an equilateral elliptical Lagrangian periodic orbit as $t \to \infty$.

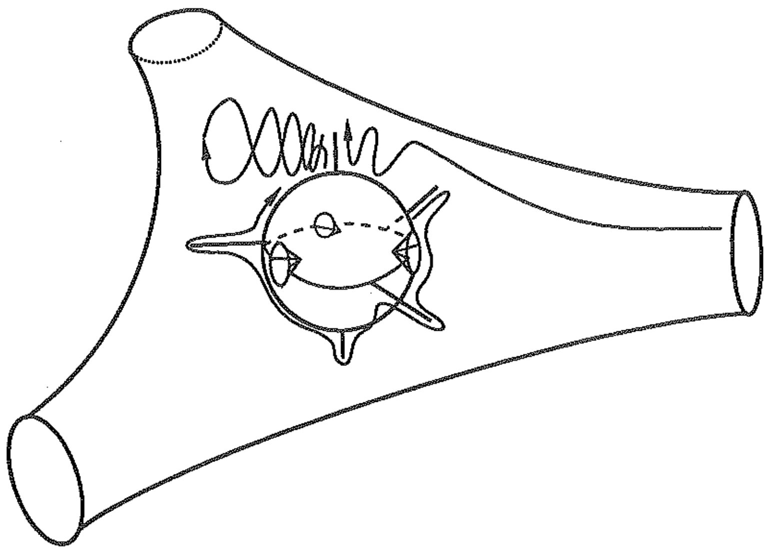

FIGURE 14 : Some Low Angular Momentum Orbits

There are transverse homoclinic and heteroclinic orbits connecting the two equilateral Lagrangian orbits. In fact, there are infinitely many distinct connecting orbits, some of which pass very near to the collinear Lagrangian orbits. The presence of homoclinic orbits produces all of the usual chaos. There are wild orbits which change shape abruptly after each close encounter with triple collision. In fact, if the masses are nearly equal, one can arrange orbits which imitate all five of the Lagrangian behaviors in turn (see figure 15); one can even arrange for such orbits to be periodic.

Roughly speaking, the reason for the existence of all of these orbits in the low angular momentum case is that the combination of the recurrence of the equilateral Lagrange orbits with the stretching and spiralling which orbits experience while passing close to triple collision produces a large invariant set describable by the methods of symbolic dynamics.

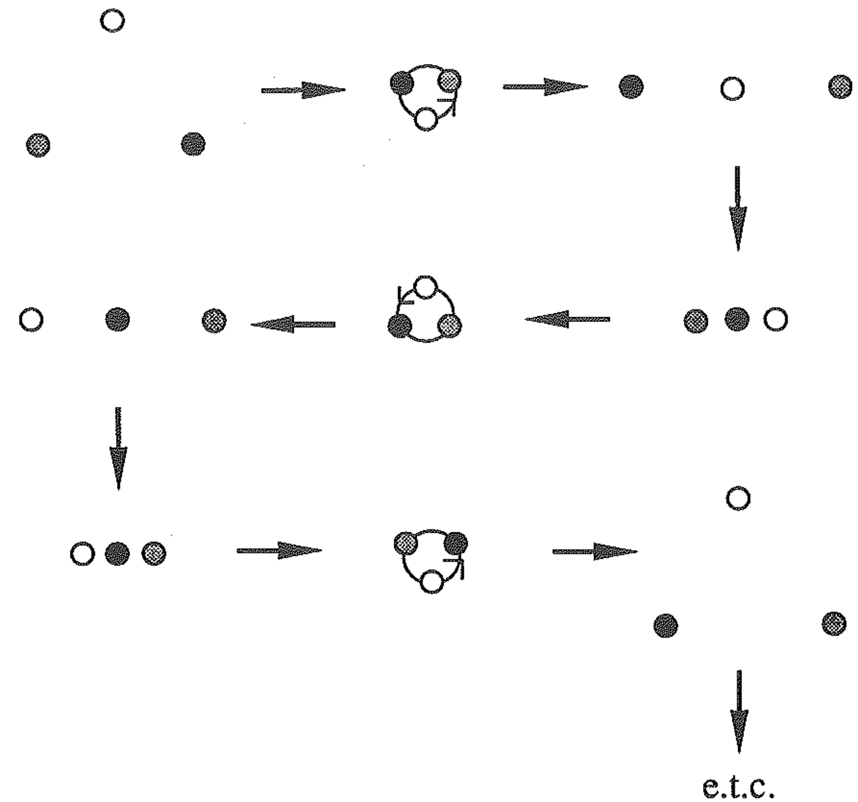

FIGURE 15 : A Low Angular Momentum Orbit

There are a number of open questions. It would be nice to incorporate the infinities into the symbolic dynamics; currently, one can get out from near triple collision to parabolic infinity but then one cannot necessarily get back for another close approach. This would enlarge the invariant set to include oscillation orbits as described in section 4. Another question concerns the collinear Lagrange orbits. These are not necessarily hyperbolic for small ω. What is going on in the neighborhood of these orbits as $\omega \to 0$? Finally, a question which could be posed for any ω is: what is happening to the angle that was quotiented out ? Do the orbits described above rotate systematically as they perform their wild changes of shape or do they emerge from the close approaches at random angles ?

9. Conclusion. We will close this tour of the three-body problem with another look at figure 1, the "big picture". There are really very few landmarks in the phase space of the three-body problem. The most important are triple collision, infinity and the periodic orbits of Lagrange. These are the elements of figure 1. A program for understanding the three-body problem is first to conduct local studies of these features and then to find out how they are connected to one another. As we have seen, a little progress has been made in the three centuries since Newton formulated the problem but a genuine understanding of what is possible remains a distant goal.

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Contemporary Mathematics
Volume 81, 1988

SYMMETRY IN n-PARTICLE SYSTEMS

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ABSTRACT. According to Noether's theorem, symmetry in Hamiltonian systems translates into integrals of motion. Some of the methods used to extract information about the dynamics from the integrals are outlined in Section 2. In this paper a conceptually simple approach is introduced to subsume and extend many of these efforts. The basic ideas are introduced with the integrals of angular momentum for n-particle systems, and the utility of this approach is indicated with some new results. This approach extends to all symmetry integrals for systems of the general form $r'' = \nabla U(r)$.

1. ANGULAR MOMENTUM. For a n-particle system in a d-dimensional physical space, $r_i \in R^d$, $i = 1, 2, .., n$, is the position vector of the i^{th} particle. Usually, $d = 2, 3$. (Treat $d = 2$ as the x-y plane in R^3.) The equations of motion are

(1.1) $m_i r_i'' = \nabla_i U(r_1,..,r_n) \quad i = 1, .. ,n$

where $m_i \neq 0$, U is defined on a domain D in $(R^d)^n$, and ∇_i is the gradient with respect to r_i. For instance, if $m_i > 0$ and $U = \Sigma_{i<j} m_i m_j / |r_i - r_j|$, then Eq. 1.1 is the Newtonian n-body problem where the domain requirements are that $r_i \neq r_j$ for $i \neq j$.

Equations (1.1) admit the energy integral

(1.2) $T = (1/2) \Sigma_i m_i v_i^2 = U + h$

where $v_i = r'_i$ is the velocity of the i^{th} particle. If U, the *self-potential*, depends on the distances between particles, then the invariance of U with respect to translations admit the 2d integrals that fix the "center of mass" of the system,

(1.3) $\Sigma_i m_i r_i = At + B, \quad \Sigma_i m_i v_i = A,$

where, if $m_i > 0$, the usual choice is $A = B = 0$. The integrals restrict the orbits to a linear subspace of $(R^d)^n x(R^d)^n$ that

1980 Mathematics Subject Classification (1985 Revision). 58F05, 70F10.
1. Supported by a grant from the NSF.