Some experiments in classical mechanics

§1: uniform gravitation. That objects fall to the earth with a constant acceleration is one of the first notable *qualitative* results in mechanics. Galileo explained an experiment to confirm this by examining objects rolling down inclined planes.

The experiment examines the following situation: to determine the relation of the *time* it takes a ball to roll down a plane at a given *inclination*.

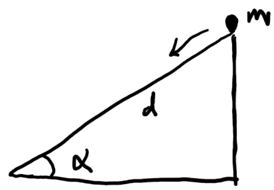

Figure 1. A plane of fixed length d is fixed at in inclined angle α from the horizontal and the time, t, it takes a point mass m to roll down the plane is measured. One finds a relationship: $\frac{d}{t^2 \sin\alpha} = cst. = g/2$.

In conventional units, the value of g is $\approx 9.8m/s^2$ (although there is small variation and dependence upon ones current location/elevation).

Exercises:

1. Suppose a point on a line moves with constant acceleration (initially being at rest). If the particle moves 1 unit of distance in the first unit of time of its motion, show it moves 3 units of distance in its second unit interval of time of its motion, and in general $2k+1$ units of distance in its n'th unit interval of time of its motion.

2. Show that a point mass, m, on a plane of length d inclined at angle α subject to constant vertical acceleration mg takes time $\sqrt{\frac{2d}{g\sin\alpha}}$ to reach the bottom (ignoring friction, etc.).

3. Suppose an infinite plane (say the xy-plane) has a uniform mass distribution of density σ. Show from Newton's inverse square law that the resulting force acting on a particle of mass m is a constant multiple of $\hat{k}$.

4. The gravitational constant g may also be related to Newton's inverse square law since close to the surface of the earth, we have the gravitational force on a particle of mass m has strength: $f_{grav} = \frac{GmM}{(R_E+h)^2}$ (directed towards the center of the Earth) where M_E is the mass of the earth and R_E say the radius of the earth at sea level and h ones current elevation. Since $h << R_E$ one has $f_{grav} \approx m\frac{GM_E}{R_E^2} = mg$, and so $g = \frac{GM_E}{R_E^2}$.

   For the true force law ($f_{grav} = \frac{GmM_E}{(R_E+h)^2}$), determine the 'escape velocity' for a vertically thrown object from the surface of the earth.

5. A pendulum might also be used to measure g. Since $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, we have for θ small that the period of oscillation is approximately $T \approx 2\pi\sqrt{\frac{\ell}{g}}$. Since T and ℓ can be measured, so too can g.

   (a) Explain how a pendulum could –in principle– also be used to measure altitudes.

   (b) Explain why a pendulum altimeter as in part (a), would not be very practical by showing that a small error in the measurement of T may result in a large error in the measured altitude.

6. Try to measure g using an inclined plane and a pendulum!

§2: astronomical observations. We consider some coordinate systems used in astronomy. From the earth we have direct access to *line of sight* measurements of stars/planets/etc.. The great distances to the stars allows one to make the simplifying assumption that 'parallax effects' may be ignored, ie the line of sight to the star remains unchanged when the point from which it is observed changes.

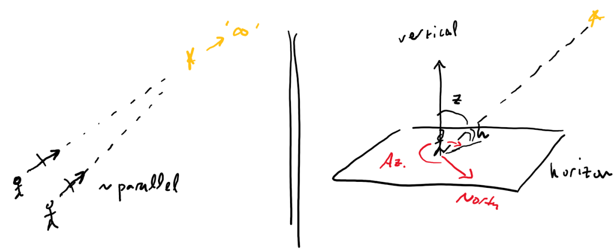

Figure 2. The lines of sight to a very distant object measured by two relatively close observers will be along approximately parallel light rays. In the approximation with the stars 'at infinity' these line of sight rays to the stars are parallel –for all but the most precise of measurements, this approximation lies well below the measurement errors (in fact the distances to the first stars where first found by W. Bessel by accounting for these 'parralax' differences). When sighting a star from the earth we may measure its line of sight with two angles (spherical coordinates): an *altitude* angle, h, measuring the angle from the horizontal plane to the line of sight to the star and an *azimuthal angle*, Az, measuring the direction to the star along the horizontal plane from a fixed direction (say due north).

Conversions between different spherical coordinates for these line of sight measurements are determined using spherical trigonometry:

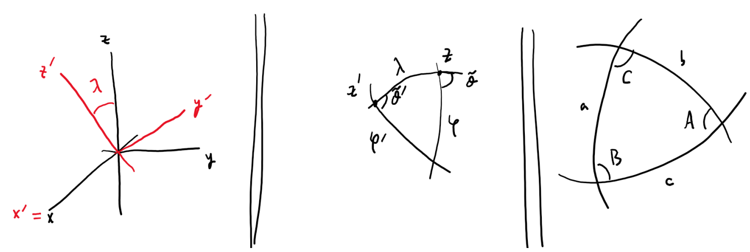

Figure 3. Given two systems of Cartesian coordinates with common x-axis and angle λ between their z-axes, the corresponding spherical coordinate systems (eg $(x, y, z) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$) may be related through: $\sin\varphi\cos\theta = \sin\varphi'\cos\theta'$, $\sin\varphi\sin\theta = \cos\lambda\sin\varphi'\sin\theta' - \sin\lambda\cos\varphi'$, $\cos\varphi = \sin\lambda\sin\varphi'\sin\theta' + \cos\lambda\cos\varphi'$. In fact these are just special case of formulas of 'spherical trigonometry', given a triangle on the unit sphere (sides arcs of great circles) as on the right, one has the relations: $\sin a\sin B = \sin b\sin A$, $\sin a\cos B = \cos b\sin c - \sin b\cos c\cos A$, $\cos a = \cos b\cos c + \sin b\sin c\cos A$

In this way one may convert altitude and azimuth measurements to coordinates for the line of sight to the star more analogous to usual latitude and longitude measurements (spherical coordinates based on the earth's axis of rotation and equatorial plane, fig. 4): the declination, δ, and hour, H, angles via the relations:

$$\sin\delta = \sin h\sin\ell + \cos h\cos\ell\cos Az$$

$$\cos\delta\sin H = \cos h\sin Az$$

$$\cos\delta\cos H = \cos h\sin\ell\cos Az - \sin h\sin\ell$$

where ℓ is one's latitude on the surface of the Earth. Note that the first determines $\delta(h, \ell, Az)$ while dividing the 2nd and 3rd gives $H(h, \ell, Az)$.

The declination of a given star is essentially a constant (changes in the axis of the earth are very small and slow), while the same can be said for *differences* in Hour angles between stars measured at the same time. In this way, one may make a rough *star chart* of the *celestial sphere*: the relative positions of the lines of sight to the 'fixed' stars, eg online here.

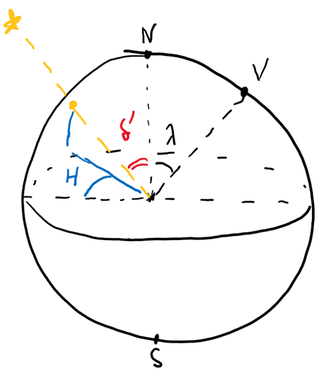

Figure 4. Lines of sight to the stars may also be coordinatized with *declination*, δ, and *Hour*, H, angles. If δ' is the angle between the line of sight to the star and the earth's axis, then $\delta + \delta' = \frac{\pi}{2}$, ie declination is the angle from the equatorial plane of the earth to the line of sight to the star. For V the vertical direction, then the great circle from V to N intersects the equatorial plane of the earth in a point and H is the angle along the equatorial plane measured from this point to the line of sight to the star.

Similarly, if one has a reliable time keeping device, one may make an *almanac* based from ones current position: record the declinations and hour angles of objects measured at given times throughout the year (better, record their predicted declinations and hour angles throughout the coming year from ones given position). The standard location upon which almanacs are based is Greenwich England, see eg here.

The regular motions of lines of sights to stars (declination angles constant, Hour angle changing uniformly), lead to definitions of time. The *sidereal time* is measured through change in hour angle of a given star (which is *not* the sun). The *solar time* is measured by change in hour angle of the sun (slightly different than sidereal time due to the motion of the earth around the sun). Finally the *mean solar time* is measured through change in hour angle of a fictitious sun whose motion is uniform (as opposed to the actual sun whose motion varies due to the small eccentricity of earth's orbit and Kepler's 2nd law).

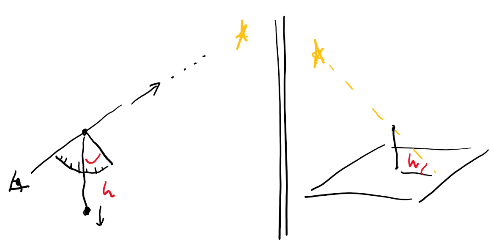

Figure 5. One may make some rough astronomical measurements with some simple instruments. To measure the altitude angle, h, of a night-time object, one may use a *quadrant* (left). To measure altitude angle of the sun one may use a sun-dial (right), the length of the shadow casting stick and shadow cast may be measured and one may find h by an arc tangent. Azimuth angles may be measured with a compass (corrected from magnetic north to true north). More accurate measurements may be made with a sextant or telescope.

Exercises:

1. The north star, Polaris,[1] has a line of sight direction essentially parallel to the earth's axis of rotation. If one is in the northern hemisphere and measures the line of sight to the star Polaris at an altitude angle h from the horizon, show that one is located at a northern latitude h.

2. Let $\hat{\xi}$ be a unit vector in $\mathbb{R}^3$ and $\hat{\eta}$ a unit vector in $\hat{\xi}^{\perp}$. For[2] $\hat{\xi}' = R_{\hat{\eta}}(\lambda)\hat{\xi}$ show that $R_{\hat{\xi}'}(\alpha) = R_{\hat{\eta}}(\lambda) R_{\hat{\xi}}(\alpha) R_{\hat{\eta}}(-\lambda)$.

[1] This star is however rather dim, and sometimes tricky to locate.

[2] We write $R_{\hat{\xi}}(\theta)$ for the rotation by angle θ around the axis $\hat{\xi}$.

3. Let $\Delta \subset S^2$ be a spherical triangle on the unit sphere labeled as in figure 3 (right) above. Show that:

$$\sin a \sin B = \sin b \sin A, \quad \sin a \cos B = \cos b \sin c - \sin b \cos c \cos A, \quad \cos a = \cos b \cos c + \sin b \sin c \cos A$$

4. (a) Suppose that at a given time an object's line of sight from Greenwich England has a declination and hour angle $(\delta^\circ, H^\circ) \in (0, 90) \times (0, 360)$. Show at this time that when located at the latitude and longitude $\delta^\circ\ N, H^\circ\ E$ the object is directly overhead.

   (b) Suppose at the same time as in (a) we measure the same object to have an altitude $h \in (0, \frac{\pi}{2})$. Show that we are located at a distance $(\frac{\pi}{2} - h)R_E$ from the point on the earth where the object is directly overhead (with R_E the radius of the earth).[3]

5. Try to make some measurements of stars and compute their declinations. Compare your results with the 'official values' one can find online.

[3] In this way, equipped with an almanac and reliable time keeping device, one may determine one's position on the earth through intersecting spherical circles.