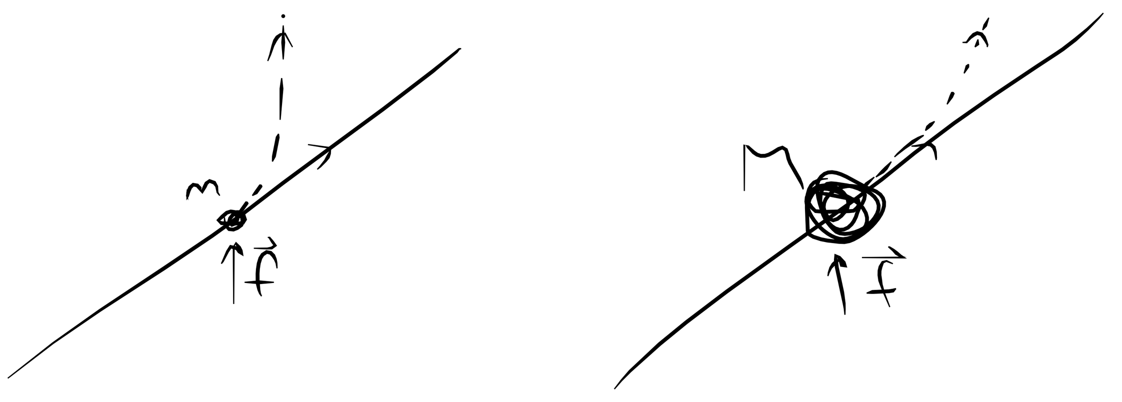
Force

a force influences a object to deviate from its free motion (straight lines in inertial frame).
it is tied to the concept of mass.



applying the same force, f, to two objects of masses $m < M$ has different effects.

Newton's 2nd law:

$$f = m\,a = m\,\ddot{x}$$

$*$ *expressions for f in certain situations are determined by experiments* $*$

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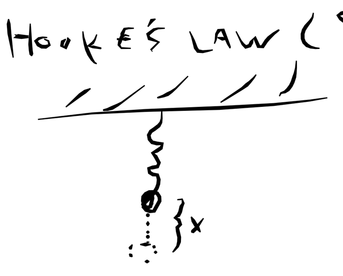
examples:

1. Hooke's law (springs)



x = *displacement from equilibrium*
$m\ddot{x} = -k\,x$, k *spring constant*

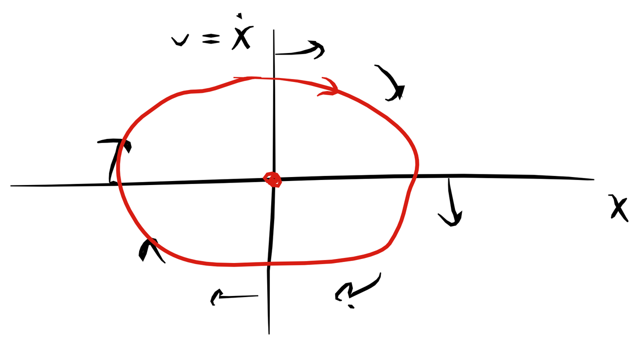
$m\ddot{x} = f(x)$, $f(0) = 0$
Taylor expansion of f around $x = 0$

$m\ddot{x} = f'(0)\,x + O(x^2)$
$f'(0) < 0$, *'stable'*
$f'(0) > 0$, *'unstable'*.

$x(t) = A\cos(\omega t + \phi)$

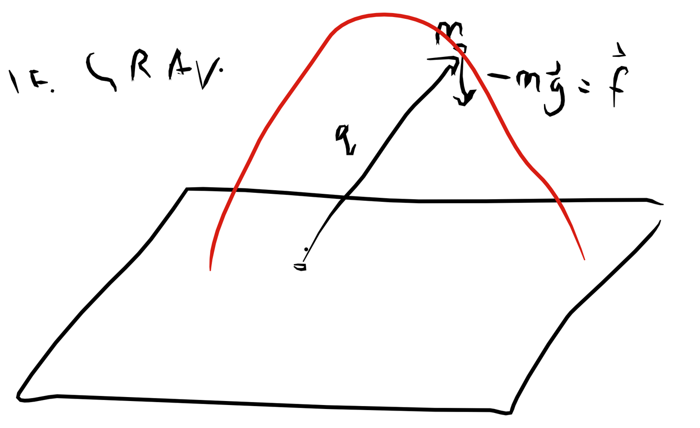
$\omega^2 = \dfrac{k}{m}$

$\dot{v} = -\frac{k}{m}x$
$\dot{x} = v$



$\ddot{x} = -c\,x$
$\ddot{x}\,\dot{x} = -c\,x\,\dot{x}$
$\dfrac{d}{dt}\dfrac{\dot{x}^2}{2} = -c\dfrac{d}{dt}\dfrac{x^2}{2}$
$\dfrac{\dot{x}^2}{2} + c\dfrac{x^2}{2} = k$

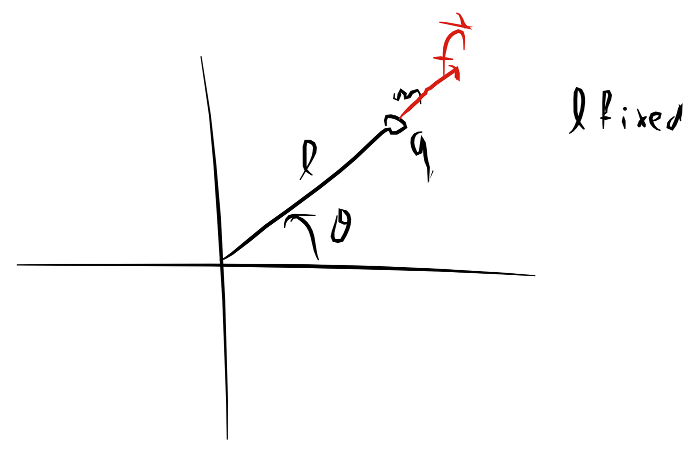
2. unif. grav.



$m\,\ddot{q} = -m\,\vec{g}$

$q(t) = q_o + t\,v_o - \dfrac{t^2}{2}\vec{g}$

Rotational motion



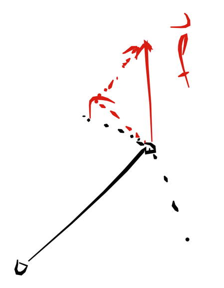
ℓ fixed

$\dot{\theta} = \omega$ (*angular velocity*)
$\ddot{\theta} = \alpha$ (*angular acc.*)

$q = \ell e^{i\theta}$
$\dot{q} = i\omega q$
$\ddot{q} = i\alpha q - \omega^2 q$

what happens if we apply $\vec{f}$ to q?

1. *when $\vec{f} \parallel q$ nothing happens*
2. *for general $\vec{f}$, only the component tangent to the circle has effect on the motion*



$m\,\ddot{q}\cdot iq = \vec{f}\cdot iq$
$m\ell^2\,\alpha = \vec{f}\cdot iq = \tau$ (*is the torque*)

$I = m\ell^2$ *is the moment of inertia*
$C = I\omega = m\,\dot{q}\cdot iq$ *is the angular momentum*

linear motion	*rotational motion*
q	θ
v	ω
a	α
f	τ
m	I
p = mv	C = Iω

$\frac{d}{dt}(\underbrace{mv}_{p}) = f$

$\frac{d}{dt}(\underbrace{I\omega}_{C}) = \tau$

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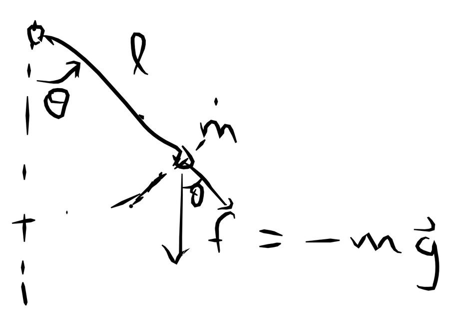
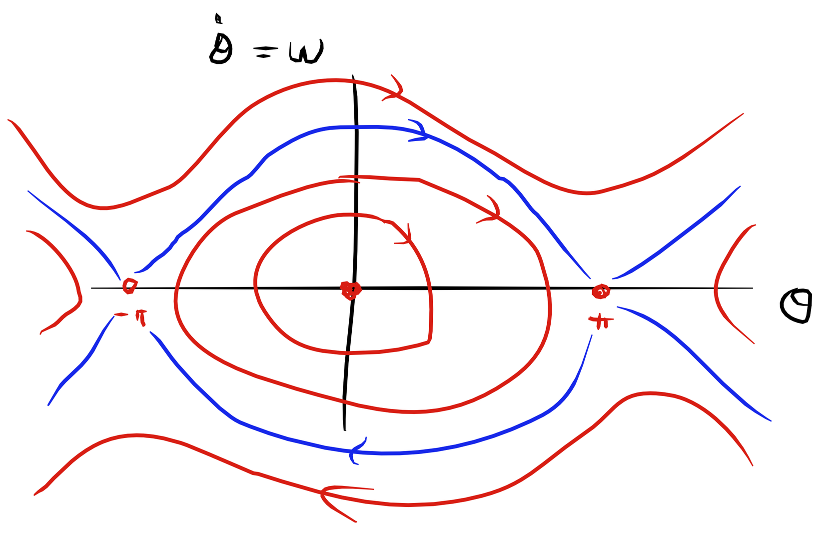
example (pendulum):



$m\ell^2\,\ddot{\theta} = -mg\ell\sin\theta$

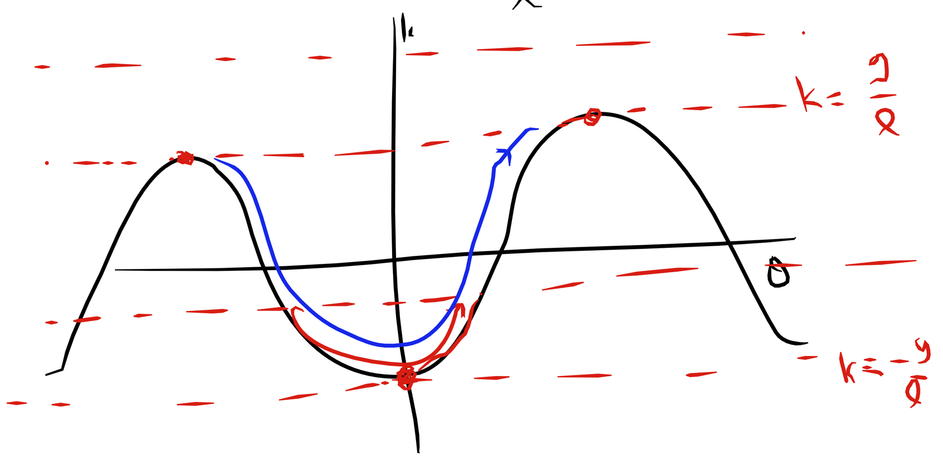
$\ddot{\theta} = -\dfrac{g}{\ell}\sin\theta$

$\dfrac{\omega^2}{2} = \dfrac{g}{\ell}\cos\theta + k$ $\geqslant 0$



position and velocity space = state space or phase space

$k \geqslant -\frac{g}{\ell}\cos\theta$



'effective potential'

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for a general particle moving, we may define its 'instantaneous angular velocity',...

in $\mathbb{R}^3$ replace $\cdot\, i\,q$ with $q\times$.

$q, \dot{q}$, subject to $\vec{f}$

$\vec{C} = q \times m\,\dot{q}$ (*angular momentum*)
$\vec{\tau} = q \times \vec{f}$ (*torque due to $\vec{f}$*)

$\mathbb{I}$ is called the inertia tensor, it is a linear map:
$\mathbb{I}\,\vec{\omega} = \vec{C}$

reminders: $(\vec{a}\times\vec{b})\cdot\vec{c} = \det(\vec{a},\vec{b},\vec{c}) = -(\vec{a}\times\vec{c})\cdot\vec{b}$
$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{c}\cdot\vec{b})\,\vec{a} - (\vec{c}\cdot\vec{a})\,\vec{b}$

$\vec{\omega}\times q = \dot{q}$