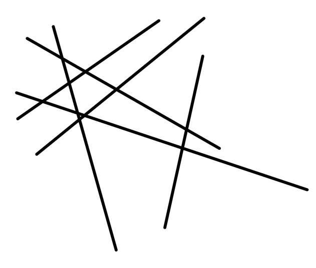
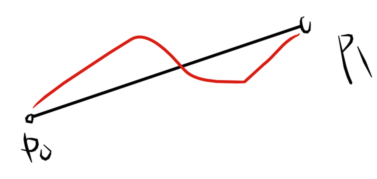
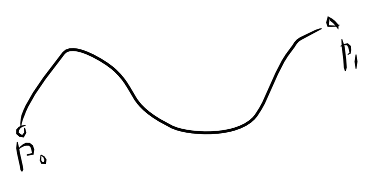
Cálculo variacional



ejemplo : lineas en el plano

podemos caracterizar lineas como :
1. $\ddot{x} = \ddot{y} = 0$
2. $ax + by + c = 0$
3. $\dfrac{d^2y}{dx^2} = 0$
4. *senderos que minimizan distancia entre cualquier dos puntos en la curva.*





$\gamma : \mathbb{R} \to \mathbb{R}^2$ *parametriza una linea*
$\Leftrightarrow$
$\forall\, t_0, t_1 \in \mathbb{R},$
$$\ell_{p_o,p_1}(\gamma) = \int_{t_0}^{t_1} |\dot{\gamma}(t)|\, dt$$
es minimal entre curvas que reunen $p_0 = \gamma(t_0)$ *y* $p_1 = \gamma(t_1)$.

Considera $\gamma \in \Gamma := \left\{ \gamma : [0,1] \to \mathbb{R}^2 \text{ t.q. } \gamma(0) = p_0,\ \gamma(1) = p_1 \right\}$ *y*
$$\gamma \mapsto E(\gamma) := \int_0^1 |\dot{\gamma}(t)|^2\, dt.$$
Entonces :
γ *minimiza* $E \Leftrightarrow \gamma$ *minimiza* ℓ *y* $|\dot{\gamma}| = cst.$

$E(\gamma) \le E(\tilde{\gamma})$
$\forall \tilde{\gamma} \in \Gamma$

para $f,g : [0,1] \to \mathbb{R}$ *suave*
$$\text{pon } < f, g > := \int_0^1 f(t)g(t)\, dt.$$
Es un producto interior y Cauchy – Schwarz es valido :
$< f, g >^2 \le |f|^2|g|^2$
con egalidad solo cuande $f = \lambda g$ *para algún* $\lambda \in \mathbb{R}$

prf: $\ell(\gamma) = \langle |\dot{\gamma}|, 1 \rangle \le E(\gamma) \cdot 1 \qquad {=}' \qquad |\dot{\gamma}| = cst.$

$(\Leftarrow) \qquad E(\gamma) = \ell(\gamma) \le \ell(\tilde{\gamma}) \le E(\tilde{\gamma}).$

$|\dot{\gamma}| = cst. \qquad \forall \tilde{\gamma} \in \Gamma$

$(\Rightarrow)$ ex. $\square$

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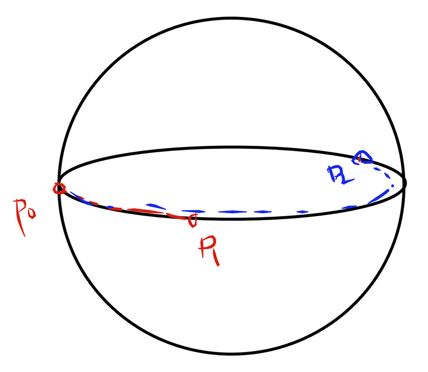
arcos de gran círculos en la esfera



Situación general

Consideramos :

$\Gamma = \{$*algún clase de curvas*$\}$

$A : \Gamma \to \mathbb{R}$ *(una funcional)*

e.g. fixed end pts. fixed time
$\Gamma = \{ \gamma : [0,T] \to \mathbb{R}^n \mid \gamma(0) = p_0,\ \gamma(T) = p_1 \}$

e.g. $A(\gamma) = \int_0^T L(\gamma(t), \dot{\gamma}(t))\, dt$
Lagrangian

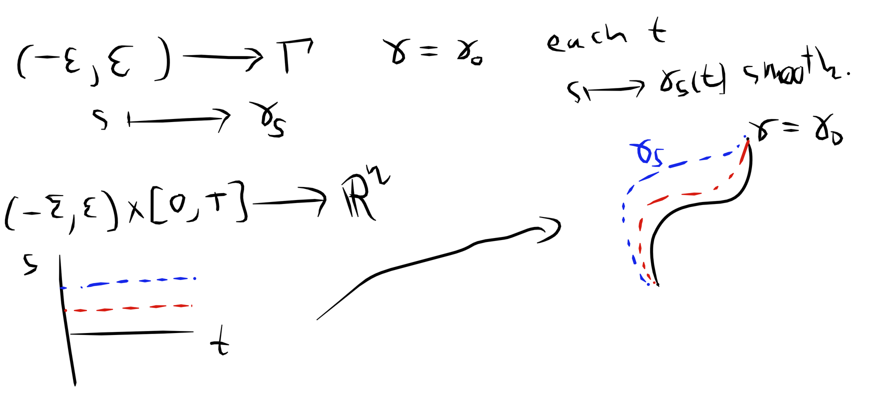
Buscamos extremales de A :

$\gamma \in \Gamma$ *t.q. para cada variación* $\gamma_s \in \Gamma$ *de* γ, *tenemos*

$$\frac{d}{ds}_{s=0} A(\gamma_s) = 0.$$

$(-\varepsilon, \varepsilon) \longrightarrow \Gamma \qquad \gamma = \gamma_0 \qquad$ each t
$s \longmapsto \gamma_s \qquad\qquad s \longmapsto \gamma_s(t)$ smooth.

$(-\varepsilon, \varepsilon) \times [0,T] \longrightarrow \mathbb{R}^n$



$\mathbb{R}^n \xrightarrow{f} \mathbb{R}$

$df_p(v) = \frac{d}{ds}\Big|_0 f(p + sv)$

$*\ \gamma$ *minimiza* $A \Rightarrow \gamma$ *es un extremal de A* $*$

$\not\Leftarrow$

Ecuaciones de Euler – Lagrange

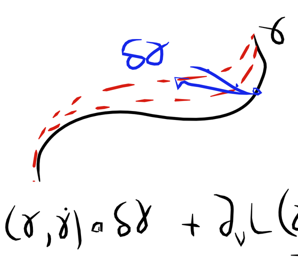
Considera extremales de
$$A(\gamma) = \int_0^T L(\gamma, \dot{\gamma})\, dt$$
sobre curvas con puntos finales y tiempo fijada
$\Gamma = \left\{ \gamma : [0,T] \to \mathbb{R}^n \text{ t.q. } \gamma(0) = p_o, \gamma(T) = p_1 \right\}.$

Entonces :
γ *es un extremal de A*
$\Leftrightarrow$
$$\frac{d}{dt}\partial_v L(\gamma, \dot{\gamma}) = \partial_q L(\gamma, \dot{\gamma})$$

$L : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$
$(q, v) \longmapsto L(q, v)$
$\ \gamma\ \ \dot{\gamma}$

prf: $\gamma_s(t) = \underbrace{\gamma_0(t)}_{\gamma(t)} + s \underbrace{\frac{d}{ds}\Big|_0 \gamma_s(t)}_{\delta\gamma(t)} + O(s^2) \qquad \dot{\gamma}_s = \dot{\gamma} + s\,\dot{\delta\gamma} + O(s^2)$

var. of γ



$\frac{d}{ds}(\gamma_s(0) = p_0)$
$\delta\gamma(0) = \delta\gamma(T) = 0$

$A(\gamma_s) = \int_0^T L(\gamma_s, \dot{\gamma}_s)\, dt$

$\frac{d}{ds}\Big|_0 A(\gamma_s) = \int_0^T \partial_q L(\gamma, \dot{\gamma}) \cdot \delta\gamma + \partial_v L(\gamma, \dot{\gamma}) \cdot \dot{\delta\gamma}\, dt$

I.B.P.

$= \int_0^T \left(\partial_q L - \frac{d}{dt}\partial_v L\right) \cdot \delta\gamma\, dt + \partial_v L \cdot \delta\gamma \Big|_0^T$

0 (fixed endpts.)

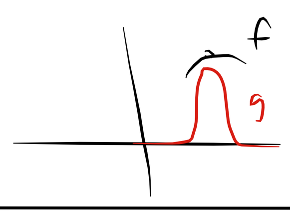
$(\Leftarrow)$ ✓

$(\Rightarrow) \quad 0 = \int_0^T \left(\partial_q L - \frac{d}{dt}\partial_v L\right) \cdot \delta\gamma\, dt \Rightarrow \partial_q L = \frac{d}{dt}\partial_v L \quad \square$

$\forall\, \delta\gamma : [0,T] \longrightarrow \mathbb{R}^n$
$\delta\gamma(0) = \delta\gamma(T) = 0$

$\delta\gamma = g(t) e_1 \qquad \int_0^T g(t) f_1(t)\, dt = 0$

(f cts. s.t. $\int f(t) g(t) = 0\ \forall g$
$\Longleftrightarrow f = 0$



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ejemplo : $\gamma = (x,y) \mapsto \int \dot{x}^2 + \dot{y}^2\, dt = E(\gamma)$

$L(q, v) = v_1^2 + v_2^2$
$\ x, y \quad \dot{x}, \dot{y}$

$\partial_v L = 2(v_1, v_2)$

$\frac{d}{dt}\partial_v L = \partial_q L = 0$
$2\frac{d}{dt}(\dot{x}, \dot{y}) \Rightarrow \ddot{x} = \ddot{y} = 0$

$\begin{cases} \frac{d}{dt}\partial_{\dot{x}} L = \partial_x L \\ 2\ddot{x} = 0 \end{cases}$

En mecánica

edo's en mecánica son de la forma :

$m\ddot{q} = \partial_q U$

$\frac{d}{dt}\partial_v L = \partial_q L$
$mv = m\dot{q}$

$L = \frac{m|v|^2}{2} + U(q)$

$\partial_v L = mv$

$E = \frac{m|v|^2}{2} - U(q)$

Son las E – L eqs. de la Lagrangiana :

$$L = m\frac{|v|^2}{2} + U(q)$$

$E = v \cdot \partial_v L - L$

$L(q, \overset{\dot{q}}{v})$

La energía de la sistema relaciona con la L como :

$$v \cdot \partial_v L - L = E$$

Para una general problema variacional con una Lagrangiana autonomo $(L(q,v))$, *la función :*
$E = v \cdot \partial_v L - L$
queda costante sobre las extremales.

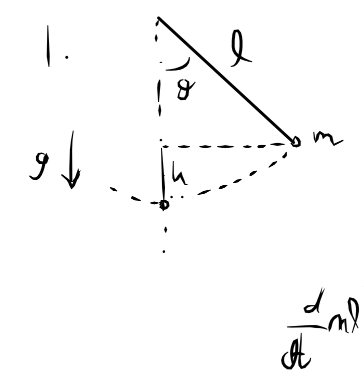
$\frac{d}{dt} E(\gamma, \dot{\gamma}) = \ddot{\gamma} \cdot \partial_v L + \dot{\gamma} \cdot \overset{\frac{d}{dt}\partial_v L}{\partial_q L} - \partial_q L \cdot \dot{\gamma} - \partial_v L \cdot \ddot{\gamma} = 0$

para una sistema $q_1, \ldots, q_N$ *de partículas*
$m_j \ddot{q}_j = \partial_{q_j} U$
son las E – L eqs. de :

$$L = \sum m_j \frac{|v_j|^2}{2} + U(q)$$

Ejemplos

1.



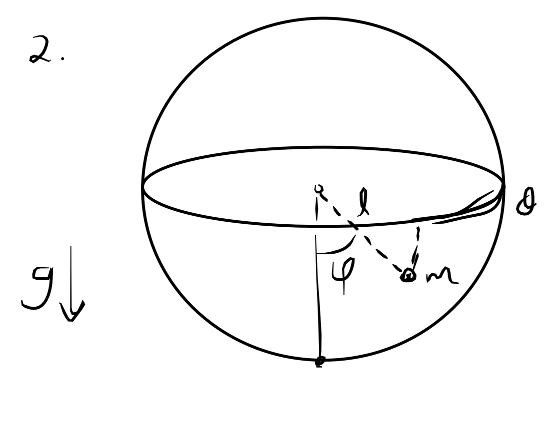
$K = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{m\ell^2}{2}\dot{\theta}^2$

$V = -U = mgh = mg\ell(1 - \cos\theta)$

$L = K + U$

$\frac{d}{dt} m\ell^2\dot{\theta} = \frac{d}{dt}\partial_{\dot{\theta}} L = \partial_\theta L = -mg\ell\, s\theta \Rightarrow \ddot{\theta} = -\frac{g}{\ell}\sin\theta$

2.



$K = \frac{m}{2}\ell^2\left(\dot{\varphi}^2 + s^2\varphi\,\dot{\theta}^2\right)$

$U = mg\ell(\cos\varphi - 1)$

$L = K + U$

$\frac{d}{dt}\partial_{\dot{\theta}} L = \partial_\theta L = 0 \Rightarrow cst. = \partial_{\dot{\theta}} L = m\ell^2 s^2\varphi\,\dot{\theta}$

$E = K - U$ eff. pot'l.