1960's 1850's 1930's

Teorema de Arnold – Liouville – Milneur

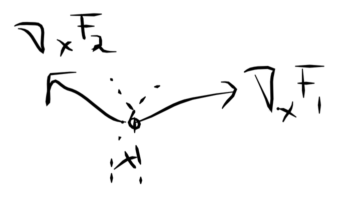
(sistemas 'integrables en el sentido de Liouville')

Consideramos una sistema mecánica, $H : \mathbb{R}^{2n} \to \mathbb{R}$
(digamos hay n – grados de libertad).

Suponemos que hay n primer integrales (incluso de la energía),

$F_1 = H, \dots, F_n$

$\{F_j = cst\} \rightsquigarrow n\text{-dim.}$



Si las integrales son

* *en involución :*
$\{F_i, F_j\} = 0 \quad$ *cada* $i,j \in \{1,\dots,n\}$

* *independientes :*
$\nabla_x F_1, \dots, \nabla_x F_n \in \mathbb{R}^{2n} \quad$ *indep. cada* $x \in \mathbb{R}^{2n}$

* *con niveles compactos y conectados :*
$\Sigma_f = \{ F_j = f_j = cst. \} \subset \mathbb{R}^{2n}$ *es compacto y conectado*

Entonces :

1. Σ_f *son toros n – dimensional* ('*parametrizados' por los valores de los integrales,* $f = (f_1,\dots,f_n) \in \mathbb{R}^n$).

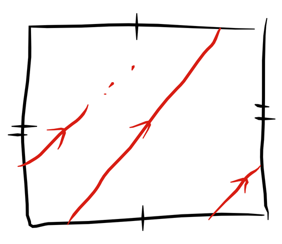
2. *Existe (locales) coordenadas simplecticas,* $(I, \theta) \in \mathbb{R}^n \times \mathbb{T}^n$ *para que* $H(I) = F_1$ *solo depende de I.*

$\omega = dI_j \wedge d\theta^j$:

$\dot{I} = -\partial_\theta H = 0$

$\dot{\theta} = \partial_I H = \omega(I)$

$n=2$



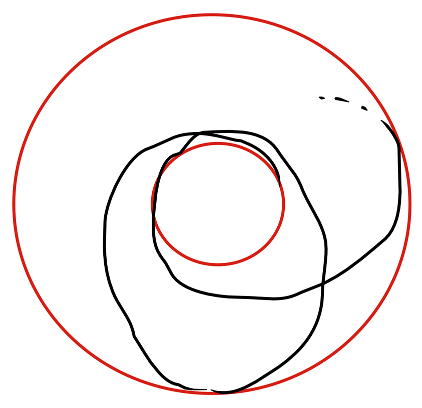
$\begin{cases} I = I_0 \\ \theta = \theta_0 + t\omega(I_0) \end{cases}$

Ejemplos

1. Fuerzas centrales

2 d.o.f.

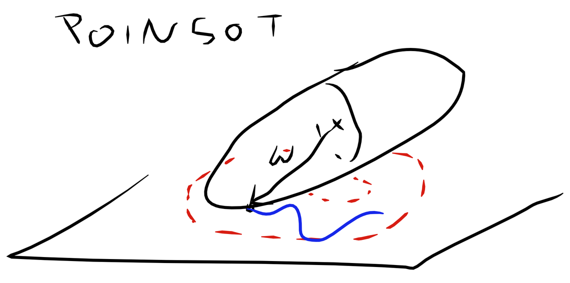
$F_1 = H, \ F_2 = C$ (a.m.)

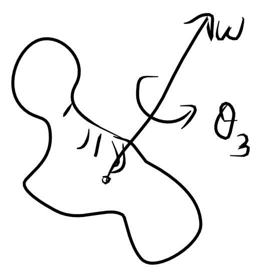


towards

away

POINSOT





$\mathbb{T}^{2\prime} + \theta_3'$

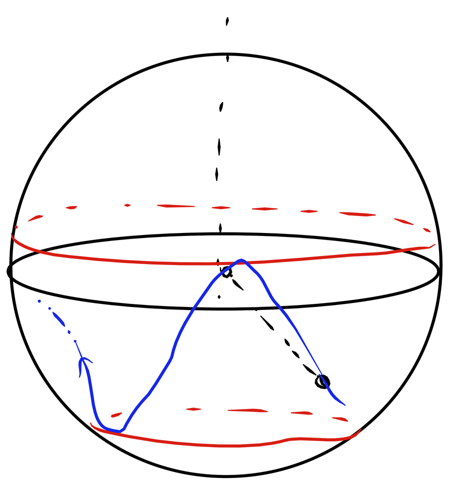
$\mathbb{T}^3$.

2. péndulo esférical

2 d.o.f.

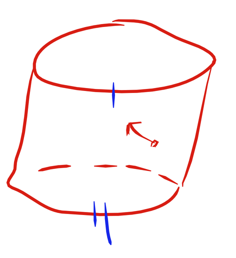
$F_1 = H$

$F_2 = M_z$

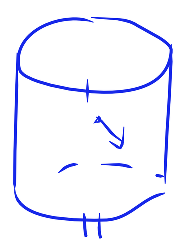


$\downarrow g$

'up'

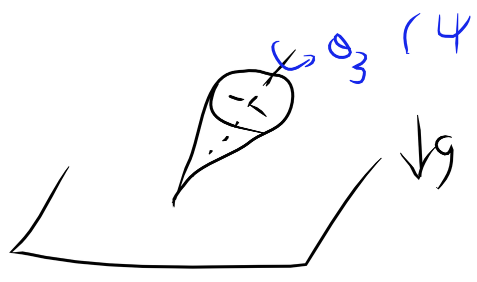


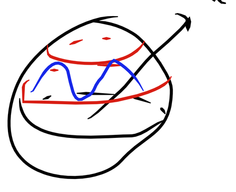
'down'



LAGRANGE TOP: H, M_{ax}, M_z

$\mathbb{T}^{2\prime} + \theta_3' \rightsquigarrow \mathbb{T}^3$.





3. Cuerpos rígidos (libre, trompo de Lagrange)

3 d.o.f. (EULER angles)

SO_3

H, M_1, M_2, M_3

$M = \vec{p} \times \vec{q} = (M_1, M_2, M_3)$

$\{M_j, H\} = 0$

$H, M_1, M_1^2 + M_2^2 + M_3^2 = M^2$

$\{M_1, M^2\} = 2M_2\{M_1, M_2\} + 2M_3\{M_1, M_3\}$

$= 2M_2M_3 - 2M_3M_2 = 0$

$* \ \{M_1, M_2\} = M_3 \quad (\{M_2, M_3\} = M_1, \dots) \ *$

Demonstración

1. *Por la independencía de* ∇F_j,
$\Sigma_f = \{ F_j = f_j = cst.\} \subset \mathbb{R}^{2n}$
es un subvariedad (compacto) de dimension n.

(assump.)

2. *Debido al hipotesis de involución, los campos vectoriales*

$X_j := X_{F_j}$

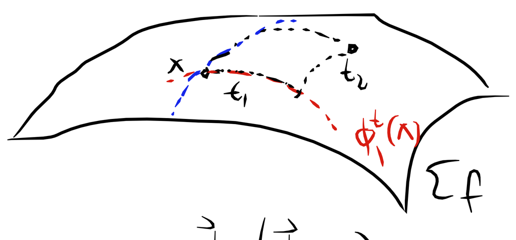
son tangentes a Σ_f. *Ademas, se comutan :*

$[X_j, X_k] = 0.$

$* \ 0 = dF_k(X_j) = \{F_k, F_j\}$

$* \ [X_j, X_k] = X_{\underbrace{\{F_k, F_j\}}_{0}} = 0.$

Vamos a usar las lineas de flujo de los campos X_j *para construir coordenadas sobre* Σ_f *en estas coordenadas vamos a ver que* Σ_f *es un toro.*



Es decir, ponemos ϕ_j^t *para el flujo de* X_j, *y tenemos un acción de* $\mathbb{R}^n$ *sobre* Σ_f :

$\vec{t} = (t_1, \dots, t_n)$ *actua en* $x \in \Sigma_f$ *por*

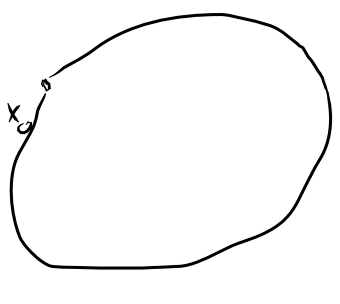
$x \mapsto \phi_1^{t_1} \circ \dots \circ \phi_n^{t_n}(x) =: \vec{t} \cdot x$

$(\vec{t}_1 + \vec{t}_2) \cdot x = \vec{t}_1 \cdot (\vec{t}_2 \cdot x)$
$= \vec{t}_2 \cdot (\vec{t}_1 \cdot x)$

* *debido a que* Σ_f *es compacto, existe* $\varepsilon > 0$ *para que*

$\vec{t} \cdot x \neq x \quad$ *cada* $x \in \Sigma_f$

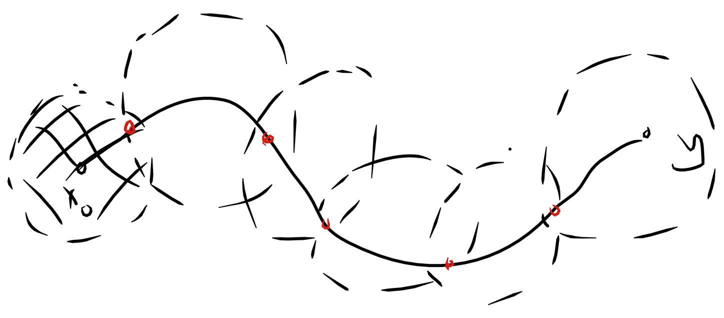
cada $\vec{t}$ *con* $0 < |\vec{t}| < \varepsilon$



* *en fijando un 'punto de base'* $x_o \in \Sigma_f$, *consideramos*

$\vec{t} \leftrightarrow \vec{t} \cdot x_o = x$

$\mathbb{R}^n \to \Sigma_f$ *es sobreyectiva…pero ¡no inyectiva!*



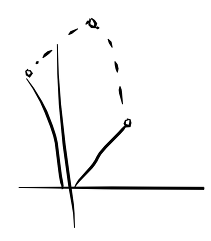
Ponemos
$\Gamma = \left\{ \vec{t} \in \mathbb{R}^n \ t.q. \ \vec{t} \cdot x_o = x_o \right\}$

entonces

* $\Gamma \subset \mathbb{R}^n$ *es un subgrupo*
* Γ *es discreta*

$\vec{t}_1, \vec{t}_2 \in \Gamma \Rightarrow \vec{t}_1 + \vec{t}_2 \in \Gamma$

$\vec{t} \in \Gamma \quad \vec{t} + \delta\vec{s} \notin \Gamma$ δ small enough



$\Rightarrow \Gamma = span_{\mathbb{Z}} \{ \vec{v}_1, \dots, \vec{v}_n \}$

El aplicación,

$\Phi : (\theta_1, \dots, \theta_n) \mapsto \vec{v} = \theta_1 \vec{v}_1 + \dots + \theta_n \vec{v}_n \mapsto \vec{v} \cdot x_o \in \Sigma_f$

identifique Σ_f *con* $\mathbb{T}^n$.

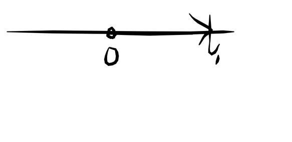
Debido a que $F_1 = H$, *el flujo Hamiltoniana es lineal en estas coordenadas :*

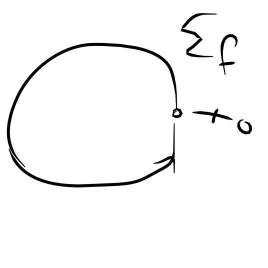
$\theta(t) = \theta_o + \omega_o t$

donde ω_o *depende solo de los valores, f, de F.*

$\mathbb{T}^k \times \mathbb{R}^{n-k}$

$\{ n = 1$





$n = 2$