Final (classical mechanics 2021)

Each problem counts for 20 pts.

1. Let $P : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map with $\det P = 1$. Show that:

   (i) P has eigenvalues $\lambda = e^{i\theta}, \bar{\lambda} \in \mathbb{C}\backslash\mathbb{R}$ iff $|tr(P)| < 2$

   (ii) P has eigenvalues, $\lambda, \lambda^{-1} \in \mathbb{R}\backslash 0$ iff $|tr(P)| \geq 2$.

2. For the Lagrange top, given by the Lagrangian system:

$$L = \frac{I}{2}\left(\dot{\theta}^2 + \dot{\varphi}^2 \ \sin^2\theta\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\varphi} \ \cos\theta\right)^2 - gM\ell\cos\theta,$$

   determine the Hamiltonian, $H(\theta, \varphi, \psi, p_\theta, p_\varphi, p_\psi)$, and give the equations of motion in Hamiltonian form.

3. Let $\omega(u, v) = u \cdot Jv$ be the standard symplectic form on $\mathbb{R}^{2n}$.

   (a *(10 pts.)*) For $V \subset \mathbb{R}^{2n}$ a k-dimensional vector subspace, let

$$V' := \{w \in \mathbb{R}^{2n} : \omega(v, w) = 0, \ \forall v \in V\}$$

   be the ω-orthogonal complement of V. Show that V' has dimension $2n - k$.

   (b *(4 pts.)*) Give an example when $V \cap V' \neq \{0\}$.

   (c *(6 pts.)*) Suppose $V \subset V'$. Show that $dim(V) \leq n$. Give an example when $dim(V) = dim(V') = n$.

4. Consider three bodies in the plane, $q_o, q_1, q_2 \in \mathbb{R}^2$, with masses $m_o = 1, m_1 = \varepsilon\mu_1, m_2 = \varepsilon\mu_2$. Let

$$Q_o = q_o, \quad Q_1 = q_1 - q_o, \quad Q_2 = q_2 - q_o.$$

   Be 'heliocentric' coordinates. Show that the 'lift'

$$P_o := p_o + p_1 + p_2, \quad P_1 := p_1, \quad P_2 := p_2$$

   gives rise to a transformation[1] $(q, p) \mapsto (Q, P)$ preserving the standard symplectic form on $\mathbb{R}^{12}$.

5. Let $q(t) \in \mathbb{R}^2$ be an elliptic Kepler orbit in the plane, with energy $\frac{|\dot{q}|^2}{2} - \frac{1}{|q|} = E < 0$. Show that the total action over one period of the orbit is given by:

$$\int_0^T \frac{|\dot{q}(t)|^2}{2} + \frac{1}{|q(t)|} \ dt = \frac{3\pi}{\sqrt{-2E}}$$

   where T is the period of the orbit.

[1] Taking center of mass zero coordinates –as usual– translates to setting $P_o = 0$, and the Hamiltonian for the Newtonian 3-body problem in these heliocentric coordinates has the form: $\frac{1}{\varepsilon}H = \frac{|P_1|^2}{2\mu_1} - \frac{\mu_1}{|Q_1|} + \frac{|P_2|^2}{2\mu_2} - \frac{\mu_2}{|Q_2|} + \varepsilon\left(\frac{|P_1+P_2|^2}{2} - \frac{\mu_1\mu_2}{|Q_1-Q_2|}\right)$.