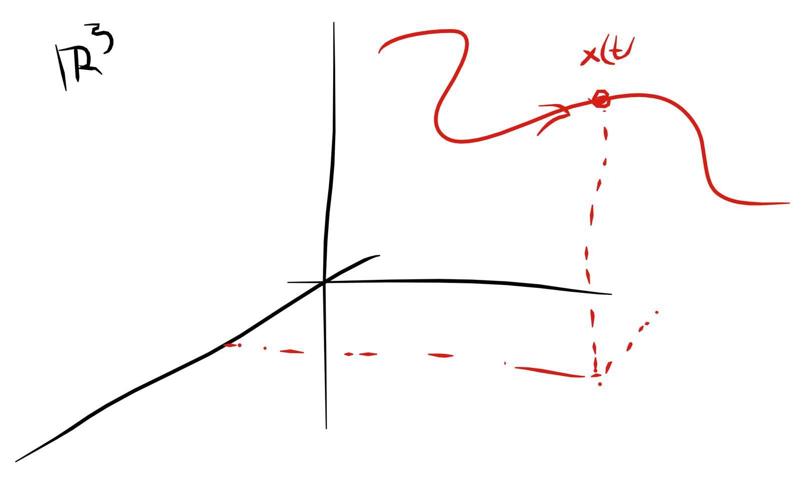
Coordinates, frames

Lunes : 14 – 15
Miercoles : 15 – 16 office hours

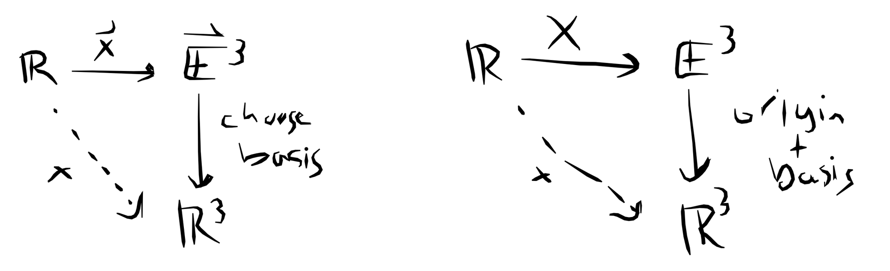
at the end of the day we consider parametrized subsets of $\mathbb{R}^3$
$A_t \subset \mathbb{R}^3 \ \ t \in \mathbb{R}$.

example (particle) $x : \mathbb{R} \to \mathbb{R}^3$



this description depends on various arbitrary choices :

* units of measurement (meters, seconds, smoots, ...)
* axes = orthon. basis
* origin



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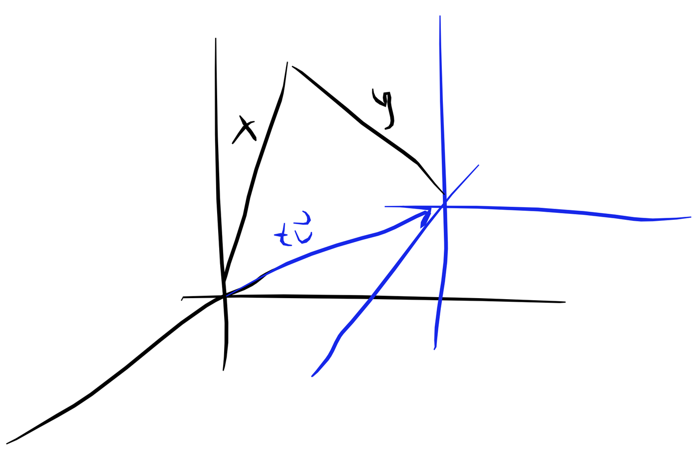
choice of origin + axes may depend on time.

example (uniform translation) :



$$x = y + t\vec{v}$$
$$\dot{x} = \dot{y} + \vec{v}$$
$$\ddot{x} = \ddot{y}$$

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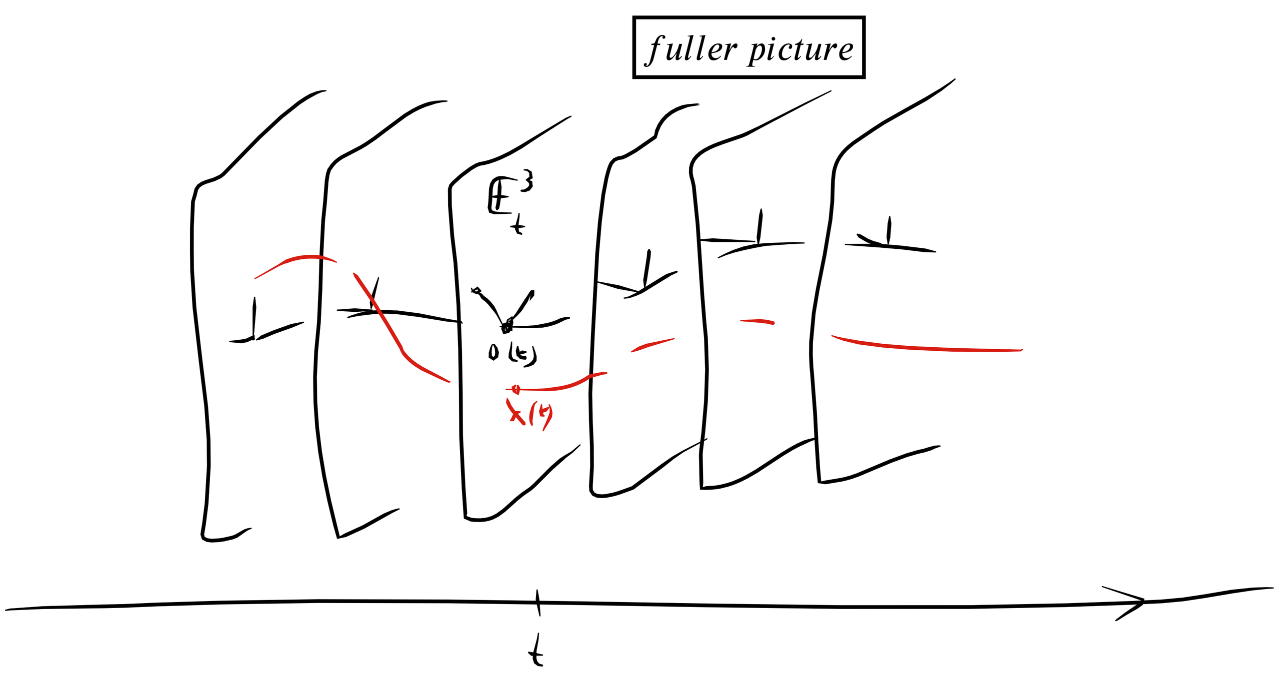
fuller picture



time line, T

The 'universe' or 'space – time' is :
$$M := \sqcup_{t\in T} \mathbb{E}^3_t$$

where $\mathbb{E}^3_t$ is simultaneous events at time t.
We think of M as a pile of Euclidean parametrized by T.
* we assume M has a smooth structure.

A frame consists of :
(i) origin $o(t) \in \mathbb{E}^3_t$ for each $t \in T$
(ii) $e_1(t), e_2(t), e_3(t) \in \mathbb{E}^3_t$ s.t. $e_j(t) - o(t)$ are orth. basis
(iii) depending smoothly on t.

A frame identifies $M \cong T \times \mathbb{R}^3$
choosing as well origin and unit of time identifies $M \cong \mathbb{R}\times\mathbb{R}^3$.
A particle is then $(t, x(t))$ where $x : \mathbb{R} \to \mathbb{R}^3$

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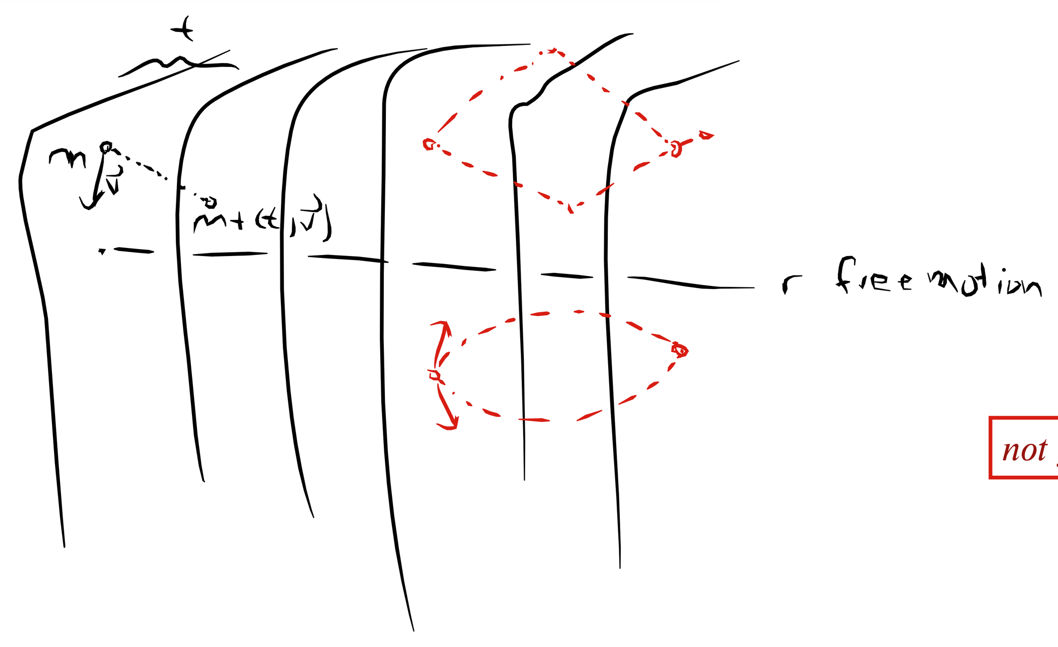
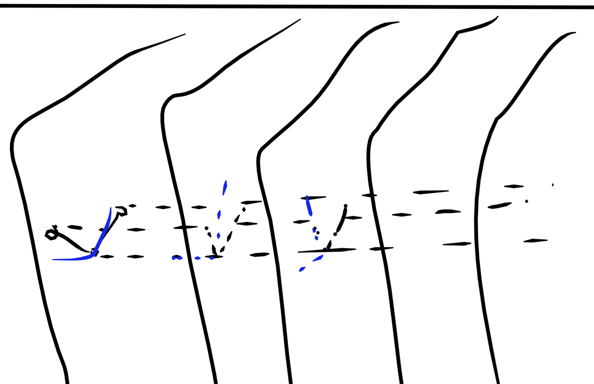
additional structure from idea of free motion



not free motions

we assume that this equips M with an affine structure Galileo's principle of inertia (Newton's 1st law) :
given $m \in M$ and $(t, \vec{v}) \in \mathbb{R}\times\mathbb{R}^3$
$m + (t, \vec{v})$
is defined as the point where m would be if given an initial velocity $\vec{v}$ and let move freely for time t.

we come to the definition in Arnold's book.
The universe or space – time M is :
(i) an affine space $\mathbb{A}^4 \cong M$
(ii) with a map $s : \mathbb{A}^4 \to T \ (\cong \mathbb{R})$
(iii) each $s^{-1}(t) \subset \mathbb{A}^4$ is a Euclidean space.



an inertial frame is one in which free motions are straight lines.
They are obtained by translating an initial frame.

The Galilean group are transformations of M preseving :
simultaneous events
Euclidean structure
free motions (affine str.)

in the coordinates $M \cong \mathbb{R}\times\mathbb{R}^3$ from an inertial frame :
$$(t,x) \in \mathbb{R}\times\mathbb{R}^3 \to \left(t + t_0, gx + a + t\,\vec{v}\right)$$
where g is a rotation, $a, \vec{v} \in \mathbb{R}^3$ $t_o \in \mathbb{R}$.

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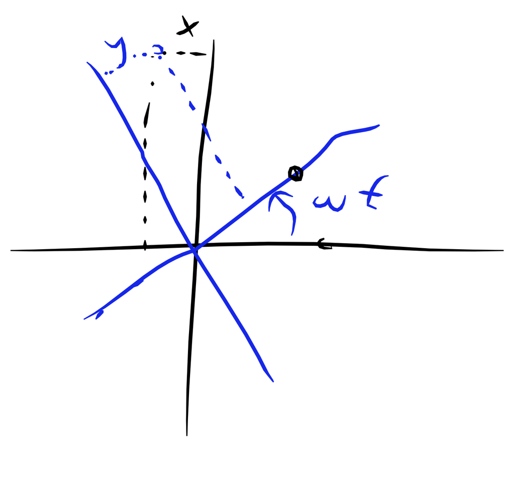
example (uniformly rotating frame) :



$$x, y \in \mathbb{C}$$

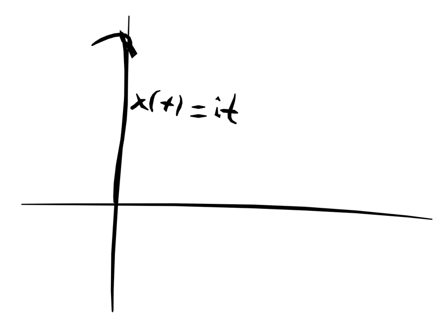
$$x = e^{i\omega t}\, y$$

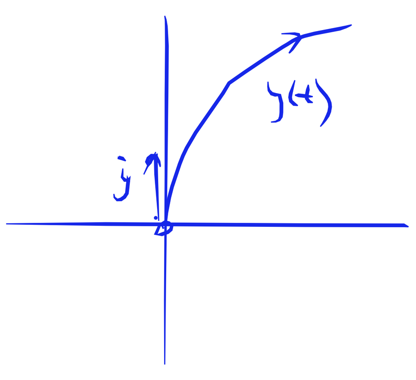
$$\dot{x} = i\omega x + e^{i\omega t}\dot{y}$$
$$\dot{y} = -i\omega y + e^{-i\omega t}\dot{x}$$

$$\ddot{x} - 2i\omega\dot{x} - \omega^2 x = e^{i\omega t}\ddot{y}$$
$$\ddot{y} + 2i\omega\dot{y} - \omega^2 y = e^{-i\omega t}\ddot{x}$$

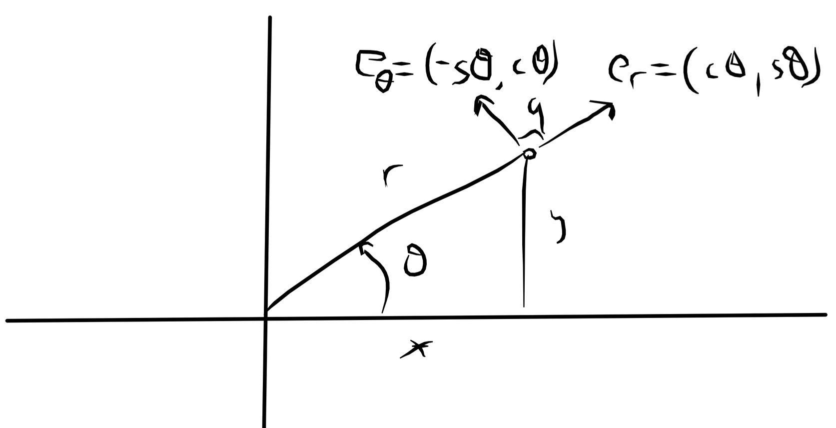
Coriolis force

Centrifugal force





example (polar coordinates) :



$$q = (x,y) = r(\cos\theta, \sin\theta)$$

$$\dot{q} = \dot{r}\, e_r + r\,\dot{\theta}\, e_\theta = \dot{r}\,\partial_r + \dot{\theta}\,\partial_\theta$$

$$\ddot{q} = \left(\ddot{r} - r\dot{\theta}^2\right)e_r + (2\dot{r}\,\dot{\theta} + r\ddot{\theta})\, e_\theta$$

$$C := r^2\dot{\theta}$$

$$\ddot{q} = \left(\ddot{r} - \frac{C^2}{r^3}\right)e_r + \frac{\dot{C}}{r}e_\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2$$