Midterm (classical mechanics 2021)

Each problem counts for 20 pts (the parts (a), (b) each count for 10 pts. respectively).

1. Consider a central force problem of the following form: $\ddot{q} = -\frac{q}{r^{a+1}} = F(q)$ with $q \in \mathbb{R}^2$ and $r = |q|$.

   (a) Fix $q_o \in \mathbb{R}^2 \backslash 0$. Show the work $W(q) = \int_{q_o}^{q} F(q) \cdot dq$ over a path from q_o to q is independent of the path.

   (b) Determine the potential energy.

2. Consider unit speed ($|\dot{q}| = 1$) geodesics (free motions), $q(t)$, on a surface of revolution $\Sigma \subset \mathbb{R}^3$.

   (a) For $q(t)$ a unit speed geodesic on Σ, let $r(t)$ be the distance of $q(t)$ to the axis of rotation and $\alpha(t)$ the angle between $\dot{q}(t)$ and the latitude containing $q(t)$. Show that $r(t)\cos\alpha(t) = c$ is constant over the motion.

   (b) Let Σ be obtained by revolving the arc-length parametrized curve $s \mapsto (x(s), 0, z(s))$ (with $x(s) > 0$) about the z-axis. For θ the angle around the axis of revolution, show that the geodesics are the curves $s \mapsto (x(s)\cos\theta(s), x(s)\sin\theta(s), z(s))$ with $\theta(s)$ defined through the integral:

   $$\theta = \int \frac{c\ ds}{x(s)\sqrt{x(s)^2 - c^2}}$$

   where c is the constant of motion from part (a).

3. Consider a double pendulum, with say lengths and masses both equal to 1 (see figures).

   Determine the equilibrium points and describe the trajectories of the linearized system around at least 2 equilibrium points.

4. Consider a homogeneous (constant density) solid ball. Drill a narrow straight tube through the ball and drop a point mass from the surface of the sphere into the tube (see figures). Describe the resulting motion of the point mass, in particular give its period of oscillation.

5. Consider N particles $q_1, ..., q_N$ in space with masses $m_1, ..., m_N$ and subject to mutually attracting 'spring' forces: $F_{jk} = m_j m_k (q_j - q_k)$ is the force on particle q_k due to particle q_j.

   Describe the trajectories of the system (hint: when the center of mass is zero, the system is exactly solvable).

Figure 1. A double pendulum (left). A narrow tube drilled through a homogeneous ball (right).